Functional Analysis

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1 Metric Spaces

Definition (Metric Space). The pair $(X, d)$, where $X$ is a set and the function

$$d : X \times X \to \mathbb{R}$$

is called a metric space if

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ iff $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) \leq d(x, z) + d(z, y)$

Example 1.1 (Metric Spaces).

1. $d(x, y) = |x - y|$ in $\mathbb{R}$.
2. $d(x, y) = [\sum_{i=1}^{n}(x_i - y_i)^2]^{\frac{1}{2}}$ in $\mathbb{R}^n$.
3. $d(x, y) = \|x - y\|$ in a normed space.
4. Let $(X, \rho)$, $(Y, \sigma)$ be metric spaces and define the Cartesian product $X \times Y = \{(x, y)|x \in X, y \in Y\}$. Then the product measure $\tau((x_1, y_1), (x_2, y_2)) = [\rho(x_1, x_2)^2 + \sigma(y_1, y_2)^2]^{\frac{1}{2}}$.
5. (Subspace) $(Y, \bar{d})$ of $(X, d)$ if $Y \subset X$ and $\bar{d} = d_{|Y \times Y}$.
6. $l^\infty$. Let $X$ be the set of all bounded sequences of complex numbers, i.e., $x = (\xi_i)$ and $|\xi_i| \leq c_x$, $\forall i$. Then

$$d(x, y) = \sup_{i \in \mathbb{N}} |\xi_i - \eta_i|$$

defines a metric on $X$.

7. $X = C[a, b]$ and

$$d(x, y) = \max_{t \in [a,b]} |x(t) - y(t)|.$$ 

8. (Discrete metric)

$$d(x, y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}.$$
9. $l^p$. $x = (\xi_i) \in l^p$ if $\sum_{i=1}^{\infty} |\xi_i|^p < \infty$, ($p \geq 1$, fixed),

$$d(x, y) = \left( \sum_{i=1}^{\infty} |\xi_i - \eta_i|^p \right)^{\frac{1}{p}}.$$ 

**Problem 1.**

1. Show that $\tilde{d}$ is a metric on $C[a, b]$, where

$$\tilde{d}(x, y) = \int_a^b |x(t) - y(t)| dt.$$

2. Show that the discrete metric is a metric.

3. Sequence space $s$: set of all sequences of complex numbers with the metric

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|}. \tag{1}$$

**Solution.**

1. $\tilde{d}(x, y) = 0$ iff $|x(t) - y(t)| = 0$ for all $t \in [a, b]$ because of the continuity. We have $\tilde{d}(x, y) \geq 0$ and $\tilde{d}(x, y) = \tilde{d}(y, x)$ trivially. We can argue the triangle inequality as follows:

$$\tilde{d}(x, y) = \int_a^b |x(t) - y(t)| dt \leq \int_a^b |x(t) - z(t)| dt + \int_a^b |z(t) - y(t)| dt = \tilde{d}(x, z) + \tilde{d}(z, y).$$

2. Left as an exercise.

3. We show only the triangle inequality. Let $a, b \in R$. Then we have the inequalities

$$\frac{|a + b|}{1 + |a + b|} \leq \frac{|a|}{1 + |a|} + \frac{|b|}{1 + |b|},$$

where in the first step we have used the monotonicity of the function

$$f(x) = \frac{x}{1 + x} = 1 - \frac{1}{1 + x}, \text{ for } x > 0.$$ 

Substituting $a = \xi_i - \zeta_i$ and $b = \zeta_i - \eta_i$, where $x = (\xi_i)$, $y = (\eta_i)$, and $z = (\zeta_i)$ we get

$$\frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|} \leq \frac{|\xi_i - \zeta_i|}{1 + |\xi_i - \zeta_i|} + \frac{|\zeta_i - \eta_i|}{1 + |\zeta_i - \eta_i|}.$$ 

If we multiply both sides by $\frac{1}{2^i}$ and sum over from $i = 1$ to $\infty$ we get the stated result. 

$\square$
1.1 Open Sets, Closed Sets

Definition (Open Ball, Closed Ball, Sphere).

1. \( B(x_0, r) = \{ x \in X | d(x, x_0) < r \} \)
2. \( \bar{B}(x_0, r) = \{ x \in X | d(x, x_0) \leq r \} \)
3. \( S(x_0, r) = \{ x \in X | d(x, x_0) = r \} \)

Definition (Open, Closed, Interior).

1. \( M \) is open if contains a ball about each of its points.
2. \( K \subset X \) is closed if \( K^c = X - K \) is open.
3. \( B(x_0; \varepsilon) \) denotes the \( \varepsilon \) neighborhood of \( x_0 \).
4. \( \text{Int}(M) \) denotes the interior of \( M \).

Remark 1.2 (Induced Topology). Consider the set \( X \) with the collection \( \tau \) of all open subsets of \( X \). Then we have

1. \( \emptyset \in \tau \), \( X \in \tau \).
2. The union of any members of \( \tau \) is a member of \( \tau \).
3. The finite intersection of members of \( \tau \) is a member of \( \tau \).

We call the pair \((X, \tau)\) a topological space and \( \tau \) a topology for \( X \). It follows that a metric space is a topological space.

Definition (Continuous). Let \( X = (X, d) \) and \( Y = (Y, \bar{d}) \) be metric spaces. The mapping \( T : X \rightarrow Y \) is continuous at \( x_0 \in X \) if for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that

\[
\bar{d}(Tx, Tx_0) < \varepsilon, \quad \forall x \text{ such that } d(x, x_0) < \delta.
\]

Theorem 1.3 (Continuous Mapping). \( T : X \rightarrow Y \) is continuous if and only if the inverse image of any open subset of \( Y \) is an open subset of \( X \).

Proof.

