Space-time Codes in Keyhole Channels: Analysis and Design

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***Abstract*— The keyhole condition, where the MIMO channel has only one degree of freedom, impairs the performance of MIMO systems. In cases that this condition is likely, one may wish to design codes that are robust to this condition. So far, a general analysis of space-time codes in keyhole conditions has not been available (except in the special case of orthogonal space-time block codes). In this work, we provide pairwise error probabilities for general space-time codes in keyhole condition. We present design criteria in high SNR, providing guidelines for codes that are robust to keyhole conditions. We also prove the intuitive result that the diversity under keyhole condition is min(M,N), with a slightly unexpected twist in the case of M=N.**

I. Introduction

It has been demonstrated recently [1], [2] that multiple-input multiple-output (MIMO) fading channels can experience a condition known as *keyhole* or pinhole condition, where despite rich local scattering and uncorrelated transmit and receive signals, the system has only one degree of freedom. Under this condition, space-time coding will have diminished performance.

Analysis and design of codes under this condition has not been studied extensively. Shin and Lee [3] studied the effect of keyholes on the performance of space-time block codes. Nasri *et al* [4] designed a space-time trellis code for the keyhole channel, but they limited themselves to the special case of two transmit and one receive antennas.

In this work, we undertake a general analysis of space-time codes in keyhole channels. The special case of this analysis for orthogonal space-time block codes [3] was made tractable by the helpful structure of these codes that creates an equivalent SISO channel. The general analysis, unfortunately, is not that easy.

Furthermore, we present design guidelines for the keyhole condition. Denoting the number of transmit antennas with M and receive antennas with N, we report that for $M \leq N$, the design criteria are similar to non-keyhole condition, but for $M > N$, the design criteria are different.

We also prove a result mentioned in [2], that the diversity order of the keyhole channel is $\min(M, N)$. Although this result is intuitive, to our knowledge no proof of it exists in prior literature.

The notation in this paper is as follows: $\mathbb{E}[\ ]$ refers to expected value of a random variable, $\gamma \approx 0.57721566$ is the Euler-Mascheroni constant and $e \approx 2.718281828$ is the base of natural logarithm. Gamma function is defined as: $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$. The functions $f(\cdot)$ and $g(\cdot)$ are said to be asymptotically equivalent if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$; this is denoted by $f(x) \overset{\circ}{=} g(x)$.

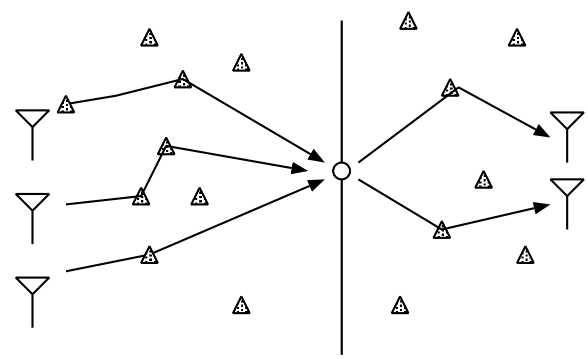

Fig. 1. Keyhole channel.

II. System Model

We consider a multiple antenna system with M transmit and N receive antennas. We assume a frequency non-selective linear time invariant fading channel. The signal model is

$$\mathbf{Y} = \sqrt{\frac{\rho}{M}}\,\mathbf{H}\mathbf{X} + \mathbf{N} \tag{1}$$

where $\mathbf{Y}$ is an $N \times T$ matrix representing the received signal, $\mathbf{X}$ is the $M \times T$ transmitted codeword matrix, ρ is the average SNR, and N is the zero mean additive circularly symmetric complex Gaussian noise matrix with size $N \times T$, whose elements have unit variance per dimension. The $N \times M$ MIMO channel matrix is denoted by $\mathbf{H}$ whose i, j element is the channel coefficient between the j transmit antenna and i receive antenna. The channel $\mathbf{H}$ is a keyhole channel modeled as

$$\mathbf{H} = \mathbf{h}\,\mathbf{g}^T \tag{2}$$

where $\mathbf{h}$ is a $M \times 1$ vector and $\mathbf{g}$ is an $N \times 1$ vector both with zero mean complex Gaussian entries. In decoding the received codeword $\mathbf{Y}$ we employ maximum likelihood (ML) decoder, assuming the channel state information (CSI) is perfectly known at the receiver and unknown at the transmitter.

