Orthogonal Equipartitions

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Abstract

Consider two absolutely continuous probability measures in the plane. A subdivision of the plane into $k \geq 2$ regions is *equitable* if every region has weight $1/k$ in each measure. We show that, for any two probability measures in the plane and any integer $k \geq 2$, there exists an equitable subdivision of the plane into k regions using at most $k-1$ horizontal segments and at most $k-1$ vertical segments.

We also prove the existence of orthogonal equipartitions for point measures and present an efficient algorithm for computing an orthogonal equipartition.

Key words: Equipartition, point measure, absolutely continuous measure

1 Introduction

Our investigation is motivated by recent equipartitions studied by Kano, Kawano, and Uno [6]. Consider a finite set of red points and a finite set of blue points on the grid $\mathbb{Z}^2$. A *semi-vertical* (*semi-horizontal*) segment consists of two vertical (horizontal) line segments connected by a horizontal (vertical) line segment of length one. A *semi-rectangular chain* consists of two segments, one semi-vertical and one semi-horizontal, emanating from the same point. They proved that, if both the number of red points and the number of blue points is even, then there exists a semi-rectangular bisector avoiding red and blue points, see Fig. 1.

In this paper we study orthogonal equipartitions of both continuous and point measures. In the continuous case, we make some assumptions about the measures (that will be discussed in Section 2) to make proofs easier. For a measure μ in the plane, we denote by $\mu(A)$ the μ-measure of a measurable set $A \subseteq \mathbb{R}^2$. The main result is the following theorem.

Theorem 1 (Equitable orthogonal partition, continuous version) *For any integer $k \geq 2$ and any two absolutely continuous (with respect to the*

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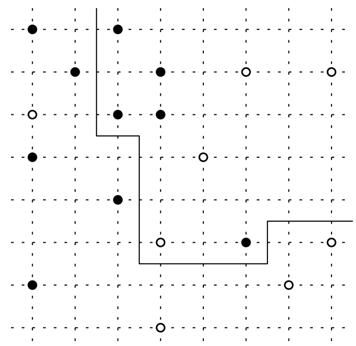


(a)

Fig. 1. A semi-rectangular bisector of red and blue points on the plane.

Lebesgue measure) finite Borel measures μ_1 and μ_2 in the plane, there is a subdivision of the plane into k regions $R_1, R_2, \ldots, R_k$ using at most $k-1$ horizontal segments and at most $k-1$ vertical segments such that, for every $i = 1, 2$, μ_i-measures of all regions are equal, i.e. $\mu_i(R_j) = \mu_i(\mathbb{R}^2)/k$ for all $j = 1, 2, \ldots, k$.

The special case for $k = 2$ is easy to prove by showing that two measures can be simultaneously halved by an “L”-shaped bisector. The proof is similar to the proof of the ham sandwich theorem in the plane [7] in the fact that it follows from applications of the intermediate value theorem for continuous functions. The difference is that the ham sandwich theorem guarantees a single line which simultaneously bisects both the measures. The result for $k = 2$ is also related to equipartitions of three measures using 2-fans [2; 4]. The bisector here can be viewed as a 2-fan where the two semi-infinite rays are horizontal/vertical.

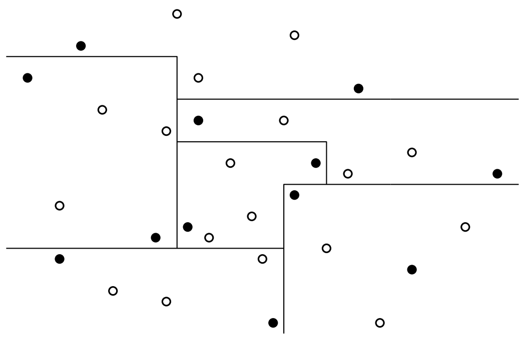


Fig. 2. An equitable orthogonal partition of 12 black points and 18 white points. The partition into $k = 6$ regions uses 5 horizontal and 3 vertical segments.

For point measures we consider two cases: arbitrary point sets and points in general position. In both cases we count points in the interior of every region. For two arbitrary point sets S_1 and S_2 we show that the existence of a partition such that the number of points from $S_i, i = 1, 2$ in each region is bounded from above by $\lfloor |S_i|/k \rfloor$.

Theorem 2 (Equitable orthogonal partition, discrete version) *For any integer $k \geq 2$ and any set of $n \geq 1$ red points and any set of $m \geq 1$ blue points in the plane, there exists a subdivision of the plane into k regions $R_1, R_2, \ldots, R_k$ using at most $k-1$ horizontal segments and at most $k-1$ vertical segments such that the interior (open set) of every $R_i, i = 1, 2, \ldots, k$ contains at most $\lfloor n/k \rfloor$ red points and $\lfloor m/k \rfloor$ blue points. If all $n + m$ points are in general position, then the subdivision has the property that the interior of every $R_i, i = 1, 2, \ldots, k$ contains exactly $\lfloor n/k \rfloor$ red points and $\lfloor m/k \rfloor$ blue points.*

If the number of red/blue points is multiple of k then we have the following.