1. Suppose that \( T \) is continuous. Let \( S \subset Y \) be open \( S_0 \) the inverse image of \( S \). Let \( S_0 \neq \emptyset \) and take \( x_0 \in S_0 \). We have \( Tx_0 = y_0 \in S \). Since \( S \) is open there exists an \( \varepsilon \)-neighborhood of \( y_0 \), say \( N \subset S \) such that \( y_0 \in N \). The continuity of \( T \) implies that \( x_0 \) has a \( \delta \)-neighborhood \( N_0 \) which is mapped into \( N \). Since \( N \subset S \) we get that \( N_0 \subset S_0 \), and it follows that \( S_0 \) is open.
2. Assume that the inverse image of every open set in $Y$ is an open set in $X$. Then
\[ \forall \ x_0 \in X, \text{ and } N \ (\varepsilon\text{-neighborhood of } Tx_0) \text{ the inverse image } N_0 \text{ of } N \text{ is open.} \]
Therefore $N_0$ contains a $\delta$-neighborhood of $x_0$. Thus $T$ is continuous.

Some more definitions:

**Definition (Accumulation Point).** $x \in M$ is said to be an accumulation point of $M$ if
\[ \exists (x_n) \subset M \text{ s.t. } x_n \to x. \]

**Definition (Closure).** $\overline{M}$ is the closure of $M$.

**Definition (Dense Set).** $M \subset X$ is in $X$ dense if $\overline{M} = X$.

**Definition (Separable Space).** $X$ is separable if there is a countable subset which is dense in $X$.

**Remark 1.4.**

1. If $M$ is dense, then every ball in $X$ contains a point of $M$.

2. $R$, $C$ are separable.

3. A discrete metric space is separable if and only if it is countable.

**Theorem 1.5.** $\ell^\infty$ is not separable.

**Proof.** Let $y = (\eta_i)$ where $\eta_i = 0, 1$. There are uncountably many $y$’s. If we put small balls with radius $\frac{1}{3}$ at the $y$’s they will not intersect. It follows that if $M \subset \ell^\infty$ is dense in $\ell^\infty$, then $M$ is uncountable. Therefore $\ell^\infty$ is not separable.

**Problem 2.** Show that $l^p$, $1 \leq p < \infty$ is separable.

**Solution.** Let $M$ the set of all sequences of the form $x = (\xi_1, \xi_2, ..., \xi_n, 0, 0, ...)$, where $n$ is any positive integer and the $\xi$’s are rational. $M$ is countable. We argue that $M$ is dense in $l^p$ as follows. Let $y = (\eta_i) \in l^p$ be arbitrary. Then for every $\varepsilon > 0$ there is an $n$ such that
\[
\sum_{i=n+1}^{\infty} |\eta_i|^p < \frac{\varepsilon^p}{2}.
\]
Since the rationals are dense in $R$, for each $\eta_i$ there is a rational $\xi_i$ close to it. Hence there is an $x \in M$ such that
\[
\sum_{i=1}^{n} |\eta_i - \xi_i| < \frac{\varepsilon^p}{2}.
\]

It follows that $d(y, x) < \varepsilon$.  \qed
1.2 Convergence, Cauchy Sequence, Completeness

**Definition** (Convergent Sequence). We say that the sequence \((x_n)\) is convergent in the metric space \(X = (X, d)\) if

\[
\lim_{n \to \infty} d(x_n, x) = 0 \text{ for some } x \in X.
\]

We shall also use the notation \(x_n \to x\).

**Definition** (Diameter). For a set \(M \subset X\) we define the diameter \(\delta(M)\) by

\[
\delta(M) = \sup_{x,y \in M} d(x, y).
\]

\(M\) is said to be **bounded** if \(\delta(M)\) is finite.

**Lemma 1.6.**

1. A convergent sequence \((x_n)\) in \(X\) is bounded and its limit \(x\) is unique.
2. If \(x_n \to x\) and \(y_n \to y\) in \(X\), then \(d(x_n, y_n) \to d(x, y)\).

**Problem 3.** Prove Lemma 1.6.

**Solution.**

1. Suppose that \(x_n \to x\). Then, taking \(\varepsilon = 1\) we can find an \(N\) such that \(d(x_n, x) < 1\) for all \(n > N\). By the triangle inequality we have

\[
d(x_n, x) < 1 + \max d(x_1, x), d(x_2, x), ..., d(x_N, x).
\]

Therefore \((x_n)\) is bounded. If \(x_n \to x\) and \(x_n \to z\), then

\[
0 \leq d(x, z) \leq d(x_n, x) + d(x_n, z) \to 0 \text{ as } n \to \infty
\]

and uniqueness of the limit follows.

2. We have

\[
d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n),
\]

and hence

\[
d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y_n, y).
\]

Interchanging \(x_n\) and \(x\), \(y_n\) and \(y\), and multiplying by \(-1\) we get

\[
d(x, y) - d(x_n, y_n) \leq d(x_n, x) + d(y_n, y).
\]

Combining the two inequalities we get

\[
|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y) \to 0 \text{ as } n \to \infty.
\]
**Definition** (Cauchy Sequence). \((x_n)\) in \((X, d)\) is a Cauchy sequence if \( \forall \varepsilon > 0 \ \exists N = N (\varepsilon) \) s.t.
\[
m, n > N \Rightarrow d(x_m, x_n) < \varepsilon.
\]

**Definition** (Complete). \(X\) is complete if every Cauchy sequence in \(X\) converges in \(X\).

**Theorem 1.7.** \(\mathbb{R}, \mathbb{C}\) are complete metric spaces.

**Theorem 1.8.** Every convergent sequence in a metric space is a Cauchy sequence.

**Theorem 1.9.** Let \(M \subset X = (X, d)\), \(X\) is a metric space and let \(\overline{M}\) denote the closure of \(M\) in \(X\). Then

1. \(x \in \overline{M}\) iff \(\exists (x_n) \in M\) s.t. \(x_n \to x\).

2. \(M\) is closed iff \(x_n \in M\) and \(x_n \to x\) imply that \(x \in M\).

**Theorem 1.10.** Let \(X\) be a complete metric space and \(M \subset X\). \(M\) is complete iff \(M\) is closed in \(X\).

**Theorem 1.11.** Let \(T\) be a mapping from \((X, d)\) into \((Y, \tilde{d})\). \(T : X \to Y\) is continuous at \(x_0 \in X\) iff \(x_n \to x_0\) implies \(T x_n \to T x_0\).

**Problem 4.**

1. Prove Theorem 1.10

2. Prove Theorem 1.11

**Solution.**

1. Let \(M\) be complete. Then for every \(x \in \overline{M}\) there is a sequence \((x_n)\) which converges to \(x\). Since \((x_n)\) is Cauchy and \(M\) is complete \(x_n \to x \in M\), therefore \(M\) is closed.

