The Lax–Friedrichs scheme (1.3.5) and the backward-time central-space scheme (1.6.1) and the (2, 2) implicit scheme (4.2.3) are nondissipative but not strictly so. For example, the Lax–Friedrichs scheme has

$$|g(\theta)|^2 = \cos^2\theta + a^2\lambda^2\sin^2\theta$$

(see (2.2.12)), and since $|g(\pi)| = 1$, this scheme is not dissipative.

For the Lax–Friedrichs and backward-time central-space schemes, $|g(\theta)|$ is less than 1 for most values of θ. These schemes reduce the magnitude of most frequencies but not the highest frequency on the grid.

Adding Dissipation to Schemes

Dissipation can be added to any nondissipative scheme, as we will show, and this provides us with some control over the properties of the scheme. In adding dissipation to a nondissipative scheme, we must be careful not to affect the order of accuracy adversely. For example, the modified leapfrog scheme

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\delta_0 v_m^n + \frac{\varepsilon}{2k}\left(\frac{1}{2}h\delta\right)^4 v_m^{n-1} = f_m^n \qquad (5.1.6)$$

is a second-order accurate scheme for $u_t + au_x = f$ for small values of ε. Notice that $(\sin\frac{1}{2}\theta)^{2r}$ is the symbol of $(\frac{1}{2}ih\delta)^{2r}$.

The amplification factors are

$$g_\pm(\theta) = -ia\lambda\sin\theta \pm \sqrt{1 - a^2\lambda^2\sin^2\theta - \varepsilon\sin^4\tfrac{1}{2}\theta}.$$

If ε is small enough, then the scheme is stable and dissipative of order 4 (see Exercise 5.1.2) and satisfies

$$|g_\pm(\theta)|^2 = 1 - \varepsilon\sin^4\tfrac{1}{2}\theta\ .$$

Similarly, the Crank–Nicolson scheme (3.1.3) can be modified as

$$\frac{v_m^{n+1} - v_m^n}{k} + \frac{a}{2}\delta_0\left(v_m^{n+1} + v_m^n\right) + \frac{\varepsilon}{k}\left(\frac{1}{2}h\delta\right)^4 v_m^n = \frac{1}{2}\left(f_m^{n+1} + f_m^n\right). \qquad (5.1.7)$$

This scheme is second-order accurate and dissipative of order 4 for small values of ε.

Figures 5.1 and 5.2 show the effect of adding dissipation to the leapfrog scheme. The solution is the propagation of a simple piecewise linear pulse. Notice that the dissipation removes most of the oscillations to the left of the pulse. It does not remove the larger oscillation behind the pulse. This oscillation is inherent in higher order schemes, as discussed after Theorem 3.1.4.

To show that any scheme can have dissipation added to it, we consider the amplification polynomial and modify it as in formula (4.3.14). To be more precise, the scheme corresponding to

$$\Phi^\varepsilon(g,\theta) = \Phi(g,\theta) + \varepsilon(\sin\tfrac{1}{2}\theta)^{2r}\, g\Phi'(g,\theta) \qquad (5.1.8)$$