III. PAIRWISE ERROR PROBABILITY

Approximating FER and BER is usually through *pairwise error probability* leading to a union bound over the probability of error.

$$P_e \leq \sum_{\mathbf{c}\in\mathcal{C}} P(\mathbf{c}) \sum_{\mathbf{c}'\neq\mathbf{c}} P(\mathbf{c}\to\mathbf{c}')$$

where $P(\mathbf{c}\to\mathbf{c}')$ is the pairwise error probability (PEP). In this section we investigate the pairwise error probability for the keyhole channel and drive expressions for PEP. We use these PEP expressions to calculate bounds for the probability of error of space time codes. Moreover these bound provide insight into designing better codes when keyhole condition exits. We use the popular techniques of Simon and Alouini [5] which incorporates the use of moment generating function (MGF). Consequently we calculate the MGF for the keyhole channel in general case. For a given channel realization $\mathbf{H}$ the PEP of a Gaussian channel can be calculated as:

$$P(\mathbf{c}\to\mathbf{c}'|\mathbf{H}) = Q\left(\sqrt{\frac{E_s}{2N_0}\|\mathbf{H}\boldsymbol{\Delta}\|^2}\right) \tag{3}$$

where $\boldsymbol{\Delta} = \mathbf{c}-\mathbf{c}'$ is the code word difference matrix. We use the alternative definition of Q-function by a finite integral [5]

$$Q(x) = \int_0^{\pi/2} \exp\left(-\frac{x^2}{2\sin^2\theta}\right) d\theta \ . \tag{4}$$

Using Eq. (4) and by integrating over all channel realizations we obtain the unconditional PEP as follows:

$$P(\mathbf{c}\to\mathbf{c}') = \int_0^{\pi/2} \psi\left(-\frac{E_s}{4N_0\sin^2\theta}\right) d\theta \tag{5}$$

where $\psi(.)$ is the MGF of the random variable $\|\mathbf{H}\boldsymbol{\Delta}\|^2$.

A. Calculation of MGF for the Keyhole Channel

In this section we calculate $\psi(.)$ in closed form. We have:

$$\begin{aligned}
\|\mathbf{H}\boldsymbol{\Delta}\|^2 &= \|\mathbf{h}\mathbf{g}^T\boldsymbol{\Delta}\|^2 \\
&= \mathrm{tr}(\mathbf{h}\mathbf{g}^T\boldsymbol{\Delta}\boldsymbol{\Delta}^T\mathbf{g}\mathbf{h}^T) \\
&= \mathrm{tr}(\mathbf{h}^T\mathbf{h}\mathbf{g}^T\boldsymbol{\Delta}\boldsymbol{\Delta}^T\mathbf{g}) \\
&= \|\mathbf{h}\|^2\|\mathbf{g}^T\boldsymbol{\Delta}\|^2
\end{aligned}$$

since $\mathbf{h}$ is a Gaussian vector with jointly independent elements the random variable $X \triangleq \|\mathbf{h}\|^2$ is distributed as χ^2_{2N}. Let $\boldsymbol{\Delta}\boldsymbol{\Delta}^T = \mathbf{U}^T\mathbf{D}\mathbf{U}$ be the eigen decomposition of $\mathbf{A} \triangleq \boldsymbol{\Delta}\boldsymbol{\Delta}^T$ in which $\mathbf{U}$ is unitary matrix and $\mathbf{D} = \mathrm{diag}(d_1,\ldots,d_M)$ where $d_1,\ldots,d_M$ are the eigen values of $\mathbf{A}$ (since $\mathbf{A}$ is symmetric positive definite d_i's are all non-negative). The elements of the vector $\mathbf{g}$ are circularly symmetric Gaussian random variables and are assumed to be jointly independent, thus $\hat{\mathbf{g}} = \mathbf{U}\mathbf{g}$ is distributed the same as $\mathbf{g}$ and we have

$$Y \triangleq \|\mathbf{g}^T\boldsymbol{\Delta}\|^2 = \sum_{i=1}^{M} d_i|\hat{g}_i|^2 \ . \tag{6}$$