Conversely, let \(M\) be closed and \((x_n)\) be Cauchy in \(M\). Then \(x_n \to x \in X\), which implies \(x \in \overline{M}\), and therefore \(x \in M\). Thus \(M\) is complete.

2. Assume that \(T\) is continuous at \(x_0\). Then for \(\varepsilon > 0 \ \exists \delta > 0\) such that
\[
d(x, x_0) < \delta \text{ implies } \tilde{d}(T x, T x_0) < \varepsilon.
\]

Take a sequence \((x_n)\) such that \(x_n \to x_0\). Then \(\exists N\) s.t. \(\forall n > N\) we have \(d(x_n, x_0) < \delta\) and hence
\[
\forall n > N, \ \tilde{d}(T x_n, T x) < \varepsilon.
\]

Conversely, assume that \(x_n \to x\) \(\Rightarrow T x_n \to T x_0\) and show that \(T\) is continuous at \(x_0\). Otherwise, \(\exists \varepsilon > 0\) such that \(\forall \delta > 0, \ \exists x \neq x_0\) satisfying \(d(x, x_0) < \delta\) but \(\tilde{d}(T x, T x_0) \geq \varepsilon\). Let \(\delta = \frac{1}{n}\). Then there is an \(x_n\) such that \(d(x_n, x_0) < \frac{1}{n}\) but \(\tilde{d}(T x_n, T x_0) \geq \varepsilon\) contrary to our assumption.
1.3 Completeness Proofs

**Theorem 1.12.** $\mathbb{R}^n, C^n$ are complete.

**Theorem 1.13.** $\ell^\infty$ is complete.

*Proof.* Let $(x_m)$ be a Cauchy sequence in $\ell^\infty$.

\[ d(x_m, x_n) = \sup_i |x_m^{(i)} - x_n^{(i)}| < \varepsilon, \text{ for } m, n > N(\varepsilon). \]

$\Rightarrow$ for fixed $i$, the sequence $(\xi_i^{(m)})$ is Cauchy in $R$, and therefore we have that $\xi_i^{(m)} \to \xi_i$, for $i = 1, 2, 3, \ldots$ Let $x = (\xi_i)$. It follows easily that $x \in \ell^\infty$ and $x_m \to x$. \hfill $\Box$

**Theorem 1.14.** The space of convergent sequences $x = (\xi_i)$ of complex numbers with the metric induced from $\ell^\infty$ is complete.

**Theorem 1.15.** $\ell^p$ is complete if $1 \leq p < \infty$.

**Theorem 1.16.** $C[a, b]$ is complete.

*Proof.* Let $(x_m)$ be a Cauchy sequence in $C[a, b]$. Then

\[ \forall \varepsilon > 0 \exists N \text{ s.t. for } m, n > N \Rightarrow d(x_m, x_n) = \max_{t \in [a, b]} |x_m(t) - x_n(t)| < \varepsilon. \]  \hfill (2)

$\Rightarrow$ for any fixed $t_0 \in [a, b]$ we have that $|x_m(t_0) - x_n(t_0)| < \varepsilon$. It follows that $x_m(t_0)$ is Cauchy and therefore $x_m(t_0) \to x(t_0)$. Show that $x(t) \in C[a, b]$ and $x_m \to x$. From (2) with $n \to \infty$ we get

\[ \max_{t \in [a, b]} |x_m(t) - x(t)| \leq \varepsilon. \]

Therefore $x_m(t)$ converges uniformly to $x(t)$ on $[a, b]$. Since the $x_m$’s are continuous on $[a, b]$ and the convergence is uniform $x(t)$ is continuous, and thus belongs to $C[a, b]$. \hfill $\Box$

**Remark 1.17.** Note that in $C[a, b]$ convergence is uniform convergence; we also use the terminology uniform metric for the metric generated by the “sup”-norm.

**Example 1.18** (Incomplete Metric Spaces).

1. $\mathbb{Q}$

2. The polynomials

3. $C[a, b]$ with the $\| \cdot \|_2$ norm
1.4 Completion of Metric Spaces

**Definition** (Isometry). Let \( X = (X, d) \) and \( \bar{X} = (\bar{X}, \bar{d}) \) be metric spaces. The mapping 
\[ T : X \rightarrow \bar{X} \]
is an isometry if 
\[ \bar{d}(Tx, Ty) = d(x, y). \]
The metric spaces \( X \) and \( \bar{X} \) are isometric if there is a bijective isometry of \( X \) onto \( \bar{X} \).

**Theorem 1.19.** Let \( X = (X, d) \) be a metric space. Then there exists a complete metric space \( \hat{X} = (\hat{X}, \hat{d}) \) which has a subspace \( W \) that is isometric with \( X \) and is dense in \( \hat{X} \). \( \hat{X} \) is unique except for isometries.

**Proof.** This proof is divided into steps.

1. Construction of \( \hat{X} \). Let \( (x_n) \) and \( (x'_n) \) be equivalent Cauchy sequences, i.e.,
\[ \lim_{n \to \infty} d(x_n, x'_n) = 0. \]
Define \( \hat{X} \) to be the set of all equivalence classes \( \hat{x} \) of Cauchy sequences. Define now
\[ \hat{d}(\hat{x}, \hat{y}) = \lim_{n \to \infty} d(x_n, y_n) \text{ where } (x_n) \in \hat{x} \text{ and } (y_n) \in \hat{y}. \] (3)
We show that limit in (3) exists and independent of the particular choice of the representatives. We have using the triangle inequality
\[ d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n). \]
It follows that
\[ d(x_n, y_n) - d(x_m, y_m) \leq d(x_n, x_m) + d(y_m, y_n). \]
Exchanging the role of \( n \) and \( m \) we get
\[ d(x_m, y_m) - d(x_n, y_n) \leq d(x_n, x_m) + d(y_m, y_n). \]
The two inequality together yield
\[ |d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_m, y_n). \]
It follows that \( (d(x_n, y_n)) \) is a Cauchy sequence in \( R \) and therefore it converges, i.e., the limit in (3) exists.