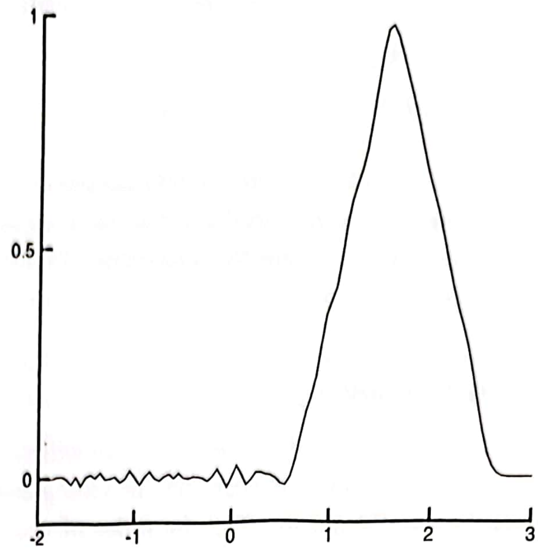

Figure 5.1. *The leapfrog scheme with no dissipation added.*

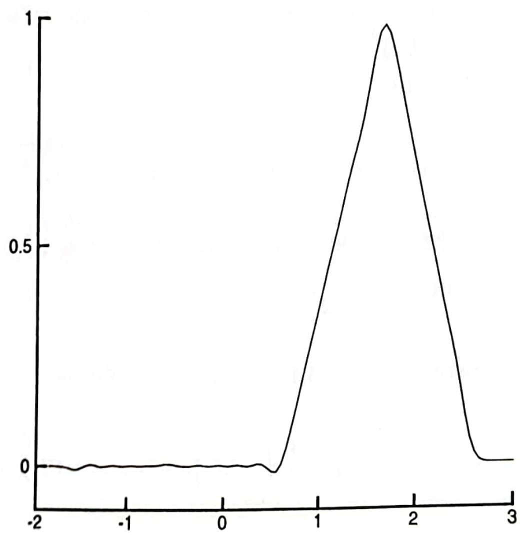

Figure 5.2. *The leapfrog scheme with dissipation of* $\varepsilon = 0.5$.

will have all roots inside the unit circle except at θ equal to 0. ($\Phi'(g,\theta)$ is the derivative of Φ with respect to g.) Another choice for a dissipative scheme is

$$\Phi^{\varepsilon}(g,\theta) = \Phi(g,\theta) + \varepsilon(\sin\tfrac{1}{2}\theta)^{2r}\left[g\Phi'(g,\theta) - d\Phi(g,\theta)\right], \qquad (5.1.9)$$

where d is the degree of $\Phi(g,\theta)$. The preceding general procedures are not always advisable to use, but they do give one guidance in adding dissipation to a scheme (see Exercises 5.1.3 and 5.1.4).

If we use the methods of Section 4.3, then we can determine if the scheme is dissipative by checking if the amplification polynomial is a Schur polynomial for values of θ other than θ equal to 0. For a dissipative scheme the amplification polynomial is a Schur polynomial when θ is not equal to zero.

Exercises

5.1.1. Show that the scheme (5.1.7) is dissipative of order 4 and stable if $0 < \varepsilon < 2$.

5.1.2. Show that the modified leapfrog scheme (5.1.6) is stable for ε satisfying

$$0 < \varepsilon \leq 1 \quad \text{if} \quad 0 < a^2\lambda^2 \leq \tfrac{1}{2}$$

and

$$0 < \varepsilon \leq 4a^2\lambda^2\left(1 - a^2\lambda^2\right) \quad \text{if} \quad \tfrac{1}{2} \leq a^2\lambda^2 < 1.$$

Note that these limits are not sharp. It is possible to choose ϵ larger than these limits and still have the scheme be stable.

5.1.3. Construct the modified scheme corresponding to formula (5.1.8) using the multistep scheme (4.2.3). Compare this scheme with

$$\frac{3v_m^{n+1} - 4v_m^n + v_m^{n-1}}{2k} + a\delta_0 v_m^{n+1} = \frac{\varepsilon}{2k}\left(\frac{i}{2}h\delta\right)^{2r} v_m^{n-1}.$$

5.1.4. Construct the leapfrog scheme with added dissipation using the method given by formula (5.1.9). Compare this scheme with the scheme (5.1.6).

5.1.5. Construct the Crank–Nicolson scheme with added dissipation using the method given by formulas (5.1.8) and (5.1.9). Compare these schemes with each other and with the scheme (5.1.7).

5.1.6. Show that the scheme of Exercise 3.3.5 is dissipative of order 6 for

$$0 < |a\lambda| < \left(\frac{\sqrt{17} - 1}{6}\right)^{1/2}.$$

5.1.7. Show that the scheme (3.3.16) is dissipative of order 4 if $0 < |a\lambda| < 3$.

5.2 Dispersion

To introduce the idea of dispersion we look again at equation (1.1.1) and notice that we can write the solution as

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} e^{-i\omega at} \hat{u}_0(\omega)\, d\omega. \tag{5.2.1}$$

From this we conclude that the Fourier transform of the solution satisfies

$$\hat{u}(t+k,\omega) = e^{-i\omega ak}\hat{u}(t,\omega). \tag{5.2.2}$$