Math 6340 Midterm Exam 3/7/18
Dr. Minkoff

Instructions: You may not use notes or books on this exam. Don't spend too much time on any one problem. Show your work!

NAME:

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[10 pts](1a) For which numbers b is the matrix

$$A = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix}$$

symmetric positive definite?

All upper left submatrices must have positive determinants. $1>0$.

$$(1)(9) - b^2 > 0 \Rightarrow 9 - b^2 > 0 \Rightarrow 9 > b^2 \Rightarrow \boxed{|b| < 3}$$

(b) If $b = 2$ in the matrix above, find the Cholesky factor R in the decomposition $A = R^t R$.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix}$$

$$\boxed{R = \begin{bmatrix} 1 & 2 \\ 0 & \sqrt{5} \end{bmatrix}}$$

Check: $R^t R = \begin{bmatrix} 1 & 0 \\ 2 & \sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & \sqrt{5} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix}$ ✓

[20 pts](2) Let

$$A = \begin{bmatrix} 1 & 1000 \\ 0 & 1 \end{bmatrix}.$$

(a) Calculate A^{-1} and the condition number $k_\infty(A)$.

$$A^{-1} = \begin{bmatrix} 1 & -1000 \\ 0 & 1 \end{bmatrix}$$

$\|A\|_\infty =$ max row sum (A)

so $\|A\|_\infty = 1001$

$\|A^{-1}\|_\infty = 1001$

$$K_\infty(A) = (1001)^2 = 1{,}002{,}001$$

so $K_\infty(A) \approx 1 \times 10^6$

(b) Find the solution to the system $Ax = b$ for the two right hand sides:

$$b = \begin{bmatrix} 1000 \\ 1 \end{bmatrix}$$

and

$$b + \delta b = \begin{bmatrix} 1000 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 & -1000 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1000 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x + \delta x = \begin{bmatrix} 1 & -1000 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1000 \\ 0 \end{bmatrix} = \begin{bmatrix} 1000 \\ 0 \end{bmatrix}$$

(c) Calculate $\|\delta b\|_\infty/\|b\|_\infty$ and $\|\delta x\|_\infty/\|x\|_\infty$ for the two right hand sides in Part (b) above. Carefully explain Part (b) in light of Part (a).

$$\|\delta b\|_\infty = \left\|\begin{bmatrix}0\\1\end{bmatrix}\right\|_\infty = 1 \qquad \Rightarrow \quad \frac{\|\delta b\|_\infty}{\|b\|_\infty} = \frac{1}{1000}$$

$$\|b\|_\infty = \left\|\begin{bmatrix}1000\\1\end{bmatrix}\right\|_\infty = 1000$$

$$\|\delta x\|_\infty = \left\|\begin{bmatrix}1000\\-1\end{bmatrix}\right\|_\infty = 1000 \qquad \Rightarrow \quad \frac{\|\delta x\|_\infty}{\|x\|_\infty} = 1000$$

$$\|x\|_\infty = \left\|\begin{bmatrix}0\\1\end{bmatrix}\right\|_\infty = 1$$

So $$\frac{\|\delta x\|}{\|x\|} \le K(A)\,\frac{\|\delta b\|}{\|b\|}$$

$$\Rightarrow \quad 1000 \le 10^6\left(10^{-3}\right) = 10^3 \checkmark$$

(d) If we solve the system $Ax = b$ using Gaussian Elimination with partial pivoting on a machine with 8 digits of accuracy, what does the condition number in Part (a) tell us about the number of digits we can trust in the solution x?

$s = 8$ digits accuracy

$K(A) = 10^6 \Rightarrow 6 = t$

$s - t$ accurate digits

$\Rightarrow$ $\boxed{\text{we can trust 2 digits in soln}}$

[9 pts](3) Show that the following operation in the back substitution algorithm for solving $Rx = b$ is backwards stable:

$$\tilde{x_1} = b_1 \oslash r_{11}.$$

Specifically you should show that the floating point algorithm gives exactly the right answer to a nearby problem $(R + \delta R)x = b$. (Here $\oslash$ is a floating point divide.)

By the Fundamental Axiom of Floating Point Arithmetic, every floating point arithmetic operation is exact up to a relative error of at most ϵ_{mach}. So $\tilde{x}_1 = \frac{b_1}{r_{11}}(1+\varepsilon_1)$ where $|\varepsilon_1| \le \epsilon_{mach}$. We want all the "data perturbations to occur on the matrix entries of R. So use a truncated Taylor series approximation to define $\varepsilon_1' = \frac{-\varepsilon_1}{1+\varepsilon_1}$ [Aside: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $-1 < x < 1$. If $x = -\varepsilon_1$ then $\sum_{n=0}^{\infty}(-\varepsilon_1)^n = \frac{1}{1+\varepsilon_1}$ or $-\varepsilon_1 \sum_{n=0}^{\infty}(-\varepsilon_1)^n = \frac{-\varepsilon_1}{1+\varepsilon_1} \Rightarrow -\varepsilon_1 \approx \frac{-\varepsilon_1}{1+\varepsilon_1} \equiv \varepsilon_1'$.] Then $\frac{b_1}{r_{11}}(1+\varepsilon_1) = \frac{b_1}{r_{11}\left(\frac{1}{1+\varepsilon_1}\right)}$

Please sign the following honor statement: *On my honor, I pledge that I have neither given nor received any aid on this exam.*

$$= \frac{b_1}{r_{11}\left(\frac{1+\varepsilon_1-\varepsilon_1}{1+\varepsilon_1}\right)} = \frac{b_1}{r_{11}\left(1-\frac{\varepsilon_1}{1+\varepsilon_1}\right)} = \frac{b_1}{r_{11}(1+\varepsilon_1')}$$

So $\tilde{x}_1$ is exactly the correct solution to the Perturbed Problem $(r_{11} + \delta r_{11})\tilde{x}_1 = b_1$

[12 pts](4) Apply the Gram-Schmidt process to

$$a_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \qquad a_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

and write the result in the form $A = QR$.