We show now that the limit in (3) is independent of the particular choice of representatives. Let \( (x_n), (x'_n) \in \hat{x} \) and \( (y_n), (y'_n) \in \hat{y} \). Then
\[ |d(x_n, y_n) - d(x'_n, y'_n)| \leq d(x_n, x'_n) + d(y_n, y'_n) \rightarrow 0 \text{ as } n \to \infty, \]
which implies
\[ \lim_{n \to \infty} d(x_n, y_n) = \lim_{n \to \infty} d(x'_n, y'_n). \]
2. Show \( \hat{d} \) is a metric on \( \hat{X} \). Left as an exercise.

3. Construction of an isometry \( T : X \to W \subseteq \hat{X} \). With each \( b \in X \) we associate the class \( b \in \hat{X} \) which contains the constant Cauchy sequence \( (b, b, b, ...) \). This defines the mapping \( T : X \to W \) onto the subspace \( W = T(X) \subseteq \hat{X} \). The mapping \( T \) is given by \( b \to \hat{b} = T(b) \), where \( (b, b, ...) \in \hat{b} \). According to (3) \( \hat{d}(\hat{b}, \hat{c}) = \hat{d}(b, c) \), so \( T \) is an isometry. \( T \) is onto \( W \) since \( T(X) = W \). (\( W \) and \( X \) are isometric.) Show that \( W \) is dense in \( \hat{X} \). Let \( \hat{x} \in \hat{X} \) be arbitrary and let \( (x_n) \in \hat{x} \). For every \( \varepsilon > 0 \) there is an \( N \) such that \( d(x_n, x_N) < \frac{\varepsilon}{2} \), for \( n > N \). Let \( (x_N, x_N, ...) \in \hat{x}_N \). Then \( \hat{x}_N \in W \), and by (3)
\[
\hat{d}(\hat{x}, \hat{x}_N) = \lim_{n \to \infty} d(x_n, x_N) \leq \frac{\varepsilon}{2} < \varepsilon.
\]

4. Completeness of \( \hat{X} \). Let \( (\hat{x}_n) \) be an arbitrary Cauchy sequence in \( \hat{X} \). Since \( W \) is dense in \( \hat{X} \) for every \( \hat{x}_n \) there is a \( \hat{z}_n \in W \) such that \( \hat{d}(\hat{x}_n, \hat{z}_n) < \frac{1}{n} \). It follows using the triangle inequality and the Cauchyness of \( (\hat{x}_n) \) that \( (\hat{z}_n) \) is Cauchy in \( X \). Let \( \hat{x} \in \hat{X} \) be the class to which \( (\hat{z}_n) \) belongs. We have
\[
\hat{d}(\hat{x}_n, \hat{x}) \leq \hat{d}(\hat{x}_n, \hat{z}_n) + \hat{d}(\hat{z}_n, \hat{x}) < \frac{1}{n} + \hat{d}(\hat{z}_n, \hat{x}) < \varepsilon
\]
for sufficiently large \( n \).

5. Uniqueness of \( \hat{X} \). If \( (\hat{X}, \hat{d}) \) is another complete metric space with a subspace \( \hat{W} \) dense in \( \hat{X} \) and isometric with \( X \), then \( \hat{W} \) is isometric with \( W \) and the distances on \( \hat{X} \) and \( \hat{X} \) must be the same. Hence \( X \) and \( \hat{X} \) are isometric.

\( \square \)

**Problem 5.** Show now that \( \hat{d} \) is a metric on \( \hat{X} \).

**Solution.** Clearly, \( \hat{d}(\hat{x}, \hat{y}) \geq 0 \) because of the definition (3). Also, \( \hat{d}(\hat{x}, \hat{x}) = 0 \), and \( \hat{d}(\hat{x}, \hat{y}) = \hat{d}(\hat{y}, \hat{x}) \), because of the properties of \( d \).
\[
\hat{d}(\hat{x}, \hat{y}) \leq \hat{d}(\hat{x}, \hat{z}) + \hat{d}(\hat{z}, \hat{y})
\]
because we have
\[
d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n) \text{ for all } n.
\]

\( \square \)
Problem 6. A homeomorphism is a continuous bijective mapping $T : X \to Y$ whose inverse is continuous. If such mapping exists, then $X$ and $Y$ are homeomorphic.

1. Show that if $X$ and $Y$ are isometric, then they are homeomorphic.

2. Illustrate with an example that a complete and an incomplete metric space may be homeomorphic.

Solution. 1. If $f : (X, d) \to (Y, \hat{d})$ is an isometry, then $\hat{d}(f(x), f(y)) = d(x, y) < \epsilon$ whenever $d(x, y) < \delta$, so $f$ is continuous. If $(X, d)$ and $(Y, \hat{d})$ are isometric, then there exists a bijective isometry $f : X \to Y$. It follows that $f^{-1} : Y \to X$ is also an isometry, and that both $f$ and $f^{-1}$ are continuous. Therefore $X$ and $Y$ are homeomorphic.

2. Let $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ be given by $f(x) = \tan x$. $f$ is continuous and bijective. Moreover, $f^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ given by $f^{-1}(x) = \tan^{-1}(x)$ is also continuous. So, $\mathbb{R}$ is homeomorphic to $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Problem 7. If $(X, d)$ is complete, show that $(X, \bar{d})$, where $\bar{d} = \frac{d}{1+d}$ is complete.

Solution. It is easy to see that if $(x_n)$ is Cauchy in $(X, \bar{d})$, then $(x_n)$ is Cauchy in $(X, d)$. Since $(X, d)$ is complete, $x_n \to x$, but then we also have $x_n \to x$ in $(X, \bar{d})$. It follows that $(X, \bar{d})$ is complete.
2 Normed Spaces

Vector spaces, linear independence, finite- and infinite-dimensional vector spaces.

**Definition (Basis).** If $X$ is any vector space, and $B$ is a linearly independent subset of $X$ which spans $X$, then $B$ is called a basis (or Hamel basis) for $X$.