Corollary 3 *For any kn red and km blue points in general position in the plane, there exists a subdivision of the plane into k regions with at most $k-1$ horizontal segments and at most $k-1$ vertical segments such that every region contains n red points and m blue points.*

Figure 2 illustrates such a subdivision. It should be noted that the horizontal and vertical segments in Theorems 1 and 2 and Corollary 3 are not crossing, i.e. the intersection of any two segments s_1 and s_2 is either the empty set or a vertex of s_1 or s_2. The general position assumption in the second claim of Theorem 3 can be weakened to the condition of distinct x-coordinates and distinct y-coordinates of the points.

The orthogonality is not essential in Theorems 1 and 2. For any two linearly independent vectors u and v in the plane, there exist equipartitions as in Theorems 1 and 2 using line segments parallel to u and v. This can be shown by changing measures using a coordinate transformation.

In Section 3 we prove the existence of a balanced partition of two measures into two regions. It is used then to prove Theorem 1. In Section 4 we consider point measures and follow the ideas from Section 1. We prove the existence of a balanced partition in two cases: arbitrary point sets and points in general position. Then we prove Theorem 2. We also show that an equitable orthogonal subdivision can be computed in $O((n+m)\log k)$ time. Finally we discuss the degenerate case of point layout and show the existence of a partition with count as for points in general position.

2 Preliminaries

In this paper we assume that the continuous measures are absolutely continuous Borel probability measures in the plane. Following [2; 3] we make a stronger assumption in the proof of Theorem 1 that the measures are *nice*, i.e., the measures are absolutely continuous with respect to the Lebesgue measure and such that any nonempty open set has a strictly positive measure. This simplifies the proof and the restrictions can be justified using standard arguments (measure approximation and compactness), see for example [2, Lemma 3.1] and [3].

The following combinatorial result is useful for partitioning measures into many regions.

Theorem 4 ([5, Theorem 9]) *For any map* $s : \{1, 2, \dots, k-1\} \to \{\pm 1\}$, *there exists a pair* (k_1, k_2) *or a triple* (k_1, k_2, k_3) *with sum* k *and the same signs* $s(k_i)$ *such that all* $k_i \leq \lfloor 2k/3 \rfloor$.

For an efficient computation, we need a slightly stronger result [1].

Theorem 5 *For any map* $s : \{1, 2, \dots, k-1\} \to \{\pm 1\}, k \geq 2$, *one of the following holds*

(i) *There exists a pair* (a, b) *such that* $a + b = k, s(a) = s(b)$ *and* $a, b \leq \lfloor 2k/3 \rfloor$.
(ii) *There exists a triple* (a, b, c) *such that* $a + b + c = k, s(a) = s(b) = s(c)$ *and* $\lceil (k-1)/6 \rceil \leq a, b, c \leq \lfloor 2k/3 \rfloor$.

PROOF. If k is even then the couple $(k/2, k/2)$ satisfies (i). If k is multiple of 3 then the triple $(k/3, k/3, k/3)$ satisfies (ii). It remains to consider two cases $k = 6m + 1$ and $k = 6m - 1$.

Case 1. Suppose that $k = 6m + 1$. Without loss of generality we assume that $s(2m) = -1$.

Suppose that $s(m) = s(m+1) = \cdots = s(2m) = -1$ and $s(2m+1) = s(2m+2) = \cdots = s(3m) = 1$. If $s(4m) = 1$ then the couple $(2m+1, 4m)$ satisfies (i). Otherwise, the triple $(m, m+1, 4m)$ satisfies (ii).

In the remaining case there is an $i \in \{1, 2, \dots, m\}$ such that

(a) $s(2m - i + 1) = s(2m - i + 2) = \cdots = s(2m) = -1$ and $s(2m+1) = s(2m+2) = \cdots = s(2m + i - 1) = 1$, and
(b) $s(2m - i) = 1$ or $s(2m + i) = -1$.

If $s(2m + i) = -1$ then the triple $(2m - i + 1, 2m, 2m + i)$ satisfies (ii). If $s(2m - i) = 1$ and $s(2m + i) = 1$ then the triple $(2m - i, 2m + 1, 2m + i)$ satisfies (ii).

[1] The algorithm in [5] computes 3-cuttings and Theorem 5 would be useful there.

Case 2. Suppose that $k = 6m - 1$. The proof is similar to the above case. Without loss of generality we assume that $s(2m) = 1$.

Suppose that $s(m) = s(m+1) = \cdots = s(2m-1) = -1$ and $s(2m) = s(2m+1) = \cdots = s(3m) = 1$. If $s(4m-1) = 1$ then the couple $(2m, 4m-1)$ satisfies (i). Otherwise, the triple $(m, m, 4m-1)$ satisfies (ii).

In the remaining case there is an $i \in \{1, 2, \ldots, m\}$ such that

(a) $s(2m-i+1) = s(2m-i+2) = \cdots = s(2m-1) = -1$ and $s(2m) = s(2m+1) = \cdots = s(2m+i-1) = 1$, and
(b) $s(2m-i) = 1$ or $s(2m+i) = -1$.

If $s(2m-i) = 1$ then the triple $(2m-i, 2m, 2m+i-1)$ satisfies (ii). If $s(2m+i) = -1$ and $s(2m-i) = -1$ then the triple $(2m-i, 2m-1, 2m+i)$ satisfies (ii). $\square$

3 Partitions of Continuous Measures

Let λ and μ be nice measures in the plane. For a measurable set $A \subset \mathbb{R}^2$, its λ-*weight* is denoted as $\lambda(A)$. In particular, $\lambda(\mathbb{R}^2) = \mu(\mathbb{R}^2) = 1$. For a non-vertical line l, we denote by l^+ and l^- the upper and the lower halfplane of l, respectively.

