A scheme that is dissipative of order $2r$ is also said to have dissipation of order $2r$. Similar to the observation at the end of Section 2.2, we note that (5.1.2) is equivalent to

$$|g_\nu(\theta)|^2 \leq 1 - c' \sin^{2r} \tfrac{1}{2}\theta. \tag{5.1.3}$$

We also note that for the most general case, in which we cannot use the restricted stability condition (2.2.7) but must use the general condition of (2.2.6), the estimate (5.1.2) must be replaced by

$$|g_\nu(\theta)| \leq \left(1 - c \sin^{2r} \tfrac{1}{2}\theta\right)(1 + Kk), \tag{5.1.4}$$

and similarly for (5.1.3).

As the definition shows, to decide if a given scheme is dissipative, we need consider only $|g(\theta)|$. For the Lax-Wendroff scheme we have

$$|g(\theta)|^2 = 1 - 4a^2\lambda^2\left(1 - a^2\lambda^2\right)\sin^4 \tfrac{1}{2}\theta.$$

So for $|a\lambda| = 1$ we have $|g(\theta)| = 1$, but for $0 < |a\lambda| < 1$ the scheme is dissipative of order 4. In fact, for $\theta = \pi$, we have $g(\theta) = 1 - 2a^2\lambda^2$, as our example showed.

Many texts give the definition of dissipation by replacing the expression $\sin^{2r}\theta$ in (5.1.2) by $|\theta|^{2r}$, with $\theta$ restricted in magnitude to less than $\pi$. The definitions are equivalent; we prefer the form (5.1.2), since that is the form that actually occurs in evaluating $|g(\theta)|$ for most schemes.

The leapfrog scheme (1.3.4) and the Crank-Nicolson scheme (3.1.3) are called *strictly* nondissipative schemes because their amplification factors are identically 1 in magnitude. The Lax-Friedrichs scheme (1.3.5) and the backward-time central-space scheme (1.6.1) and the $(2,2)$ implicit scheme (4.2.3) are nondissipative but not strictly nondissipative. For each of these schemes $|g(\pi)|$ and $|g(0)|$ are 1, but $|g(\theta)|$ is less than 1 for all other values of $\theta$. These schemes reduce the magnitude of most frequencies but not the highest frequency on the grid.

Dissipation can be added to any nondissipative scheme, as we will show, and this provides us with some control over the properties of the scheme. In adding dissipation to a nondissipative scheme, we must be careful not to affect the order of accuracy adversely. For example, the modified leapfrog scheme

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\delta_0 v_m^n + \frac{\varepsilon}{2k}\left(\frac{1}{2}h\delta\right)^4 v_m^{n-1} = f_m^n \tag{5.1.5}$$

is a second-order accurate scheme for $u_t + au_x = f$ for small values of $\varepsilon$. The amplification factors are

$$g_\pm = -ia\lambda\sin\theta \pm \sqrt{1 - a^2\lambda^2\sin^2\theta - \varepsilon\sin^4\tfrac{1}{2}\theta}.$$

If $\varepsilon$ is not greater than $1 - a^2\lambda^2$, the scheme is stable and dissipative of order 4 (see Exercise 5.1.2). Note that $\sin^{2r}\frac{1}{2}\theta$ is the symbol of $\left(\frac{1}{2}ih\delta\right)^{2r}$. Similarly, the Crank-Nicolson scheme (3.1.3) can be modified as

$$\frac{v_m^{n+1} - v_m^n}{k} + \frac{a}{2}\delta_0\left(v_m^{n+1} + v_m^n\right) + \frac{\varepsilon}{k}\left(\frac{1}{2}h\delta\right)^4 v_m^n = \frac{1}{2}\left(f_m^{n+1} + f_m^n\right). \tag{5.1.6}$$

This scheme is second-order accurate and dissipative of order 4 for small values of $\varepsilon$.

To show that any scheme can have dissipation added to it, we consider the amplification polynomial and modify it as in formula (4.3.11). To be more precise the scheme corresponding to

$$\Phi^{\varepsilon}(g,\theta) = \Phi(g,\theta) + \varepsilon \sin^{2r} \tfrac{1}{2}\theta \, g\Phi'(g,\theta) \tag{5.1.7}$$

will have all roots inside the unit circle except at $\theta$ equal to 0. ( $\Phi'(g,\theta)$ is the derivative of $\Phi$ with respect to $g$. ) Another choice for a dissipative scheme is

$$\Phi^{\varepsilon}(g,\theta) = \Phi(g,\theta) + \varepsilon \sin^{2r} \tfrac{1}{2}\theta \left[g\Phi'(g,\theta) - n\Phi(g,\theta)\right]. \tag{5.1.8}$$

The preceding general procedures are not always advisable to use, but it does give one guidance in adding dissipation to a scheme (see Exercises 5.1.3 and 5.1.4).

If we use the methods of Section 4.3, then we can determine if the scheme is dissipative by checking if the amplification polynomial is a Schur polynomial for values of $\theta$ other than $\theta$ equal to 0. For a dissipative scheme the amplification polynomial is a Schur polynomial for $\theta$ not equal to zero.

## EXERCISES 5.1

**1.** Show that the scheme (5.1.6) is dissipative of order 4 and stable if $0 < \varepsilon < 2$.

**2.** Show that the modified leapfrog scheme (5.1.5) is stable for $\varepsilon$ satisfying

$$0 < \varepsilon \le 1 \quad \text{if} \quad 0 < a^2\lambda^2 \le \tfrac{1}{2}$$

and

$$0 < \varepsilon \le 4a^2\lambda^2\left(1 - a^2\lambda^2\right) \quad \text{if} \quad \tfrac{1}{2} \le a^2\lambda^2 < 1.$$

**3.** Construct the modified scheme corresponding to formula (5.1.7) using the multistep scheme (4.2.3). Compare this scheme with

$$\frac{3v_m^{n+1} - 4v_m^n + v_m^{n-1}}{2k} + a\delta_0 v_m^{n+1} = \frac{\varepsilon}{2k}\left(\frac{i}{2}h\delta\right)^{2r} v_m^{n-1}.$$

**4.** Construct the leapfrog scheme with added dissipation using the method given by formula (5.1.8). Compare this scheme with the scheme (5.1.5).

**5.** Construct the Crank-Nicolson scheme with added dissipation using the method given by formulas (5.1.7) and (5.1.8). Compare these schemes with each other and with the scheme (5.1.6).

**6.** Show that the scheme of Exercise 3.2.6 is dissipative of order 6 for

$$0 < |a\lambda| < \left(\frac{\sqrt{17}-1}{6}\right)^{1/2}.$$