**Definition (Banach Space).** A complete normed space.

**Definition (norm).** A real-valued function $\| \cdot \| : X \to \mathbb{R}$ satisfying

1. $\| x \| \geq 0$
2. $\| x \| = 0$ iff $x = 0$
3. $\| \alpha x \| = |\alpha| \| x \|$
4. $| x + y | \leq |x| + |y|$

**Definition (Induced Metric).** In a normed space $(X, d)$ we can always define a metric by $d(x, y) = \| x - y \|$.

**Remark 2.1.** All previous results about metric spaces apply to normed spaces with the induced metric.

**Problem 8.** The norm is continuous, that is $\| \cdot \| : X \to \mathbb{R}$ by $x \mapsto \| x \|$ is a continuous.

**Solution.** The triangle inequality implies that

$$\| \| x \| - \| y \| \| \leq \| x - y \|.$$ 

**Example 2.2 (Norm Spaces).** $\mathbb{R}^n, \mathbb{C}^n, \ell^p, \ell^\infty, C[a, b], \mathcal{L}^2[a, b], \mathcal{L}^p[a, b]$.

**Remark 2.3 (Induced Norm is Translation Invariant).** A metric induced by a norm on a normed space is translation invariant, i.e., $d(x + a, y + a) = d(x, y)$ and $d(\alpha x, \alpha y) = |\alpha| d(x, y)$. For example the metric (1) is not coming from a norm.

**Theorem 2.4.** A subspace $Y$ of a Banach space $X$ is complete iff the set $Y$ is closed in $X$.

**Definition (Convergent, Cauchy, Absolutely Convergent).**

1. $(x_n)$ is convergent if $\| x_n - x \| \to 0$. ($x$ is the limit of $(x_n)$).
2. $(x_n)$ is Cauchy if $\| x_m - x_n \| < \epsilon$ for $m, n > N(\epsilon)$. 


Definition (Infinite series). Let \((x_k)\) be a sequence and consider the partial sums
\[
s_n = x_1 + x_2 + \ldots + x_n.
\]
If \(s_n\) converges, say \(s_n \to s\), then \(\sum_{k=1}^{\infty} x_k\) is convergent and
\[
s = \sum_{k=1}^{\infty} x_k.
\]
If \(\|x_1\| + \|x_2\| + \ldots + \|x_n\| + \ldots\) converges, then \(\sum_{k=1}^{\infty} x_k\) is absolutely convergent.

Remark 2.5. Absolute convergence implies convergence iff \(X\) is complete.

Definition (Schauder basis). If \(X\) contains a sequence \((e_n)\) such that for every \(x \in X\) there exists a unique sequence \((\alpha_n)\) with the property that
\[
\|x - (\alpha_1 e_1 + \ldots + \alpha_n e_n)\| \to 0 \text{ as } n \to \infty,
\]
then \((e_n)\) is called a Schauder Basis for \(X\). The representation
\[
x = \sum_{k=1}^{\infty} \alpha_k e_k
\]
is called the expansion of \(x\) with respect to \((e_n)\).

Remark 2.6. If \(X\) has a Schauder basis, then \(X\) is separable.

Theorem 2.7. Let \(X = (X, \| \cdot \|)\) be a normed space. Then there is a Banach space \(\hat{X}\) and an isometry \(A\) from \(X\) onto a subspace \(W \subset \hat{X}\) which is dense in \(\hat{X}\). \(\hat{X}\) is unique, except for isometries.

Theorem 2.8. Let \(X\) be a normed space, where the absolute convergence of any series always implies convergence. Then \(X\) is complete.

Proof. Let \((s_n)\) be any Cauchy sequence in \(X\). Then for every \(k \in N\) there exists \(n_k\) such that \(\|s_n - s_m\| < 2^{-k}, (m, n > n_k)\) and we can choose \(n_{k+1} > n_k\) for all \(k\). Then \((s_{n_k})\) is a subsequence of \((s_n)\) and is the sequence of the partial sums of \(\sum x_k\), where \(x_1 = s_{n_1}, \ldots, x_k = s_{n_k} - s_{n_{k-1}}, \ldots\). Hence
\[
\sum \|x_k\| \leq \|x_1\| + \|x_2\| + \sum 2^{-k} = \|x_1\| + \|x_2\| + 1.
\]
It follows that \(\sum x_k\) is absolutely convergent. By assumption, \(\sum x_k\) converges, say \(s_{n_k} \to s \in X\). Since \((s_n)\) is Cauchy, \(s_n \to s\) because
\[
\|s_n - s\| \leq \|s_n - s_{n_k}\| + \|s_{n_k} - s\|.
\]
Since \((s_n)\) was arbitrary, \(s_n \to s\) shows that \(X\) is complete. \(\Box\)
2.1 Finite dimensional Normed Spaces or Subspaces

**Theorem 2.9 (Linear Combinations).** Let $X$ be a normed space, and let
\[ \{x_1, \ldots, x_n\} \]
be a linearly independent set of vectors in $X$. Then $\exists c > 0$ s.t.
\[ \|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \geq c (|\alpha_1| + \cdots + |\alpha_n|). \]

**Proof.** Let
\[ s = |\alpha_1| + \cdots + |\alpha_n|. \]
If $s = 0$ we are done, so assume $s > 0$. Define
\[ \beta_i = \frac{\alpha_i}{s}. \]
Clearly we have that
\[ \sum_{i=1}^{n} |\beta_i| = 1. \]
Then we can reduce the above inequality to
\[ \left\| \sum_{i=1}^{n} \beta_i x_i \right\| \geq c. \quad (4) \]

We will prove the result for Theorem 4.