Since each of the random variables $|\hat{\mathbf{g}}_i|^2$ is exponentially distributed, the distribution of random variable Y is *non-central chi-square*, thus the moment generating of Y can be calculated as follows:

$$\psi_Y(s) = \mathbb{E}[e^{sY}] = \prod_{i=1}^{M} \frac{1}{1-d_i s} \ . \tag{7}$$

The following lemma provides a way to calculate the the MGF of the product of two random variables:

Lemma 1: Let $Z = X.Y$ where X and Y are two independent positive random variables with pdf $f_X(.)$ and $f_Y(.)$ and let $\psi_Y(s) = \mathbb{E}[e^{sY}]$ be the MGF of Y, then the MGF of Z can be calculated as follows:

$$\psi_Z(s) = \int_0^{\infty} f_X(t)\psi_Y(st)dt \tag{8}$$

Proof: See the appendix.

By direct application of Lemma 1 to the random variable $Z \triangleq \|\mathbf{H}\boldsymbol{\Delta}\|^2 = XY$, we obtain the following formula for the MGF of Z:

$$\psi_Z(-s) = \frac{1}{\Gamma(N)}\int_0^{\infty} \frac{t^{N-1}e^{-t}}{\prod_{i=1}^{M}(1+d_i st)}dt \tag{9}$$

Thus $\psi_Z(-s)|_{s=\frac{E_s}{4N_0}}$ serves an upper bound for PEP in Eq. (5).

B. Asymptotic Behavior of MGF and Diversity Order

In this section we investigate the behavior of the MGF obtained in Eq. 9 in the asymptote of large s. This corresponds to the behavior of the PEP in the asymptote of high SNR.

Theorem 1: Assuming that all d_i's are strictly positive and distinct,[1] the behavior of $\psi_Z(-s)$ in the asymptote of large s is:

$$\psi_Z(-s) \overset{\circ}{=} \begin{cases} \frac{\Gamma(N-M)}{\Gamma(N)\prod_{i=1}^{M} d_i}\frac{1}{s^M} & N > M \\ \frac{1}{\Gamma(N)\prod_{i=1}^{M} d_i}\frac{\log s}{s^M} & N = M \\ \left(\frac{(-1)^{N-1}}{\Gamma(N)}\sum_{i=1}^{M}\frac{\log d_i}{d_i^N}\alpha_i\right)\frac{1}{s^N} & N < M \end{cases}$$

where $\alpha_i \triangleq \prod_{j\neq i}\frac{d_i}{d_i-d_j}$.

Proof: See the appendix.

It is reported in [2] that the diversity order of a keyhole channel is $\min(M,N)$. This is expected because the keyhole channel can be thought of as the cascade of a MISO (with diversity order of M) and a SIMO (with diversity order of N). Theorem 1 verifies this result. Since the channel with the smaller diversity is the bottleneck for data transmission, hence the diversity order of the keyhole channel cannot be more than $\min(M,N)$. But our Theorem 1 also shows an interesting result that to our knowledge, has not been pointed out. In the case $M = N$, the MGF has the form $\log(s)/s^M$, which

[1]When d_i's are not distinct a similar result holds, but is not presented here due to lack of space.

dominates $1/s^M$ but is dominated by $1/s^{M-1}$. Therefore, the achievable diversity on the keyhole channel is better than diversity order of $M-1$ but worse than diversity order of M. Hence the diversity order of the keyhole channel when $M=N$, can not be exactly explained by an integer. Finally, Theorem 1 provides the *coding gain*, which can be used for designing better codes for the keyhole channel.

C. Code Design Criterion

For the case $N \geq M$ theorem 1 says that the same determinant criterion proposed in [6] also applies in this case and the codes reported for the Rayleigh fading channel will also perform well when $N \geq M$. However if $N < M$ the expression for coding gain changes to a new one so in this case the space-time code design criterion is based on minimizing a new metric other than the inverse of the determinant of the code matrices difference. For the simple case of $M=2$ and $N=1$ the bound on PEP reduces to:

$$PEP \leq \left(\frac{\log d_1 - \log d_2}{d_1 - d_2}\right)\frac{E_s}{4N_0} \tag{11}$$

which is the same result reported in [4]. So the code design criterion for the general case is based on minimizing the following expression:

$$(-1)^{N-1}\sum_{i=1}^{M}\frac{\log d_i}{d_i^N}\left(\prod_{j\neq i}\frac{d_i}{d_i-d_j}\right). \tag{12}$$