By Gram-Schmidt, $a = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ and so

$$q_1 = \frac{a_1}{\|a_1\|} \qquad \|a_1\|_2 = \sqrt{1^2+2^2+(-2)^2} = \sqrt{9} = 3$$

$$\text{so } q_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \text{ or } \boxed{q_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}}$$

$$a_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \qquad a_2' = a_2 - (q_1^t a_2) q_1$$

$$(q_1^t a_2) = \begin{bmatrix} 1/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = 1/3 - 2/3 - 8/3 = -9/3 = -3$$

$$\text{so } a_2' = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - (-3)\begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$\|a_2'\| = \sqrt{2^2+1^2+2^2} = \sqrt{9} = 3 \text{ so } q_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

$$\boxed{Q = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \\ -2/3 & 2/3 \end{bmatrix}} \qquad R = \begin{bmatrix} q_1^t a_1 & q_1^t a_2 \\ 0 & q_2^t a_2 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix}} = R$$

$$q_1^t a_1 = \begin{bmatrix} 1/3 & 2/3 & -2/3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{3} + \frac{4}{3} + \frac{4}{3} = 3 \qquad q_2^t a_2 = \begin{bmatrix} 2/3 & 1/3 & 2/3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = 2/3 - 1/3 + 8/3 = 3$$

[9 pts](5) Solve the least squares problem $Ax = b$ where

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}.$$

and

$$b = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

using the QR decomposition from the previous problem.

Least Squares for $Ax=b$, use normal eqns

$A^tAx = A^tb \qquad A=QR \Rightarrow (QR)^t(QR)x = (QR)^tb$

$\Rightarrow R^tQ^tQRx = R^tQ^tb \quad Q$ orthogonal

$\Rightarrow R^tRx = R^tQ^tb = \boxed{Rx = Q^tb}$

$\in \mathbb{R}^{2\times3} \qquad \in \mathbb{R}^3$

$$Q^tb = \begin{bmatrix} \frac{1}{3} & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Rx = Q^tb \Rightarrow \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow 3x_2 = 0 \Rightarrow$$

$x_2 = 0$

$3x_1 - 3x_2 = 1 \Rightarrow 3x_1 - 3(0) = 1 \Rightarrow 3x_1 = 1$

$x_1 = 1/3$ so $\boxed{\vec{x} = \begin{bmatrix} 1/3 \\ 0 \end{bmatrix}}$

[15 pts](6a) **Derive** the computational cost of the forward elimination portion of Gaussian Elimination. (Note: just stating a final cost without derivation is not sufficient.)

Forward elimination includes division by the pivot to get multiplier and "multiply-subtract" combination. Call each division & each mult-subtraction a single op. 1st eqn of length n takes n operations for every zero we achieve in 1st column. $n-1$ rows underneath the first so first stage of elimination takes $n(n-1) = n^2 - n$ ops. (or: all entries in n^2 matrix change except for n in 1st row). When elimination is down to k eqns we need $k^2 - k$ ops. Total ops is $(1^2 + 2^2 + \dots + n^2) - (1 + 2 + \dots + n) = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = \frac{n^3 - n}{3}$. If n is large count is $O(\frac{n^3}{3})$

(b) Approximately how many total steps does elimination take to solve 10 systems with the same 60x60 coefficient matrix A?

For forward elimination cost to reduce $A = LU$ is $O(n^3) \approx (60^3) = 216{,}000$ flops

Back substitution is $O(60^2) \approx 10(60^2)$
$\approx 36{,}000$

So total cost to solve 10 systems is about

$$\begin{array}{r} 216{,}000 \\ +\ 36{,}000 \\ \hline 252{,}000 \end{array}$$ flops

[10 pts](7) Prove that

$$||A||_1 = max_{1\leq j\leq n}\sum_{i=1}^{n}|a_{ij}|$$

Proof:

$$A = \left[a_1 \,\middle|\, a_2 \,\middle|\, \dots \,\middle|\, a_n\right]$$

where each a_j is a column vector of length n.

Unit ball for one norm is set $\{x \in \mathbb{R}^n : \sum_{j=1}^{n} |x_j| \leq 1\}$

Any vector Ax in image of this set satisfies

$$\|Ax\|_1 = \|\sum_{j=1}^{n} x_j a_j\|_1 \leq \sum_{j=1}^{n} |x_j| \, \|a_j\|_1$$

linear combination of columns of A.

scalar

vector

$$\leq \max \|a_j\|_1 \sum_{j=1}^{n} |x_j| \leq \max_{1\leq j\leq n} \|a_j\|_1$$

If $x = e_j$ where j maximizes $\|a_j\|_1$, then bound is attained and $\|A\|_1 = \max_{1\leq j\leq n} \|a_j\|_1$

$\blacksquare$