Suppose Theorem 4 is not true. Then $\exists (y_m)$
\[ y_m = \sum_{i=1}^{n} \beta_i^{(m)} x_i \]
such that
\[ \|y_m\| \to 0 \text{ as } m \to \infty. \]

Note that $\sum_{i=1}^{n} |\beta_i^{(m)}| = 1 \Rightarrow |\beta_i^{(m)}| \leq 1 \forall i$. Hence for each fixed $i$ the sequence
\[ (\beta_i^{(m)}) = (\beta_i^{(1)}, \beta_i^{(2)}, \ldots) \]
is bounded. Therefore $\beta_i^{(m)}$ has a convergent subsequence (B-W). Let $\beta_1$ be the limit, and let $(y_{1,m})$ be the corresponding subsequence of $(y_m)$
\[ (y_{1,m}) = \gamma_{1}^{(m)} x_1 + \sum_{i=2}^{n} \beta_i^{(m)} x_i \]
Then there is a there exist corresponding $\beta_2$, $(y_{2,m})$ where

$$\beta_2^{(m)} \to \beta_2 \text{ as } n \to \infty$$

$$(y_{2,m}) = \gamma_1^{(m)} x_1 + \gamma_2^{(m)} x_2 + \sum_{i=3}^{n} \beta_i^{(m)} x_i.$$ 

Since $n$ is finite this process will terminate with

$$(y_{n,m}) = \sum_{i=1}^{n} \gamma_i^{(m)} x_i$$

with $(y_{n,m})$ a subsequence of $(y_m)$. By construction,

$$\gamma_i^{(m)} \to \beta_i \forall i$$

$$\sum_{i=1}^{n} |\beta_i| = 1.$$ 

Therefore $(y_{n,m})$ has a limit, say

$$y = \sum_{i=1}^{n} \beta_i x_i.$$ 

Since $\|y_m\| \to 0$ by assumption we must also have $\|y_{n,m}\| \to 0$, We know $y \neq 0$ since $\sum_{i=1}^{n} |\beta_i| = 1$ and $\{x_i\}$ is a basis, a contradiction. 

**Theorem 2.10 (Completeness).** Every f.d.s.s. $Y$ of a n.s. $X$ is complete. In particular, every f.d.n.s. is complete.

**Proof.** Let $(y_m)$ be a Cauchy sequence in $Y$. We must find a limit $y$ such that

$$y_m \to y$$

$$y \in Y.$$ 

Let $n = \text{dim } Y$, and let $\{e_1, \ldots, e_n\}$ be a basis for $Y$. Then $\forall \ m$ we can represent $y_m$ as

$$y_m = \sum_{i=1}^{n} \alpha_i^{(m)} e_i.$$ 

But then for each fixed $i$ sequence $(\alpha_i^{(m)})$ is Cauchy in $\mathbb{R}$. So each of these sequence converges, say

$$\alpha_i^{(m)} \to \alpha_i.$$
Then define
\[ y = \sum_{i=1}^{n} \alpha_i e_i. \]
Clearly, \( y \in Y \) and \( y_m \to y \).

**Theorem 2.11.** Every f.d.s.s. \( Y \) of a closed n.s. \( X \) is closed in \( X \).

**Proof.** By Theorem 2.10 \( Y \) must be complete. Therefore \( Y \) is closed in \( X \). \qed

Note that finiteness in the previous proof is essential. Infinite dimensional subspaces need not be closed in \( X \). For example consider
\[
X = C[0,1] \\
Y = \text{span} (\{1, t, \ldots, t^n, \ldots\}).
\]
Then the sequence \((t^i)\) converges to
\[
f(t) = \begin{cases} \\
0 & t < 1 \\
1 & t = 1 \\
\end{cases}
\]
which is not in \( Y \).

**Definition** (Equivalent Norms). \( \|\cdot\|_1 \) and \( \|\cdot\|_2 \) are equivalent if \( \exists a, b > 0 \) s.t.
\[
a \|x\|_2 \leq \|x\|_1 \leq b \|x\|_2, \ \forall \ x \in X.
\]

**Theorem 2.12** (Finite Dimensional \( \Rightarrow \) All Norms are Equivalent). On a f.d.n.s. \( X \) every two norms \( \|\cdot\|_1, \|\cdot\|_2 \) are equivalent.

**Proof.** Let \( n = \dim X \), let \( \{e_1, \ldots, e_n\} \) be a basis for \( X \) with \( \|e_i\|_1 = 1 \), let \( x \in X \) be represented as
\[
x = \sum_{i=1}^{n} \alpha_i e_i.
\]
Then \( \exists c > 0 \) s.t.
\[
\|x\|_1 \geq c \sum_{i=1}^{n} |\alpha_i|.
\]
Also note
\[
\|x\|_2 \leq \sum_{i=1}^{n} |\alpha_i| \|e_i\|_2 \leq k \sum_{i=1}^{n} |\alpha_i|
\]
where
\[
k = \max \{\|e_i\|_2\}.
\]
Combining these inequalities we obtain
\[ \|x\|_2 \leq \frac{k}{c} \sum_{i=1}^{n} |\alpha_i| \leq \frac{1}{a} \|x\|_1, \]
where
\[ a = \frac{c}{k} \]
and we obtain the inequality involving \( a \). A similar argument yields for the inequality involving \( b \).

**Problem 9.** What is the largest possible \( c > 0 \) in
\[ \left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \geq c \sum_{i=1}^{n} |\alpha_i| \]
if
1. \( X = \mathbb{R}^2, x_1 = (1,0), x_2 = (0,1) \).
2. \( X = \mathbb{R}^3, x_1 = (1,0,0), x_2 = (0,1,0), x_3 = (0,0,1) \).

**Solution.** Find \( \min \{ \| \sum \beta_i x_i \| : \sum |\beta_i| = 1 \} \) and obtain \( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \), respectively.

**Problem 10.** Show that if \( \| \cdot \|_1, \| \cdot \|_2 \) are equivalent norms on \( X \) then the Cauchy sequences in \( (X, \| \cdot \|_1), (X, \| \cdot \|_2) \) are the same.

**Solution.** Suppose that \( (x_n) \) is a Cauchy sequence in \( \| \cdot \|_2 \). Then for every \( \epsilon > 0 \) there exists \( N \) such that \( \|x_n - x_m\|_2 < \frac{\epsilon}{b} \) for all \( m, n \geq N \), where \( b \) is a constant for which \( \|x\|_1 \leq b \|x\|_2 \). Then \( \|x_n - x_m\|_1 \leq b \|x_n - x_m\|_2 < \epsilon \) for all \( m, n \geq N \). It follows that \( (x_n) \) is Cauchy in \( \| \cdot \|_1 \). A similar argument works in the other direction.

### 2.2 Compactness and Finite Dimension

**Definition (Compactness).** A metric space \( (X, d) \) is compact if every sequence in \( X \) has a convergent subsequence. \( M \subset X \) is compact if \( M \) is compact as a subspace of \( X \), i.e., if every sequence in \( M \) has a convergent subsequence which converges to an element in \( M \).