IV. Application and Results

We employ the four-state QPSK space-time trellis (STT) code of [6]. This code is designed for $M=2$ and achieves the transmit diversity of two in i.i.d. MIMO channel. As stated earlier, in keyhole channels, the diversity of a space-time signaling is determined by $\min(M,N)$. Hence, for this code, we expect the diversity of one when $N=1$ and diversity of two when $N>2$. Figure 2 and Figure 3 show the bit error rate (BER) and frame error rate (FER) of the code with $N=1,2,4$ in keyhole channel. We observe that for $N=1$ the diversity of the code is one. The code shows the diversity of two for $N=4$. For $N=2$, the diversity is slightly less than two which explains the $\log s$ term in the MGF of Theorem 1 when $N=M$. The union bound of the FER for $N=1,2$ are also shown in Figure 3.

The design rule of (11) and (12) suggest a different coding gain design criterion compared to that of [6] when $M>N$. We would like to see if the above STT code can be redesignd accordingly to achieve better performance in keyhole channels. The trellis of the modified code is shown in Figure 4. The performance of the new code and the original code of [6], in both keyhole channel and Rayleigh fading MIMO channel with $N=1,2$, are shown in Figure 5. We observe that the new code achieves about 1 dB gain in keyhole channel with $N=1$ over the code of [6]. The new code gives slightly better performance even in quasi-static Rayleigh fading channel. The improved performance of the code in the Rayleigh channel and for $N=2$ is due to the minimum determinant criterion of [6],

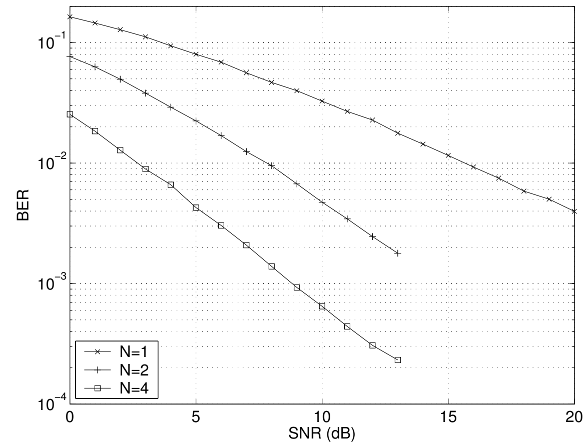

Fig. 2. Bit error rate of the four-state QPSK STT code in keyhole channel.

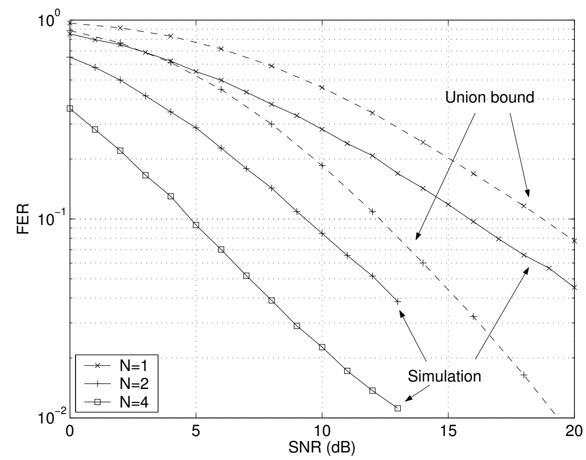

Fig. 3. Frame error rate of the four-state QPSK STT code in keyhole channel.

whose multiplicity is smaller in the modified code compared to the original code.

From above example, we realize that the reported space-time signalings can be modified to get a higher coding gain and better performance when the MIMO channel becomes rank deficient.

V. Conclusion

This paper presents a general analysis of space-time codes in keyhole channels. Using this analysis, we provide design criteria for codes that are robust to the keyhole condition. We report that for the case of $M \leq N$, the design criteria are unaffected by the keyhole condition, but for $M < N$ the design criteria are indeed affected. We also prove that the diversity order of a MIMO keyhole channel is equivalent to $\min(N,M)$.