First, we develop a tool for partitioning the measures λ and μ into two regions and then prove Theorem 1.

3.1 Γ-partition

A Γ-*partition* of the plane is formed either

(i) by a horizontal line, or
(ii) by a vertical line, or
(iii) by a horizontal ray and a downward-oriented ray emanating from a common point.

Let a and b be positive integers. A partition of the plane into two regions A and B is called (a, b)-*balanced with respect to a measure* λ if $\lambda(A) = a/(a+b)$ and $\lambda(B) = b/(a+b)$. A partition of the plane into two regions A and B is called (a, b)-*balanced* if it is (a, b)-balanced with respect to both λ and μ. A (a, b)-balanced partition is *equitable* if $a = b$.

Lemma 6 *For any two measures in the plane, there exists an equitable* Γ*-partition.*

PROOF. We define a region $R(x)$ for all $x \in \mathbb{R}$ as follows. Let $x = x_0$ be the equation of the halving line for μ, i.e. $\mu(\{p \mid p_x \geq x_0\}) = 1/2$. We assign $R(x_0) := \{p \mid p_x \geq x_0\}$. For every $x < x_0$, we assign $R(x) := \{p \mid p_x \geq x, p_y \leq y\}$ where y is chosen such that $\mu(R(x)) = 1/2$. For every $x > x_0$, we assign $R(x) := \{p \mid p_x \geq x \text{ or } p_y \geq y\}$ where y is chosen such that $\mu(R(x)) = 1/2$, see Fig. 3. Note that a unique y exists (in both cases) since μ is nice.

We define a function $f : \mathbb{R} \to [0, 1]$ as $f(x) = \lambda(R(x))$. The function $f(x)$ is continuous since μ is nice.

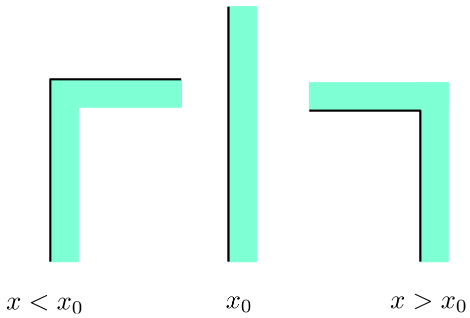


Fig. 3. Γ-partitions used to define $f(x)$. The regions $R(x)$ are shaded.

Let l be the horizontal halving line for μ, i.e. $\mu(l^+) = 1/2$. Put $t := \lambda(l^+)$. The line l makes an equitable Γ-partition if $t = 1/2$. Otherwise, there exists a $z \in \mathbb{R}$ such that $f(z) = 1/2$ since $f(x)$ is a continuous function and $\lim_{x \to -\infty} f(x) = \lambda(l^-) = 1 - t$ and $\lim_{x \to +\infty} f(x) = \lambda(l^+) = t$. Therefore the partition of the plane into $R(z)$ and $\mathbb{R}^2 - R(z)$ is an equitable Γ-partition. $\square$

In general, for a constant $r \in (0, 1)$, we define a region $R(r, x), x \in \mathbb{R}$ as follows. Let $x = x_0$ be the equation of a unique vertical line such that $\mu(\{p \mid p_x \geq x_0\}) = r$. We assign $R(r, x_0) := \{p \mid p_x \geq x_0\}$. For every $x < x_0$, we assign $R(r, x) := \{p \mid p_x \geq x, p_y \leq y\}$ where y is chosen such that $\mu(R(r, x)) = r$. For every $x > x_0$, we assign $R(r, x) := \{p \mid p_x \geq x \text{ or } p_y \geq y\}$ where y is chosen such that $\mu(R(r, x)) = r$.

We define a function $f(r, x) : \mathbb{R} \to [0, 1]$ as $f(r, x) = \lambda(R(r, x))$. Again $f(r, x)$ is a continuous function of x.

Theorem 7 *For any two measures in the plane and an integer $k \geq 2$, there exists an $(i, k - i)$-balanced Γ-partition for some integer i such that $i, k - i \leq (5k + 1)/6$.*

PROOF. If k is even then a $(k/2, k/2)$-balanced Γ-partition exists by Lemma 6. Suppose that k is odd and is at least three. We borrow some ideas from [5]: we will define a sequence of signs, find a triple (a, b, c) of numbers with the same signs (using Theorem 5), construct a 3-cutting with one downward-oriented ray such that μ is split proportional to (a, b, c).

