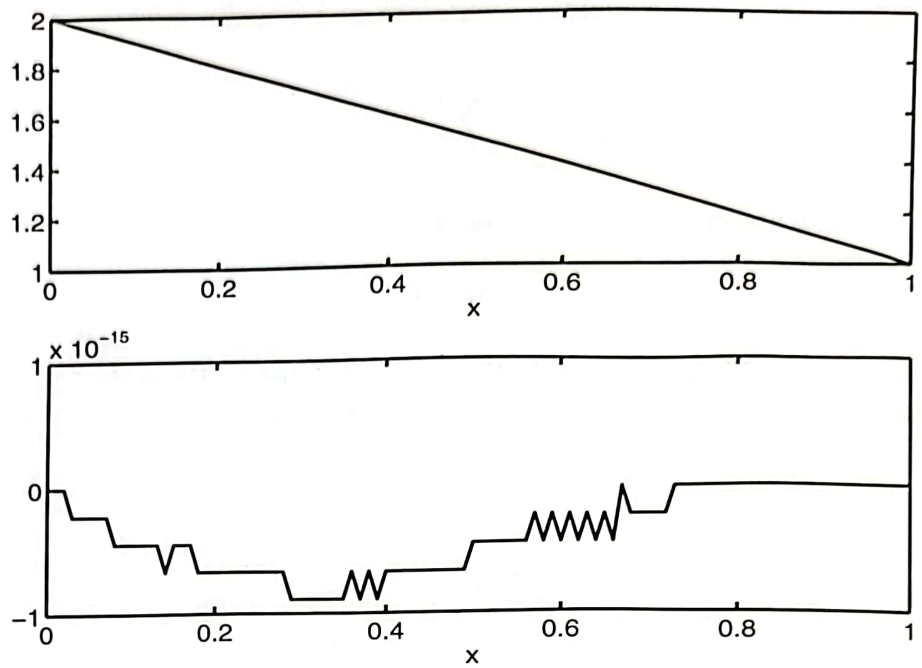

Figure 5.17. *Piecewise linear finite element approximation, with* $n = 5$ *subintervals in the mesh, to the solution of* $-\frac{d}{dx}\left((x+1)\frac{du}{dx}\right) = 1, u(0) = 2, u(1) = 1$. *The error shown in the bottom graph is due only to round-off error—the "approximation" is in fact equal to the exact solution.*

Exercises

1. (a) Using direct integration, find the exact solution to the BVP from Example 5.21.

   (b) Repeat the calculations from Example 5.21, using an increasing sequence of values of n. Produce graphs similar to Figure 5.16, or otherwise measure the errors in the approximation. Try to identify the size of the error as a function of h.[34]

2. (a) Compute the exact solution to the BVP from Example 5.22.

   (b) Explain why, for this particular BVP, the finite element method computes the exact, rather than an approximate, solution.

3. Use piecewise linear finite elements and a regular mesh to solve the following

[34] To perform these calculations for n much larger than 5 will require the use of a computer. One could, of course, write a program, in a language such as Fortran or C, to do the calculations. However, there exist powerful interactive software programs that integrate numerical calculations, programming, and graphics, and these are much more convenient to use. MATLAB is particularly suitable for the finite element calculations needed for this book, since it is designed to facilitate matrix computations. *Mathematica* and *Maple* are other possibilities.

problem:

$$-\frac{d^2u}{dx^2} = x^2,$$
$$u(0) = 0,$$
$$u(1) = 0.$$

Use $n = 10, 20, 40, \ldots$ elements, and determine (experimentally) an exponent p so that the error is

$$O\left(\frac{1}{n^p}\right).$$

Here error is defined to be the maximum absolute difference between the exact and approximate solutions. (Determine the exact solution by integration; you will need it to compute the error in the approximations.)

4. Repeat Exercise 3 for the BVP

$$-\frac{d}{dx}\left((1+x^2)\frac{du}{dx}\right) = 1,\ 0 < x < 1,$$
$$u(0) = 0,$$
$$u(1) = 0.$$

The exact solution is

$$u(x) = \frac{2\ln 2}{\pi}\tan^{-1}(x) - \frac{1}{2}\ln(1+x^2).$$

5. Derive the weak form of the BVP

$$-\frac{d}{dx}\left(k(x)\frac{du}{dx}\right) + p(x)u = f(x),\ 0 < x < \ell,$$
$$u(0) = 0,$$
$$u(\ell) = 0,$$

where k and p satisfy $k(x) > 0$, $p(x) > 0$ for $x \in [0, \ell]$. What is the bilinear form for this BVP?

6. Repeat Exercise 3 for the BVP

$$-\frac{d^2u}{dx^2} + 2u = \frac{1}{2} - x,$$
$$u(0) = 0,$$
$$u(1) = 0.$$

The results of the last exercise will be required. The exact solution is

$$u(x) = c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x} + \frac{1}{4} - \frac{x}{2},$$