VI. Appendix

Proof of Lemma 1: We know that the pdf of the product of two independent positive random variable X and Y can be

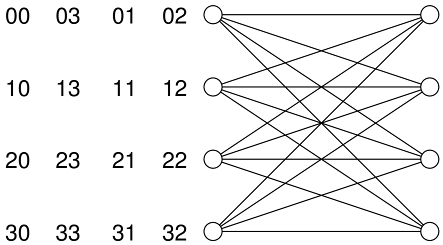

Fig. 4. An example STT code with improved performance in keyhole channel.

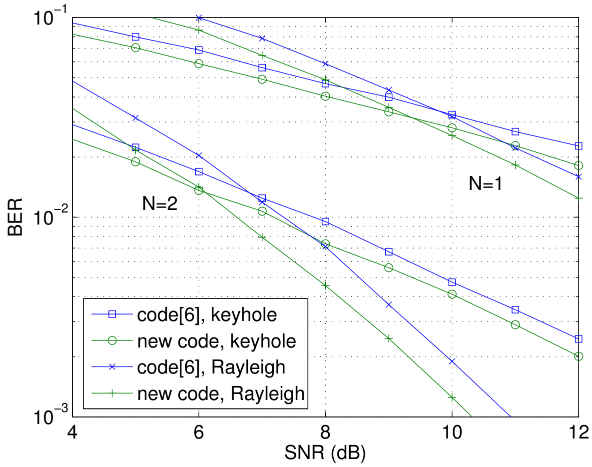

Fig. 5. Bit error rate of the four-state QPSK STT code and the modified code in i.i.d. MIMO channel and keyhole channel.

calculated as:

$$f_Z(z) = \int_0^\infty f_X(t) f_Y(\frac{z}{t}) \frac{dt}{t} \quad (13)$$

hence,

$$\begin{aligned} \psi_Z(s) &= \mathbb{E}[e^{sZ}] = \int_0^\infty e^{sz} f_Z(z) dz \\ &= \int_0^\infty \int_0^\infty e^{sz} f_x(t) f_Y(\frac{z}{t}) \frac{dt}{t} dz \\ &= \int_0^\infty f_x(t) dt \int_0^\infty e^{sz} f_Y(\frac{z}{t}) \frac{dz}{t} \end{aligned} \quad (14)$$

we make a change of variable $u = \frac{z}{t}$, hence we get:

$$\begin{aligned} \psi_Z(s) &= \int_0^\infty f_X(t) dt \int_0^\infty e^{stu} f_Y(u) du \\ &= \int_0^\infty f_X(t) \psi_Y(st) dt \,. \end{aligned} \quad (15)$$

Q.E.D.

To prove Theorem 1, we first prove the following two lemmas.

Lemma 2: Let

$$\alpha_i = \prod_{0 \le j \le M\,,\, j \ne i} \frac{d_i}{d_i - d_j}$$

given that $d_i \ne d_j$ for all $i \ne j$ and $d_i \ne 0$ for all i. We have:

$$\sum_{i=1}^{M} \frac{\alpha_i}{d_i^k} = \begin{cases} 1 & k = 0 \\ 0 & 0 \le k \le M-1 \\ \frac{(-1)^{M-1}}{\prod_{i=1}^M d_i} & k = M\,. \end{cases} \quad (16)$$

Proof: Assuming d_i's are distinct, α_i is the residue of the simple pole $s = -1/d_i$ of the complex function $f(s)$, defined below, and we have

$$f(s) \triangleq \prod_{i=1}^{M} \frac{1}{1 + d_i s} = \sum_{i=1}^{M} \frac{\alpha_i}{1 + d_i s}\,. \quad (17)$$

We notice that $f(s)$ has a zero at $s = \infty$ of multiplicity M, thus the Laurent expansion of $f(s)$ starts with $1/s^M$, i.e.

$$f(s) = \sum_{k=M}^{\infty} \frac{f_k}{s^k} \quad , \quad |s| > \max_{1 \le i \le M} d_i^{-1}\,. \quad (18)$$

On the other hand :

$$\begin{aligned} \frac{\alpha_i}{1 + d_i s} &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \alpha_i}{d_i^k} \frac{1}{s^k} \quad , \quad |s| > d_i^{-1} \\ \sum_{i=1}^{M} \frac{\alpha_i}{1 + d_i s} &= \sum_{i=1}^{M} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \alpha_i}{d_i^k} \frac{1}{s^k}\,. \end{aligned} \quad (19)$$