Let $y = y_i$, $i = 1, 2, \ldots, k-1$ be the equation of the horizontal line l_i such that $\mu(l_i^+) = 1/k$. The theorem follows if, for some $\lceil (k-1)/6 \rceil \leq i \leq \lfloor 2k/3 \rfloor$, the partition by the line l_i is $(i, k-i)$-balanced. We assume that, for all these i, the partition by l_i is not $(i, k-i)$-balanced. Then, for each i, the λ-weight of l_i^+ is less than or greater than i/k. We assign the sign $s(i) := -1$ or 1, respectively. By Theorem 5, there is a pair (a, b) or a triple (a, b, c) with sum k and the same signs. Suppose that there is a pair (a, b) with $a + b = k$, $s(a) = s(b)$ and $a, b \leq \lfloor 2k/3 \rfloor$. Without loss of generality $s(a) = s(b) = 1$. Thus $\lambda(l_a^+) > a/k$ and $\lambda(l_b^+) > b/k$. Note that $\lambda(l_b^-) = 1 - \lambda(l_b^+) < a/k$. Since

$$\lim_{x \to +\infty} f\left(\frac{a}{k}, x\right) = \lambda(l_a^+) > \frac{a}{k} \text{ and } \lim_{x \to -\infty} f\left(\frac{a}{k}, x\right) = \lambda(l_b^-) < \frac{a}{k}$$

there is a real number x such that $f(a/k, x) = a/k$. Therefore a $(a, k-a)$-balanced Γ-partition exists. The theorem follows since $a, k - a \leq (5k+1)/6$.

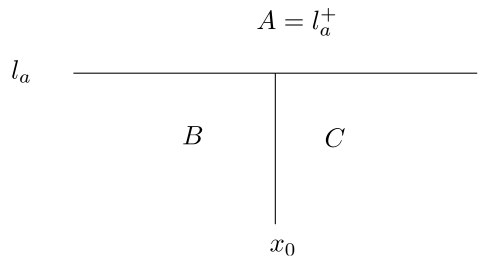


Fig. 4. A 3-cutting of the plane with μ-weights $a/k, b/k$, and c/k.

Suppose that there is a triple (a, b, c) with $a + b + c = k$, the same signs and $\lceil (k-1)/6 \rceil \leq a, b, c \leq \lfloor 2k/3 \rfloor$. Without loss of generality $s(a) = s(b) = s(c) = 1$. Consider a 3-cutting of the plane into three regions $A = l_a^+, B, C$ using the line l_a and a downward-oriented ray along the line with the equation $x = x_0$ defined as follows. The halfplane l_a^- has μ-weight $1 - a/k = (b+c)/k$. The vertical ray of the 3-cutting divides it into two regions B and C with $\mu(B) = b/k$ and $\mu(C) = c/k$, see Fig. 4. Since $\lambda(A) + \lambda(B) + \lambda(C) = \lambda(\mathbb{R}^2) = 1$ and $\lambda(A) > a/k$ one of the following inequalities holds: $\lambda(B) < b/k$ or $\lambda(C) < c/k$.

If $\lambda(B) < b/k$ then $f(1 - b/k, x_0) = 1 - \lambda(B) > 1 - b/k$. Since

$$\lim_{x \to -\infty} f(1 - b/k, x) = \lambda(l_b^-) = 1 - \lambda(l_b^+) < 1 - b/k$$

there is a real number $x \in (-\infty, x_0)$ such that $f(1 - b/k, x) = 1 - b/k$. Therefore a $(b, a+c)$-balanced Γ-partition exists. The theorem follows since $b, a + c \leq (5k+1)/6$.

If $\lambda(C) < c/k$ then $f(c/k, x_0) < c/k$. Since $\lim_{x \to +\infty} f(c/k, x) = \lambda(l_c^+) > c/k$ then there is a real number $x \in (x_0, \infty)$ such that $f(c/k, x) = c/k$. Therefore a $(c, a+b)$-balanced Γ-partition exists. The theorem follows since $c, a+b \leq (5k+1)/6$. $\square$

3.2 Subdivision into Many Regions

We recursively apply Theorem 7 to obtain an orthogonal equitable partition. The key idea is to show the existence of a shape invariant.

An orthogonal polygonal chain infinite in both directions is *descending* if it can be traversed using only downward and rightward directions. An orthogonal polygonal chain infinite in both directions is *ascending* if it can be traversed using only upward and rightward directions. Let d and a be descending and ascending orthogonal chains, respectively. Let d^+ and a^+ be the locus of points above the corresponding chains (above and to the right of d; above and to the left of a). Let l be a horizontal line. A *V-region* is defined as the intersection of any subset of $\{d^+, a^+, l^-\}$. In particular, d^+, a^+ and l^- are V-regions.

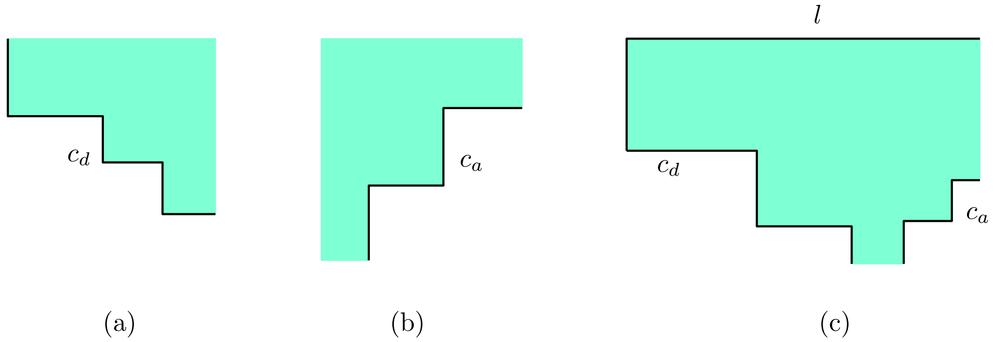


Fig. 5. (a) A V-region d^+. (b) A V-region a^+. (c) A V-region bounded by the chains d and a and the line l.

