SOLUTION FOR HOMEWORK 11, STAT 5352

Welcome to your last homework devoted to nonparametric tests.

As usual, try to find mistakes (and get extra points) in my solutions. Typically they are silly arithmetic mistakes (not methodological ones). They allow me to check that you did your HW on your own. Please do not e-mail me about your findings — just mention them on the first page of your solution and count extra points.

Now let us look at your problems.

1. Problem 16.16. a) Here $H_0: \mu = 19.4$ versus $H_a: \mu \neq 19.4$. Because the distribution of X is symmetric about μ, under H_0 we have $P(X > 19.4|\mu = 19.4) =: \theta = 1/2$. Thus the problem is equivalent to testing $H_0' : \theta = 1/2$ versus $H_a : \theta \neq 1/2$. Let us consider a Bernoulli random variable $Y := I(X > 19.4)$. Note that then $W := \sum_{l=1}^{n} Y_l$ is $Binomial(\theta, n)$. Further, instead of working with Y's we can work with random signs $Z_l := sign(X_l - 19.4)$, and this is plainly a one-to-one transformation of Y_l.

Now let us count the number of pluses in the given sample: I counted only 4, corresponding to 20.3, 22.5, 19.5 and 20, (we can also say that we observed only four successes in $n = 20$ Bernoulli trials, and thus $W = 4$). Then we can calculate the observed level of significance (p-value) using the Binomial Table:

$$p - value = 2[P(W \leq 4|\theta = 1/2, n = 20)]$$

$$= 2[0 + 0 + 0.0002 + 0.0011 + 0.0046] = 0.0118.$$

This is an obvious rejection of the null hypothesis if $\alpha = .05$.

Remark: Please note that $E(W|\theta = 1/2) = (1/2)n = 10$. This fact warns us that it is unlikely that $\theta = 1/2$, it should be smaller (and thus μ is also smaller than 19.4).

2. Problem 16.16.b) Let us use normal approximation. First, following the text, let us do not use the correction factor:

$$Z = \frac{W - \theta n}{[n\theta(1-\theta)]^{1/2}} = \frac{4 - 10}{5^{1/2}} = -6/5^{1/2} = -2.68. \quad (1)$$

Further, $z_{\alpha/2} = z_{.025} = 1.96$, and thus we reject H_0. Let us also compare this approximation with the correct answer based on the Binomial Table. Using the approximation we get

$$p - value = 2P(Z < -2.68) = 2[.5000 - 0.4963] = 0.0074.$$

Note that the answer is fairly far from the one obtained in part (a).

If we do the approximation correctly, using our approximation rule, then:

$$P(W \leq 4) = P(X_{normal} \leq 4.5) = P(Z \leq \frac{4.5 - n\theta}{[n\theta(1-\theta)]^{1/2}}) = P(Z < -5.5/5^{1/2}).$$

Note that the difference between the correct approach and (1) is that in the correct approach the probability is larger (and this is always the case regardless of left/right tail). Thus, the p-value is

$$p - value = 2P(Z < -5.5/5^{1/2}) = 2P(Z < -2.46) = 2(0.5000 - 0.4931) = 0.0138.$$

Please note that this p-value is closer to the one obtained in part (a). Absolutely similarly you can note that instead of (1) a correctly calculated z-score is $Z = -2.46$ and not -2.68.

Here this makes no difference, so the text's approach is OK, but in a more close situation the right approximation of a binomial distribution by normal can make a difference.

3. Problem 16.18.a) Here $n = 16$, $H_0 : \mu = 19$ versus $H_a : \mu > 19$. A calculation yields $y = \sum_{l=1}^{16} I(x_l > 19) = 12$, or equivalently we have 12 pluses and 4 minuses. This statistic justifies the alternative hypothesis.

Using the Binomial Table for $Y \sim Binom(.5, 16)$ we get

$$P(Y \geq 12|\theta = .5, n = 16) = .0278 + .0085 + .0018 + .0002 = .0383.$$

With $\alpha = .05$ we reject H_0. Please note that the p-value is not far from the level of significance, so we need to be accurate in our calculations.

4. Problem 16.18.b) Without taking into account the correct approximation, we get (following the text)

$$z = \frac{y - \theta n}{[n\theta(1-\theta)]^{1/2}} = \frac{12 - 8}{4^{1/2}} = 2. \qquad (2)$$

From the Normal Table we get $z_{.05} = 1.645$, so we cleanly reject H_0.

At the same time, you can recall that the correction factor (in the normal approximation of binomial probability) always increases p-value. Thus we need to be very accurate in this particular case. With the correct approximation, as I explained in the previous problem,

$$P(Y \geq 12) = P(X_{normal} \geq 12 - .5) = P(Z \geq \frac{11.5 - 8}{4^{1/2}})$$
$$= P(Z \geq 1.75).$$

Please note that this yields that the corrected z-statistics is $z' = 1.75$, and it is much closer to the critical value $z_{.05} = 1.645$ than $z = 2$ calculated without the correct approximation.

The answer is, of course, reject the null hypothesis.

5. Problem 16.20. Here $n = 10$, $\alpha = .05$. Then according to the Rule on page 326 we reject $H_0 : \mu = \mu_0$ versus:

$H_a : \mu \neq \mu_0$ if $T = \min(T^+, T^-) \leq T_\alpha = T_{.05} = 8$;
$H_a : \mu > \mu_0$ if $T^- \leq T_{(2\alpha)} = T_{.1} = 11$;
$H_a : \mu < \mu_0$ if $T^+ \leq T_{(2\alpha)} = T_{.1} = 11$.
Here values of T_β are from Table X.

6. Problem 16.23. First, we pool together the two data sets, and then order them (in what follows I mark Springs data): 15, 18, 20, 22, 25(S), 27, 28(S), 29, 32(S), 35(S), 36(S), 38(S).

We test $H_a : \mu_S = \mu_F$ versus $H_a : \mu_F < \mu_S$. According to the Rule on page 531, we need to find (here index 2 corresponds to the Fall)

$$U_2 = W_2 - n_2(n_2 + 1)/2 = [1 + 2 + 3 + 4 + 6 + 8] - (6)(7)/2 =$$
$$= 24 - 21 = 3.$$

From Table XI we get $U_{2\alpha} = U_{.02} = 3$. Because $U_2 \leq U_{.02}$, we reject H_0.