Since (18) and (19) are equal we have for all $1 \le k \le M-1$:

$$\begin{aligned} \sum_{i=1}^{M} \frac{\alpha_i}{d_i^k} &= 0 \\ \sum_{i=1}^{M} \frac{\alpha_i}{d_i^M} &= \frac{(-1)^{M-1}}{\prod_{i=1}^M d_i}\,. \end{aligned}$$

If in (17) we put $s = 0$ we get $\sum \alpha_i = 1$ and this completes the proof. Q.E.D.

Lemma 3: The integral $\int_0^\infty \frac{t^{N-1}}{1 + d_i st} e^{-t}\, dt$ behaves asymptotically as shown in (20), on top of next page.

Proof: Let $I_k = \int_0^\infty \frac{t^{k-1}}{1 + d_i st} e^{-t}\, dt$. We have:

$$\begin{aligned} I_k &= \int_0^\infty \frac{t^{k-1} + \frac{t^{k-2}}{d_i s}}{1 + d_i st} e^{-t}\, dt - \int_0^\infty \frac{\frac{t^{k-2}}{d_i s}}{1 + d_i st} e^{-t}\, dt \\ I_k &= \frac{1}{d_i s} \int_0^\infty t^{k-2} e^{-t}\, dt - \frac{1}{d_i s} \int_0^\infty \frac{t^{k-2}}{1 + d_i st} e^{-t}\, dt \\ I_k &= \frac{\Gamma(k-1)}{d_i s} - \frac{1}{d_i s} I_{k-1}\,. \end{aligned}$$

Multiplying both sides of the last equation by $(-1)^k d_i^k s^k$, we get

$$\begin{aligned} (-1)^k d_i^k s^k I_k &= (-1)^k d_i^{k-1} s^{k-1} \Gamma(k-1) \\ &\quad + (-1)^{k-1} d_i^{k-1} s^{k-1} I_{k-1} \end{aligned}$$

$$\sum_{k=2}^{N} (-1)^k d_i^k s^k I_k - (-1)^{k-1} d_i^{k-1} s^{k-1} I_{k-1} = \sum_{k=2}^{N} (-1)^k d_i^{k-1} s^{k-1} \Gamma(k-1)\,.$$

$$\int_0^\infty \frac{t^{N-1}}{1+d_i st} e^{-t}\,dt \mathrel{\mathring{=}} \sum_{k=1}^{N-1} \frac{(-1)^{k-1}\Gamma(N-K)}{d_i^k s^k} + \frac{(-1)^{N-1}(\log s+\gamma)}{d_i^N s^N} + \frac{(-1)^{N-1}\log d_i}{d_i^N s^N} \tag{20}$$

$$\psi_Z(-s) \mathrel{\mathring{=}} \frac{1}{\Gamma(N)} \sum_{i=1}^{M} \alpha_i \left(\sum_{k=1}^{N-1} \frac{(-1)^{k-1}\Gamma(N-k)}{d_i^k s^k} + \frac{(-1)^{N-1}(\log s+\gamma)}{d_i^N s^N} + \frac{(-1)^{N-1}\log d_i}{d_i^N s^N} \right) \tag{21}$$

The left-hand side of the above equation is a telescopic sum, hence we obtain:

$$I_N = \sum_{k=2}^{N} (-1)^{N-k} \frac{\Gamma(k-1)}{d_i^{N-k+1} s^{N-k+1}} - \frac{(-1)^N}{d_i^{N-1} s^{N-1}} I_1 \,.$$

We calculate I_1 using integration by parts:

$$I_1 = \int_0^\infty \frac{e^{-t}}{1+d_i st}\,dt = \frac{1}{d_i s}\int_0^\infty \frac{e^{-t}}{\frac{1}{d_i s}+t}\,dt$$
$$= \frac{1}{d_i s}\log\left(\frac{1}{d_i s}+t\right)e^{-t}\Big|_0^\infty + \frac{1}{d_i s}\int_0^\infty \log\left(\frac{1}{d_i s}+t\right)e^{-t}dt$$

hence,

$$I_1 \mathrel{\mathring{=}} -\frac{\log s+\gamma}{d_i s} - \frac{\log d_i}{d_i s}$$

where in the last step we used the fact that $\lim_{\epsilon\to 0}\int_0^\infty \log(\epsilon+t)e^{-t} = \Gamma'(1) = -\gamma$. Hence we arrive at (20). Q.E.D.