One property of V-regions is their connectivity – every V-region is a connected set. We show another useful property.

Lemma 8 *The intersection of two V-regions is a V-region.*

PROOF. It is straightforward to check that, for any two descending orthogonal chains d_1 and d_2, there a descending orthogonal chain d such that the intersection $d_1^+ \cap d_2^+ = d^+$. Similarly, for any two ascending orthogonal chains a_1 and a_2, there an ascending orthogonal chain a_3 such that the intersection $a_1^+ \cap a_2^+ = a^+$.

In general, let R_1 and R_2 be two V-regions. Let d_i, a_i, and l_i be the descending/ascending orthogonal chain and the horizontal line of R_i for each $i = 1, 2$.

Let d and a be the descending and ascending orthogonal chains such that $d^+ = d_1^+ \cap d_2^+$ and $a^+ = a_1^+ \cap a_2^+$. Let l be the horizontal line such that $l^- = l_1^- \cap l_2^-$. Then the intersection $R_1 \cap R_2 = d^+ \cap a^+ \cap l^-$ is a V-region. □

Lemma 9 *Let R be a V-region and (A, B) be a Γ-partition of the plane. Then $V \cap A$ and $V \cap B$ are V-regions.*

PROOF. It follows from Lemma 8 and the fact that A and B are V-regions. □

The main result, Theorem 1, follows from Theorem 7 and Lemma 9. It can be shown by induction on k. It is obvious if $k = 1$. Suppose that it holds for any number of regions from 1 to $k - 1$. We show that it is true for k regions. By Theorem 7 there exists an $(i, k - i)$-balanced Γ-partition of the plane into V-regions A and B. By the induction hypothesis there exists an equitable partition of the plane into i regions for the measures reduced to A. By Lemma 9 they intersect A by V-regions. Similarly we partition B into $k-i$ regions. The total number of V-regions is k and the theorem follows.

4 Point Measures

We will use point measures in this Section and consider two cases of any point sets and points in general position. First, we prove Theorem 2 and then provide an efficient algorithm for computing an equitable orthogonal partition.

4.1 Proof of Theorem 2

Theorem 2 deals with two finite sets of points colored in red and blue. Let a and b be positive integers. A partition of the plane into two regions A and B is called *(a, b)-balanced with respect to a finite set S* if $|A \cap S|/|S| \leq \lfloor a/(a + b) \rfloor$ and $|B \cap S|/|S| \leq \lfloor b/(a+b) \rfloor$. A partition of the plane into two regions A and B is called *(a, b)-balanced* if it is (a, b)-balanced with respect to the set of each color (red and blue). An (a, b)-balanced partition is *equitable* if $a = b$.

Using the idea of replacing points by disks in the plane (see for example [8] and [7], page 49), one can reduce Theorem 7 to its discrete version.

Lemma 10 (A balanced Γ-partition for point measures) *For any integer $k \geq 2$, any set of $n \geq 1$ red points and any set of $m \geq 1$ blue points in the plane, there exists a $(i, k - i)$-balanced Γ-partition for some integer i such that $i, k - i \leq (5k + 1)/6$.*

The first claim of Theorem 2 follows from the inequality

$$\left\lfloor \left\lfloor \frac{na}{k} \right\rfloor \frac{b}{a} \right\rfloor \le \left\lfloor \frac{nb}{k} \right\rfloor \tag{1}$$

where $a \in \{1, 2, \dots, k-1\}$ and $b \in \{1, 2, \dots, a-1\}$. Indeed, suppose that a region R obtained by a sequence of Γ-partitions and corresponds to the (a/k)-portion. If we use a Γ-partition for b next and obtain a region R' corresponding to b, then the upper bound of the number of red points in the interior of R' is the same as b is used in one Γ-partition for all red and blue points (by Inequality (1)). We apply a Γ-partition for every region corresponding to $b > 1$. In the end, the number of red points in the interior of every region is at most $\lfloor n/k \rfloor$. Inequality (1) follows from $\lfloor x \rfloor \le x$.

To prove the second claim about points in general position, first we show the existence of a Γ-partition into open regions A and B with the property that (i) A contains exactly $\lfloor na/k \rfloor$ red points and $\lfloor ma/k \rfloor$ blue points, and (ii) B contains exactly $\lfloor nb/k \rfloor$ red points and $\lfloor mb/k \rfloor$ blue points for some a such that a and $b = k - a$ are at most $(5k+1)/6$. Consider a Γ-partition produced by the reduction from absolutely continuous measures. Let A and B be the regions of the partition corresponding to a and b, respectively. Let L be the line (horizontal or vertical) or two rays separating A and B. In the first case, L contains at most one given point. In the second case each ray contains at most one given (red/blue) point.