**Theorem 2.13.** If \( M \) is compact then \( M \) is closed and bounded.

**Proof.** First we show closed. For all \( x \in \overline{M} \exists (x_n) \subset M \) s.t. \( x_n \to x \). \( M \) is compact \( \Rightarrow \) \( x \in M \), hence \( M \) is closed.

To show boundedness assume it is not true. Then fix \( b \in M \) and choose \( y_n \in M \) s.t. \( d(y_n, b) > n \). Then \( (y_n) \subset M \) is a sequence with no convergent subsequence, contradicting the assumption that \( M \) is compact. Thus \( M \) is bounded.

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Note the converse of this theorem is false. Consider $M \subset \ell^2$ as
\[ M = \{ x_n \in \ell^2 \text{ s.t. } x_n = (e_n), (e_n)_i = \delta_{ni} \} . \]
Then $M$ is bounded since $\|x_n\| = 1$ and $M$ is closed because there are no accumulation points
\[ \|x_n - x_m\| = \sqrt{2}. \]
But the sequence $(x_n)$ has no convergent subsequence, so $M$ is not compact. In some sense, there is too much room in the infinite dimensional setting.

**Theorem 2.14** (Compactness). Let $X$ be a f.d.n.s. . Then $M \subset X$ is (sequentially) compact iff $M$ is closed and bounded.

**Proof.** The “$\Rightarrow$” case was proved by the previous theorem. For the other direction, assume $M$ is closed and bounded.

For the other direction (“$\Leftarrow$”), assume $n = \dim X, \{e_1, \ldots, e_n\}$ is a basis for $X$, and $(x_m) \subset M$ is an arbitrary sequence. Then there is a representation
\[ x_m = \sum_{i=1}^{m} \xi^{(m)}_i e_i , \forall m . \]
Since $M$ is bounded, $(x_m)$ is bounded so $\exists k \geq 0$ s.t.
\[ \|x_m\| \leq k . \]
So then
\[
\begin{align*}
k & \geq \|x_m\| \\
& = \left\| \sum_{i=1}^{n} \xi^{(m)}_i e_i \right\| \\
& \geq c \sum_{i=1}^{n} |\xi^{(m)}_i|
\end{align*}
\]
and therefore $\forall i, (\xi^{(m)}_i) \subset \mathbb{R}$ is bounded and therefore $\forall i, (\xi^{(m)}_i)$ has an accumulation point $z$. Since $M$ is closed, $z \in M$ and there is a subsequence $(z_m) \subset (x_m)$ s.t. $z_m \to z$, say,
\[ z = \sum_{i=1}^{n} \xi_i e_i . \]
Since $(x_m)$ was arbitrary, $M$ is compact.

We shall need the following technical result to prove subsequent theorems.
Theorem 2.15 (Riesz). Let $Y, Z$ be subspaces of $X$, and suppose $Y$ is closed and is a proper subspace of $Z$. Then $\forall \theta \in (0, 1) \exists z$ s.t.

\[\|z\| = 1\]
\[\|z - y\| \geq \theta, \; \forall y \in Y.\]

Proof. Let $v \in Z - Y$ be arbitrary and let

\[a = \inf_{y \in Y} \|v - y\|.\]

Note if $a = 0$ then $v \in Y$ since $Y$ is closed. Therefore $a > 0$.

Choose $\theta \in (0, 1)$. Then $\exists y_0 \in Y$ s.t.

\[a \leq \|v - y_0\| \leq \frac{a}{\theta}.\]

Let

\[z = c(v - y_0)\text{ where } c = \frac{1}{\|v - y_0\|}.\]

By construction, $\|z\| = 1$. We want to show $\|z - y\| \geq \theta, \; \forall y \in Y$. So

\[\|z - y\| = \|c(v - y_0) - y\|
= c\|v - y_0 - \frac{y}{c}\|
= c\|v - y_1\|\]

for some $y_1 \in Y$, $\Rightarrow \|v - y_1\| \geq a \Rightarrow$

\[\|z - y\| = \|v - y_1\|
\geq ca
= \frac{a}{\theta}
\geq \frac{a}{\theta}.
= \theta.\]

\[\square\]

Theorem 2.16 (Finite Dimension and Compactness). Let $X$ be a normed space. Then the closed unit ball
\[M = \{x \text{ s.t. } \|x\| \leq 1\}\]

is compact iff $\text{dim } X < \infty$. 

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Proof. “⇒”. Suppose $M$ is compact, but $\dim X = \infty$. Pick $x_1, \|x_1\| = 1$. Then $x_1$ generates a closed, 1-dimensional subspace $X_1 \subseteq X$. Then also $\exists x_2 \in X$ s.t. $\|x_2\| = 1$ and

$$\|x_2 - x_1\| \geq \theta = \frac{1}{2}.$$  

Then $x_1, x_2$ generate a closed, 2-dimensional subspace $X_2 \subseteq X$. Then $\exists x_3 \in X$ s.t. $\|x_3\| = 1$ and

$$\|x_3 - x_i\| \geq \theta = \frac{1}{2}, \forall i = 1, 2.$$  

In this way construct $x_n$ s.t.

$$\|x_n - x_i\| \geq \theta = \frac{1}{2}, \forall i = 1, \ldots, n - 1.$$  

Then $(x_n) \subset X$ is a sequence such that $\|x_m - x_n\| \geq \frac{1}{2}, \forall m, n$. Therefore $(x_n)$ has no accumulation point, contradicting that $M$ was compact.

“⇐” was proved by Theorem 2.14.

**Theorem 2.17** (Continuous Preserves Compact). Let $T : X \to Y$ be continuous. Then $M \subset X$ compact $\Rightarrow T(M) \subset Y$ compact.

**Proof.** Let $(y_n) \subset T(M)$ be arbitrary. Then $\forall n \exists x_n$ s.t. $y_n = Tx_n$. Then $(x_n) \subset M$ has a convergent subsequence $(x_{n_k})$. Since $T$ is continuous, the image $(y_{n_k}) = T(x_{n_k})$ also converges. Therefore $T(M)$ is compact.

