Hwk 1 solutions, out of 44 marks

**Question 1 – 5 marks**

\[ \sum_{i=1}^{N} h(i) \chi_i(x_1) \chi_j(x_2) \cdots \chi_k(x_N) \]  

\[ = [h(1) \chi_1(x_1)] \chi_j(x_2) \cdots \chi_k(x_N) + \chi_i(x_1)[h(2) \chi_j(x_2)] \chi_l(x_3) \cdots \chi_k(x_N) + \chi_i(x_1) \chi_j(x_2) \cdots [h(N) \chi_k(x_N)] \]  

\[ = [\epsilon_i \chi_i(x_1)] \chi_j(x_2) \cdots \chi_k(x_N) + \chi_i(x_1) [\epsilon_j \chi_j(x_2)] \chi_l(x_3) \cdots \chi_k(x_N) + \chi_i(x_1) \chi_j(x_2) \cdots [\epsilon_k \chi_k(x_N)] \]  

\[ = (\epsilon_i + \epsilon_j + \cdots + \epsilon_k) \chi_i(x_1) \chi_j(x_2) \cdots \chi_k(x_N) \]  

**Question 2 – 8 marks**

**a) – 5 marks** We must begin with the definition of a derivative:

\[ f'(x_0) \equiv \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \]  

To show linearity, we must establish that

\[ \hat{A}(af + bg) = a\hat{A}f + b\hat{A}g \]  

where \( \hat{A} = d/dx \), and where a, b are constants, and f, g are functions f(x), g(x). Consider the point \( x = x_0 \).  

\[ \frac{d}{dx} (af + bg) = \lim_{h \to \infty} \frac{(af + bg)(x_0 + h) - (af + bg)(x_0)}{h} \]  

\[ = \lim_{h \to \infty} \frac{1}{h} (af(x_0 + h) + bg(x_0 + h) - af(x_0) - bg(x_0)) \]  

\[ = \lim_{h \to \infty} \frac{1}{h} (af(x_0 + h) - af(x_0) + bg(x_0 + h) - bg(x_0)) \]  

\[ = a \lim_{h \to \infty} \frac{f(x_0 + h) - f(x_0)}{h} + b \lim_{h \to \infty} \frac{g(x_0 + h) - g(x_0)}{h} \]  

\[ = af'(x_0) + bg'(x_0) \]  

which completes the proof.

**b – 2 marks** We must show that, for \( A = V \),

\[ \hat{A}(af + bg) = a\hat{A}f + b\hat{A}g \]
But

\[ \hat{V}(af + bg) = V \ast (af + bg) \]  

(13)
since \( \hat{V} \) is just multiplication by \( V \).

\[ = aVf + bVg \]  

(14)
since multiplication is commutative.

**c - 1 mark** We must show that, for \( A = B + C \),

\[ \hat{A}(af + bg) = a\hat{A}f + b\hat{A}g \]  

(15)

This is straightforward. Let us start with the right hand side:

\[
a(B + C)f + b(B + C)g = a(Bf + Cf) + b(Bg + Cg) = aBf + aCf + bBg + bCg \\
= (B + C)(af) + (B + C)(bg) = (B + C)(af + bg) \]  

(16)

3) – 18 points

3a) (3 points)

In \( R^2 \), consider the basis \( \{ e_1 = (1, 0), \ e_2 = \frac{1}{\sqrt{2}}(-1, 1) \} \).

**Solution:** This basis is normalized but not orthogonal. Need \( x = c_1(1, 0) + c_2(1/\sqrt{2})(-1, 1) = (a, b) \). Forming the dot product on both sides with \( e_1 \) yields \( c_1 = a + c_2/\sqrt{2} \).

Forming the dot product with \( e_2 \) yields \(-a/\sqrt{2} + b/\sqrt{2} = -c_1/\sqrt{2} + c_2 \). Plugging in \( c_1 \) from the first expression gives \( c_2 = \sqrt{2}b \). Finally, \( c_1 = a + b \). Putting \( a = 2, b = 1 \) gives \( (2, 1) = 3e_1 + \sqrt{2}e_2 \)

![FIG. 1:](image-url)
3b) (3 points)

In $\mathbb{R}^2$, consider the basis $\{e_1 = \frac{1}{\sqrt{2}}(1, 1), \ e_2 = \frac{1}{\sqrt{2}}(1, -1)\}$.