Suppose that na is not a multiple of k. Then L contains at least one red point since

$$\left\lfloor \frac{na}{k} \right\rfloor + \left\lfloor \frac{nb}{k} \right\rfloor = \left\lfloor \frac{na}{k} \right\rfloor + \left\lfloor n - \frac{na}{k} \right\rfloor = n - 1.$$

If L contains only one red point then A (resp. B) contains exactly $\lfloor na/k \rfloor$ (resp. $\lfloor nb/k \rfloor$) red points. If L contains two red points then one of the regions, say A, contains $\lfloor na/k \rfloor - 1$ red points and we translate slightly one of the rays of L so that A contains exactly $\lfloor na/k \rfloor$ red points. The same argument is applied if nb is not a multiple of k.

Suppose that na and nb are divisible by k. If L does not contain a red point then A and B contain the required number of red points. If L contains one or two red points then we translate slightly these rays so that A and B contain the required number of red points.

Similarly we can achieve the required number of blue points in both A and B. We now apply Γ-partitions recursively.

The second claim would follow if the bound in Inequality (1) is tight. Indeed, consider a region of the subdivision produced by a sequence of cuts corresponding to a decreasing sequence $k > k_1 > k_2 > \cdots > k_l = 1$. The tight bound of (1) would imply that the number of red points in these regions is equal to $\lfloor nk_1/k \rfloor, \lfloor nk_2/k \rfloor, \ldots$ and finally $\lfloor n/k \rfloor$. Unfortunately the bound is not tight. For example, if $n = 100, k = 13, k_i = 5$, and $k_j = 3$, then $\lfloor \lfloor nk_i/k \rfloor k_j/k_i \rfloor = 22 < 23 = \lfloor nk_j/k \rfloor$.

We prove another bound: the number of red points in these regions is at least $k_1\lfloor n/k \rfloor, k_2\lfloor n/k \rfloor, \ldots$ and finally $\lfloor n/k \rfloor$. The first bound follows from $\lfloor nk_1/k \rfloor \geq k_1\lfloor n/k \rfloor$ (since $k_1\lfloor n/k \rfloor$ is an integer less than or equal to nk_1/k). The remaining bounds can be shown by induction. Suppose that $n_i \geq k_i\lfloor n/k \rfloor$ red points are partitioned using a $(k_{i+1}, k_i - k_{i+1})$-split. Then the number of red points in the interior of the region corresponding to k_{i+1} is at least

$$\left\lfloor \frac{n_i k_{i+1}}{k_i} \right\rfloor \geq \left\lfloor k_i \left\lfloor \frac{n}{k} \right\rfloor \frac{k_{i+1}}{k_i} \right\rfloor \geq k_{i+1} \left\lfloor \frac{n}{k} \right\rfloor .$$

The theorem follows.

4.2 Algorithm

We call a subdivision provided by Theorem 2 *equitable orthogonal*. Note that there are two cases of point configuration: any point set and points in general position. An equitable subdivision of points in general position is more restrictive (each open region contains equal number of red (or blue) points). The main result here is the following theorem.

Theorem 11 *For any set of $n \geq 1$ red points and any set of $m \geq 1$ blue points in the plane and any integer $k \geq 2$, an equitable orthogonal subdivision of the plane into k regions can be computed in $O((n+m)\log k)$ time.*

PROOF. We prove the theorem for points in general position. In the degenerate case, one can perturb the input using for example the symbolic perturbation. We also can use the perturbation from Theorem 12 since distinct x-coordinates and distinct y-coordinates suffice.

The algorithm essentially follows the proof of the existence of an equitable orthogonal subdivision. The basic step of the construction is the computation of a balanced Γ-partition. We describe the algorithm for finding a $(a, k-a)$-balanced Γ-partition of all $n + m$ points which is applied then recursively.

Computing signs. Let $N = n + m$ be the total number of points. To find the value of a, we compute the signs $s(i)$ for $i = 1, 2, \ldots, k-1$. We define the sign $s(i)$ for points as follows. Let Y_r be the set of real numbers such that, for every $y_r \in Y_r$, the number of red points above the line with equation $y = y_r$

is exactly $\lfloor ni/k \rfloor$. Y_r is an interval. Similarly we define the interval Y_b for the blue points. If $y_r > y_b$ for all $y_r \in Y_r$ and $y_b \in Y_b$ then we assign $s(i) := -1$; otherwise $s(i) := 1$.

For $i = \lfloor k/2 \rfloor$, the intervals Y_r and Y_b and the sign $s(i)$ can be computed in $O(N)$ time by applying a linear-time selection algorithm to the y-coordinates of red and blue points. The set Y_r can be used to split the red points into two sets. We also split the blue points into two sets by Y_b. The remaining signs can be computed recursively using corresponding sets of red and blue points. The total time for computing the signs is $O(N \log k)$.

Computing a balanced Γ*-partition.* Put $b := k - a$. The existence of a balanced Γ-partition is based on Theorem 7. First, we decide what shape a balanced Γ-partition has. Similar to Y_r we find an interval X_r such that, for any line with equation $x = x_r, x_r \in X_r$, the number of red points on its right (left) side is $\lfloor na/k \rfloor$ (resp. $\lfloor nb/k \rfloor$). We also find an interval X_b for the blue points. If the intervals X_r and X_b intersect then any line with equation $x = x_0, x_0 \in X_r \cap X_b$ is the balanced Γ-partition. Otherwise, a balanced Γ-partition has two rays and we can decide whether the horizontal ray goes to $+\infty$ or $-\infty$ using the sign of a. Without loss of generality there is a balanced Γ-partition with one ray going to $+\infty$. We show that it can be found in linear time.

