2. If $\frac{1}{10}$ is correctly rounded to the normalized binary number $(1.a_1a_2\ldots a_{23})_2 \times 2^m$, what is the roundoff error? What is the relative roundoff error?

3. **a.** If $\frac{2}{5}$ is correctly rounded to the binary number $(1.a_1a_2\ldots a_{24})_2$, what is the relative roundoff error?

   b. Answer part **a** for the number $\frac{2}{7}$.

4. Is $\frac{2}{3}(1-2^{-24})$ a machine number in the `Marc-32`? Explain.

5. Let $x_1, x_2, \ldots, x_n$ be positive machine numbers in the `Marc-32`. Let S_n denote the sum $x_1 + x_2 + \cdots + x_n$, and let S_n^* be the corresponding sum in the computer. (Assume that the addition is carried out in the order given.) Prove the following: If $x_{i+1} \geq 2^{-24}S_i$ for each i, then

$$|S_n^* - S_n|/S_n \leq (n-1)2^{-24}$$

6. Prove this slight improvement of Inequality (6)

$$\left|\frac{x - x^*}{x}\right| \leq \frac{1}{1 + 2^{24}}$$

   for the representation of numbers in the `Marc-32`.

7. How many normalized machine numbers are there in the `Marc-32`? (Do not count 0.)

8. Does each machine number in the `Marc-32` have a unique normalized representation?

9. Let $x = (1.11\ldots111000\ldots)_2 \times 2^{16}$, in which the fractional part has 26 1's followed by 0's. For the `Marc-32`, determine x_-, x_+, $\text{fl}(x)$, $x - x_-$, $x_+ - x$, $x_+ - x_-$, and $|x - \text{fl}(x)|/x$.

10. Let $x = 2^3 + 2^{-19} + 2^{-22}$. Find the machine numbers on the `Marc-32` that are just to the right and just to the left of x. Determine $\text{fl}(x)$, the absolute error $|x - \text{fl}(x)|$, and the relative error $|x - \text{fl}(x)|/|x|$. Verify that the relative error in this case does not exceed 2^{-24}.

11. Find the machine number just to the right of $1/9$ in a binary computer with a 43-bit normalized mantissa.

12. What is the exact value of $x^* - x$, if $x = \sum_{n=1}^{26} 2^{-n}$ and x^* is the nearest machine number on the `Marc-32`?

13. Let $S_n = x_1 + x_2 + \cdots + x_n$, where each x_i is a machine number. Let S_n^* be what the machine computes. Then $S_n^* = \text{fl}(S_{n-1}^* + x_n)$. Prove that on the `Marc-32`,

$$S_n^* \approx S_n + S_2\delta_2 + \cdots + S_n\delta_n \qquad |\delta_k| \leq 2^{-24}$$

14. Which of these is not necessarily true on the `Marc-32`? (Here x, y, and z are machine numbers and $|\delta| \leq 2^{-24}$.)

    a. $\text{fl}(xy) = xy(1+\delta)$

    b. $\text{fl}(x+y) = (x+y)(1+\delta)$

    c. $\text{fl}(xy) = xy/(1+\delta)$

    d. $|\text{fl}(xy) - xy| \leq |xy|2^{-24}$

    e. $\text{fl}(x+y+z) = (x+y+z)(1+\delta)$

15. Use the `Marc-32` for this problem. Determine a bound on the relative error in computing $(a \cdot b)/(c \cdot d)$ for machine numbers a, b, c, d.

16. Are these machine numbers in the `Marc-32`?

    a. 10^{40}

    b. $2^{-1} + 2^{-26}$

c. $\frac{1}{5}$

d. $\frac{1}{3}$

e. $\frac{1}{256}$

17. Let $x = 2^{16} + 2^{-8} + 2^{-9} + 2^{-10}$. Let x^* be the machine number closest to x in the Marc-32. What is $|x - x^*|$?

18. Criticize the following argument: When two machine numbers are combined arithmetically in the Marc-32, the relative roundoff error cannot exceed 2^{-24}. Therefore, when n such numbers are combined, the relative roundoff error cannot exceed $(n-1)2^{-24}$.

19. Let $x = 2^{12} + 2^{-12}$.

    a. Find the machine numbers x_- and x_+ in the Marc-32 that are just to the left and right of x, respectively.

    b. For this number show that the relative error between x and $\text{fl}(x)$ is no greater than the unit roundoff error in the Marc-32.

20. What relative roundoff error is possible in computing the product of n machine numbers in the Marc-32? How is your answer changed if the n numbers are not necessarily machine numbers (but are within the range of the machine)?

21. Give examples of real numbers x and y for which $\text{fl}(x \odot y) \neq \text{fl}(\text{fl}(x) \odot \text{fl}(y))$. Illustrate all four arithmetic operations, using a machine with five decimal digits.

22. When we write $\prod_{i=1}^{n}(1+\delta_i) = 1 + \varepsilon$, where $|\delta_i| \leq 2^{-24}$, what is the range of possible values for ε? Is $|\varepsilon| \leq n2^{-24}$ a realistic bound?

23. Suppose that numbers $z_1, z_2, \ldots$ are computed from data $x, a_1, a_2, \ldots$ by means of the algorithm

$$\begin{cases} z_1 = a_1 \\ z_n = x z_{n-1} + a_n \quad (n \geq 2) \end{cases}$$

    (This is **Horner's algorithm.**) Assume that the data are machine numbers. Show that the z_n produced in the computer are the numbers that would result from applying exact arithmetic to perturbed data. Bound the perturbation in terms of the machine's unit roundoff error.

24. The quantity $(1+\varepsilon)^n - 1$ occurs in the theorem of this section. Prove that if $n\varepsilon < 0.01$, then $(1+\varepsilon)^n - 1 < 0.01006$.

25. Establish Equations (7) and (8) from the hypotheses given in the text.

26. How many floating-point numbers are there between successive powers of 2 in the Marc-32?

27. What numbers other than positive integers can be used as a base for a number system? For example, can we use π? (See, for example, Rousseau [1995].)

28. What is the unit roundoff error for a binary machine carrying 48-bit mantissas?

29. What is the unit roundoff error for a *decimal* machine that allocates 12 decimal places to the mantissa? Such a machine stores numbers in the form $x = \pm r \times 10^n$ with $\frac{1}{10} \leq r < 1$.

30. Prove that 4/5 is not representable exactly on the Marc-32. What is the closest machine number? What is the relative roundoff error involved in storing this number on the Marc-32?

31. What numbers are representable with a finite expression in the binary system but are not finitely representable in the decimal system?

32. What can be said of the relative roundoff error in adding n machine numbers? (Make no assumption about the numbers being positive because this case is covered by a theorem in the text.)