Solution: This basis is orthonormal. $x = c_1e_1 + c_2e_2 \Rightarrow c_1 = x \cdot e_1 = (a, b) \cdot 1/\sqrt{2}(1, 1) = 1/\sqrt{2}(a + b)$. Similarly for $c_2$ we get $c_2 = 1/\sqrt{2}(a - b)$. Putting $a = 2, b = 1$ gives $(2, 1) = 3/\sqrt{2}e_1 + 1/\sqrt{2}e_2$

![Diagram showing basis vectors](image)

3c) (9 points)

The eigenfunctions are

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$$

We wish to express the function $f(x)$ in terms of this basis, which means we are looking for the $c_n$ coefficients in

$$f(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$

From orthonormality we have $\psi_j(x) \cdot f(x) = c_j$, so that

$$c_n = \int_0^L f(x) \psi_n(x) dx = \int_0^{L/2} x \psi_n(x) dx + \int_{L/2}^L (-x + L) \psi_n(x) dx$$

Evaluation of these integrals yields the final result of

$$f(x) = \sum_{n=1}^{\infty} \frac{4L}{n^2 \pi^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{L}$$

See Fig. 3 for an illustration of this series.

3d) (3 points) Is the function $f(x)$ a solution to the time-independent Schrödinger equation (for the square well problem we are considering here)? (5 points)
Solution: No. Consider the first two terms in the series of equation (20). We can write these two terms as \( \psi_{12} = c_1 \psi_1 + c_2 \psi_2 \), where \( \psi_1 \) and \( \psi_2 \) are both solutions to the TISE, namely \( H \psi_1 = E_1 \psi_1 \) and \( H \psi_2 = E_2 \psi_2 \), and we note that \( E_1 \neq E_2 \). Now, for \( \psi_{12} \) to be a solution to the TISE, we would need \( H \psi_{12} = E \psi_{12} \) for some constant \( E \). But we know that \( H \psi_{12} = c_1 E_1 \psi_1 + c_2 E_2 \psi_2 \), and since \( E_1 \neq E_2 \) we can’t write this as some constant times the original function. In class we saw that if we combine more than one eigenfunction, we get a time-dependent wavefunction.

**Question 4 – 4 points**

To show something is an operator, let us demonstrate how it acts:

\[
A|\xi\rangle = |\psi\rangle \langle \phi|\xi\rangle = c|\psi\rangle
\]  

(21)

where \( c = \langle \phi|\xi\rangle \). So we give \( A \) a function, and it returns a function. This means that \( A \) is an operator.
Question 5 – 5 points

What bra should you associate with the ket $\lambda_1|\phi_1\rangle + \lambda_2|\phi_2\rangle$?

The bra of this is such that for any ket $|\psi\rangle$, the bra acting on $|\psi\rangle$ returns the scalar product

$$(\lambda_1|\phi_1\rangle + \lambda_2|\phi_2\rangle, |\psi\rangle)$$

$$= \int (\lambda_1\phi_1 + \lambda_2\phi_2)^* \psi = \int (\lambda_1^*\phi_1^*\psi + \lambda_2^*\phi_2^*\psi)$$

$$= \lambda_1^*\int \phi_1^*\psi + \lambda_2^*\int \phi_2^*\psi$$

so that the bra is

$$\lambda_1^*\langle \phi_1 | + \lambda_2^*\langle \phi_2 |$$

(25)

Question 6 – 4 points

$$\phi(r) = \sum_i b_i u_i(r) \quad (26)$$

$$\psi(r) = \sum_j c_j u_j(r) \quad (27)$$

$$(\phi, \psi) = \langle \sum_i b_i^* u_i, \sum_j c_j u_j \rangle = \sum_k b_k^* c_k$$

(28)

where we are using the orthonormality property of the basis $\{u_i\}$. 