We apply the prune-and-search technique [1]. Let x^* be the x-coordinate of the vertical ray of a balanced Γ-partition. To find x^* we use the following test. For a given $x \in \mathbb{R}$, we decide whether $x < x^*, x = x^*$, or $x > x^*$. We find y-coordinate of the horizontal ray such that two rays cut off a set S of $\lfloor na/k \rfloor + \lfloor nb/k \rfloor$ red and blue points. We count the number r of red points in S. If $r = \lfloor na/k \rfloor$ then the Γ-partition is balanced and $x = x^*$. Otherwise, we decide whether x^* is smaller or greater than x using the sign of $r - \lfloor na/k \rfloor$.

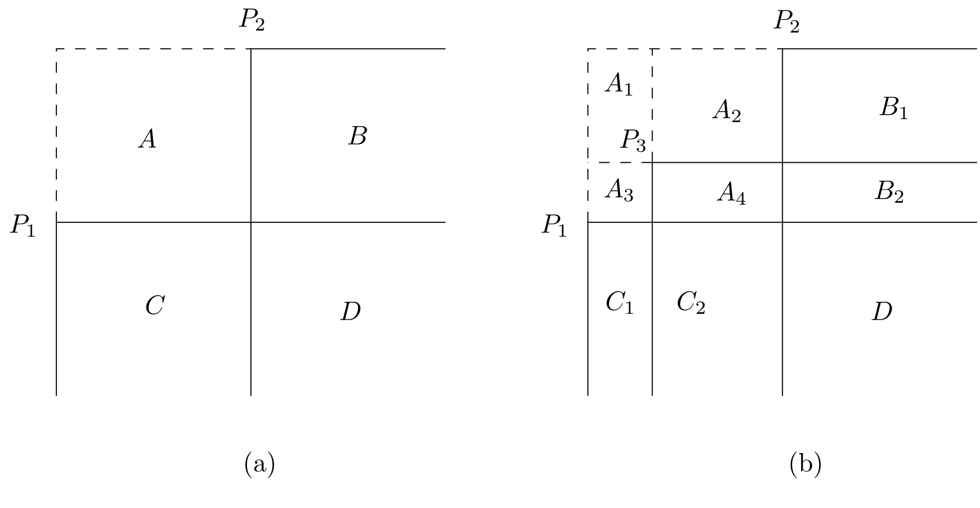


Fig. 6. A general step of the algorithm. (a) Two Γ-partitions with apexes P_1 and P_2. (b) A new Γ-partition with apex P_3.

At a general step of the search, we have two Γ-partitions with apexes P_1 and P_2 such that there exists a balanced Γ-partition with apex in A, see Fig. 6 (a). We use n_X, m_X, and N_X to denote the number of red/blue/all points in a region X. The regions $C \cup D$ and $B \cup D$ of the two Γ-partitions have the same number of points $\lfloor na/k \rfloor + \lfloor nb/k \rfloor$. Therefore $N_B = N_C$. We do not process points of D explicitly, we simply use counts n_D and m_D. The total number of points involved in the search is $N = N_A + N_B + N_C = N_A + 2N_B$. The goal of the search step is to reduce the number of points involved in the search by testing some x.

If $N_A \le N_C$ then we test the median of x-coordinates of points in C, see Fig. 6 (b). If $x^* = x$ then a balanced Γ-partition is found. If $x^* < x$ then the points in C_2 are pruned; otherwise the points of C_1 are pruned. In either case, at least $N_C/2$ points are pruned. Since $N = N_A + 2N_C \le 3N_C$, at least $N/6$ points are pruned.

If $N_A > N_C$ then we test the Mth x-coordinate of points in $A \cup C$ where $M = \lfloor (N_A + N_C)/3 \rfloor$. Note that $N = N_A + 2N_C < 3N_A$ and $M + 1 \ge (N_A + N_C)/3 = (N - N_C)/3 > N/3 - N_A/3 > N/3 - N/9 > N/6$. If $x^* > x$ then the points in $A_1 \cup A_3 \cup C_1$ are pruned; otherwise the points in $A_1 \cup A_2 \cup C_2$ are pruned. In the first case, at least $M > N/6 - 1$ points are pruned. In the second case, the number of pruned points is at least

$$N_{A_2} + N_{C_2} \ge (N_{A_2} + N_{A_4} + N_{C_2}) - N_{C_1}$$

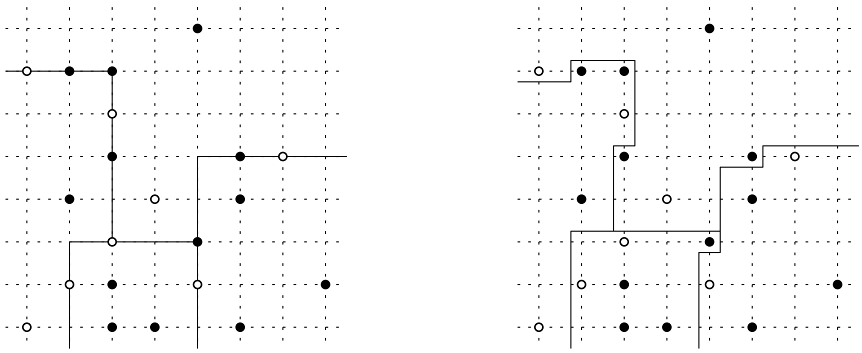
since $N_C = N_{A_4} + N_{C_2} + N_{B_2} \ge N_{A_4} + N_{C_2}$ and, thus, $N_{C_1} \ge N_{A_4}$. Since $N_{A_2} + N_{A_4} + N_{C_2} = N_A + N_C - M \ge 2(N_A + N_C)/3$ and $N_{C_1} \le M$, the number of pruned points is at least