Proof of Theorem 1:

Case I, N > M : In this case we have:

$$\left(\prod_{i=1}^{M} d_i s\right)\psi_Z(-s) = \frac{1}{\Gamma(N)}\int_0^\infty \frac{t^{N-1}e^{-t}}{\prod_{i=1}^{M}(\frac{1}{d_i s}+t)}dt$$

we have $\prod_{i=1}^{M}(\frac{1}{d_i s}+t) \geq t^M$, now using *the dominated convergence theorem* we can conclude that:

$$\begin{aligned}\left(\prod_{i=1}^{M} d_i s\right)\psi_Z(-s) &\mathrel{\mathring{=}} \frac{1}{\Gamma(N)}\int_0^\infty t^{N-M-1}e^{-t}dt \\ &= \frac{\Gamma(N-M)}{\Gamma(N)}\end{aligned}$$

hence, for $N > M$ we have

$$\psi_Z(-s) \mathrel{\mathring{=}} \frac{\Gamma(N-M)}{\Gamma(N)\prod_{i=1}^{M} d_i}\frac{1}{s^M}\,.$$

Case II and III, N ≤ M :

$$\begin{aligned}\psi_Z(-s) &= \frac{1}{\Gamma(N)}\int_0^\infty \frac{t^{N-1}e^{-t}}{\prod_{i=1}^{M}(1+d_i st)}\,dt \\ &= \frac{1}{\Gamma(N)}\int_0^\infty \sum_{i=1}^{M}\frac{\alpha_i t^{N-1}e^{-t}}{1+d_i st}\,dt \\ &= \frac{1}{\Gamma(N)}\sum_{i=1}^{M}\alpha_i\int_0^\infty \frac{t^{N-1}e^{-t}}{1+d_i st}\,dt\,.\end{aligned}$$

From Lemma 3 we obtain (21), shown on top of this page, which can be rewritten as following

$$\psi_Z(-s) \mathrel{\mathring{=}} \frac{1}{\Gamma(N)}\sum_{k=1}^{N-1}\left(\sum_{i=1}^{M}\frac{\alpha_i}{d_i^k}\right)\frac{(-1)^{k-1}\Gamma(N-k)}{s^k} \tag{22a}$$
$$+\frac{(-1)^{N-1}}{\Gamma(N)}\frac{(\log s+\gamma)}{s^N}\sum_{i=1}^{M}\frac{\alpha_i}{d_i^k} \tag{22b}$$
$$+\frac{(-1)^{N-1}}{\Gamma(N)s^N}\sum_{i=1}^{M}\frac{\alpha_i\log d_i}{d_i^N}\,. \tag{22c}$$

If $N = M$, by Lemma 2 the inner summation of the term (22a) becomes zero, the summation in term (22b) becomes $\frac{(-1)^{N-1}}{\prod_{i=1}^{N} d_i}$, and we get

$$\begin{aligned}\psi_Z(-s) &\mathrel{\mathring{=}} \frac{1}{\Gamma(N)\prod_{i=1}^{N} d_i}\frac{\log s+\gamma}{s^N} \\ &\quad+\frac{(-1)^{N-1}}{\Gamma(N)s^N}\sum_{i=1}^{N}\frac{\alpha_i\log d_i}{d_i^N} \\ &\mathrel{\mathring{=}} \frac{1}{\Gamma(N)\prod_{i=1}^{N} d_i}\frac{\log s}{s^N}\,.\end{aligned} \tag{23}$$

If $N < M$, by Lemma 2 the inner summation of the term (22a) and the summation in term (22b) become zero and we obtain:

$$\psi_Z(-s) \mathrel{\mathring{=}} \frac{(-1)^{N-1}}{\Gamma(N)}\left(\sum_{i=1}^{M}\frac{\alpha_i\log d_i}{d_i^N}\right)\frac{1}{s^N}\,. \tag{24}$$

Q.E.D.

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