$$N_{A_2} + N_{C_2} \ge \frac{2}{3}(N_A + N_C) - \frac{1}{3}(N_A + N_C) = \frac{1}{3}(N_A + N_C) > N/6.$$

Thus, a balanced Γ-partition can be found in linear time. The subdivision is then computed recursively. The total time for computing the subdivision is $O((n + m) \log k)$ since $a, b \le (5k + 1)/6$ in the balanced Γ-partition. □

4.3 Degenerate case

In this Section we consider points on the grid as in [6]. In an example shown in Fig. 7 (a), the subdivision of 8 white and 12 black points into 4 regions guaranteed by Theorem 2 may use grid lines and the number of points in the regions can be different. On the other hand they can be partitioned into regions with exactly 2 white and 3 black points in each region if the lines can avoid the grid, see Fig. 7 (b). Is it always possible to find such a subdivision? We prove that the answer is affirmative.



(a) (b)

Fig. 7. (a) An equitable orthogonal partition of 8 white and 12 black points into 4 regions using Γ-partitions. (b) An equitable orthogonal partition avoiding grid.

A *horizontal* ε*-segment* is either a horizontal segment or a polygonal chain of two horizontal segments connected by a vertical segment of length at most ε. Similarly we define a *vertical* ε*-segment*, see Fig. 8 for examples.

Fig. 8. Horizontal and vertical ε-segments.

Theorem 12 *Let R be a set of $n \geq 1$ red points and B be any set of $m \geq 1$ blue points in the plane such that all points are distinct. For any integer $k \geq 2$ and any $\varepsilon > 0$, there exists a subdivision of the plane into k regions $R_1, R_2, \ldots, R_k$ using at most $k-1$ horizontal ε-segments and at most $k-1$ vertical ε-segments such that the interior of every $R_i, i = 1, 2, \ldots, k$ contains exactly $\lfloor n/k \rfloor$ red points and $\lfloor m/k \rfloor$ blue points.*

PROOF. First, if the points are not in general position then we perturb them as follows. Let $\delta_x > 0$ be the smallest positive difference between x-coordinates of two points of $R \cup B$. Let $p_1, p_2, \ldots, p_t$ be points with the same x-coordinate. We assume that the points are sorted by y-coordinate. We change x-coordinate of p_i to $x(p_i) + i\delta$ where $\delta > 0$ is a real number smaller than $\max(\varepsilon, \delta_x)/t$. Similarly, we change y-coordinates of the points. Let R' and

B' be the perturbed point sets. Clearly, the points of $R' \cup B'$ are in general position.

By Theorem 2, there exists a subdivision of the plane into k regions $R_1, R_2, \dots, R_k$ using at most $k-1$ horizontal segments and at most $k-1$ vertical segments such that the interior of every $R_i, i = 1, 2, \dots, k$ contains exactly $\lfloor n/k \rfloor$ red points and $\lfloor m/k \rfloor$ blue points.

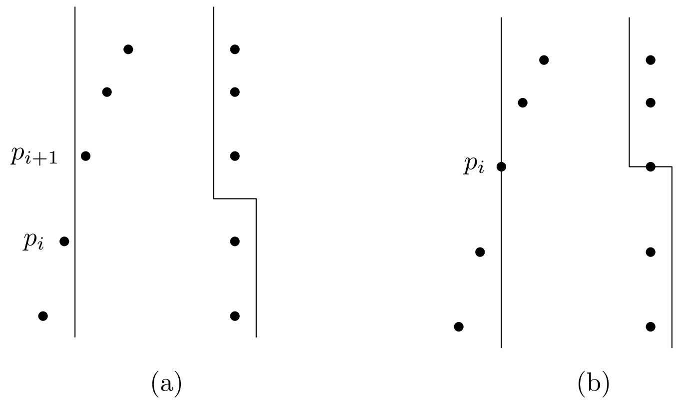


Fig. 9. The inverse perturbation.

The subdivision can be perturbed as follows. Without loss of generality we consider a vertical segment. If the segment avoids points then we introduce a horizontal segment of length at most ε between two consecutive points p_i and p_{i+1} that are split by the segment, see Fig. 9 (a) (if such a pair exists). If the segment contains a point p_i then we introduce a short horizontal segment containing it, see Fig. 9 (b). The theorem follows. $\square$

If the number of red/blue points is multiple of k then we have the following.

Corollary 13 *For any $\varepsilon > 0$ and any kn red and km blue distinct points in the plane, there exists a subdivision of the plane into k regions with at most $k-1$ horizontal ε-segments and at most $k-1$ vertical ε-segments such that every region contains n red points and m blue points.*

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