**Corollary 2.18** (Max, Min). Let $T : X \to \mathbb{R}$ be continuous and let $M \subset X$ be compact. Then $T$ obtains its max and min on $M$.

**Proof.** $T(M)$ is compact $\Rightarrow T(M)$ is closed and bounded. Therefore

$$\inf T(M) \in T(M)$$

$$\sup T(M) \in T(M).$$

Problem 11. Show that a discrete metric space with infinitely many points is not compact.

Solution. The space contains an infinite sequence $(x_n)$, $x_n \neq x_m$ ($n \neq m$) which cannot have a convergent subsequence since $d(x_n, x_m) = 1$.

**Problem 12** (Local Compactness). A metric space $X$ is said to be locally compact if every point $x \in X$ has a compact neighborhood. Show that $\mathbb{R}, \mathbb{R}^n, \mathbb{C}, \mathbb{C}^n$ are locally compact.

**Solution.** The closed ball $B(x, \epsilon)$ around an arbitrary point $x$ is compact.
2.3 Linear Operators

**Definition** (Linear, Domain and Range). Let $X, Y$ be vector spaces and let $T : X \rightarrow Y$. Then

1. If $\forall x, y \in \mathcal{D}(T)$ and scalars $\alpha, \beta$

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty$$

then $T$ is said to be a linear operator.

2. The null space of $T$ is $\mathcal{N}(T) = \{x \in \mathcal{D}(T) \text{ s.t. } Tx = 0\}$

3. The domain of $T$, $\mathcal{D}(T)$, is a vector space

4. The range of $T$, $\mathcal{R}(T)$, lies in a vector space

Note that obviously $\mathcal{D}(T) \subset X$. It is customary to write

$$T : \mathcal{D}(T) \xrightarrow{\text{onto}} \mathcal{R}(T).$$

Also, $T0 = T(0 + 0) = T0 + T0 \Rightarrow T0 = 0 \Rightarrow \mathcal{N}(T) \neq \emptyset$.

**Example 2.19** (Differentiation). Let $X$ be the space of polynomials on $[a, b]$. Then define $T : X \xrightarrow{\text{onto}} X$

$$Tx(t) = \frac{d}{dt}x(t).$$

**Example 2.20** (Integration). Let $X = C[a, b]$. Then define $T : X \xrightarrow{\text{onto}} X$ by

$$Tx(t) = \int_a^t x(s)ds.$$

**Example 2.21** (Multiplication). Let $X = C[a, b]$. Then define $T : X \xrightarrow{\text{onto}} X$ by

$$Tx(t) = tx(t).$$

**Example 2.22** (Matrices). Let $A = (a_{ij})$. Then define $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ by

$$Tx = Ax.$$

**Theorem 2.23** (Range and Null Space). Let $T$ be a linear operator.

1. $\mathcal{R}(T)$ is a vector space.

2. If $\dim \mathcal{D}(T) = n < \infty$ then $\dim \mathcal{R}(T) \leq n$.  

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3. \( \mathcal{N}(T) \) is a vector space.

**Proof.** 1. Let \( y_1, y_2 \in \mathcal{R}(T) \) and let \( \alpha, \beta \) be scalars. Since \( y_1, y_2 \in \mathcal{R}(T) \), \( \exists x_1, x_2 \in \mathcal{D}(T) \) s.t.
   \[
   Tx_1 = y_1 \\
   Tx_2 = y_2.
   \]
   Since \( \mathcal{D}(T) \) is vector space, \( \alpha x_1 + \beta x_2 \in \mathcal{D}(T) \), so
   \[
   T(\alpha x_1 + \beta x_2) \in \mathcal{R}(T).
   \]
   But
   \[
   T(\alpha x_1 + \beta x_2) = \alpha Tx_1 + \beta Tx_2 = \alpha y_1 + \beta y_2
   \]
   means \( \mathcal{R}(T) \) is a vector space.

2. Pick \( n+1 \) elements arbitrarily in \( \mathcal{R}(T) \). Then we have
   \[
   y_i = Tx_i, \; \forall \; i = 1, \ldots, n+1
   \]
   for some \( \{x_1, \ldots, x_{n+1}\} \) in \( \mathcal{D}(T) \). Since \( \dim \mathcal{D}(T) = n \), this set must be linearly dependent. Similarly, the points in the range are also linearly dependent. Therefore, \( \dim \mathcal{R}(T) \leq n \).

3. Let \( x_1, x_2 \in \mathcal{N}(T) \), i.e. \( Tx_1 = Tx_2 = 0 \), and let \( \alpha, \beta \) be scalars. But then
   \[
   T(\alpha x_1 + \beta x_2) = \alpha Tx_1 + \beta Tx_2 = 0 \Rightarrow \alpha x_1 + \beta x_2 \in \mathcal{N}(T).
   \]
   Therefore \( \mathcal{N}(T) \) is a vector space.

The next result shows linear operators preserve linear independence.

**Definition** (Injective (One-to-one)). \( T : \mathcal{D}(T) \xrightarrow{\text{into}} Y \) is injective (or one-to-one) if \( \forall \; x_1, x_2 \in \mathcal{D}(T) \) s.t. \( x_1 \neq x_2 \)
   \[
   Tx_1 \neq Tx_2.
   \]
   Equivalently,
   \[
   Tx_1 = Tx_2 \Rightarrow x_1 = x_2.
   \]
   So therefore if \( T : \mathcal{D}(T) \xrightarrow{\text{into}} Y \) is injective then \( \exists \; T^{-1} : \mathcal{R}(T) \xrightarrow{\text{onto}} \mathcal{D}(T) \)
   \[
   y_0 \mapsto x_0 \\
   y_0 = Tx_0
   \]
   and therefore
   \[
   T^{-1}Tx = x, \; \forall \; x \in \mathcal{D}(T) \\
   TT^{-1}y = y, \; \forall \; y \in \mathcal{R}(T).
   \]
Theorem 2.24 (Inverses). Let \( X, Y \) be vector spaces and \( T : D(T) \xrightarrow{\text{into}} Y \) be linear with \( D(T) \subset X \) and \( R(T) \subset Y \).

1. \( T^{-1} : R(T) \xrightarrow{\text{onto}} D(T) \) exists iff \( Tx = 0 \Rightarrow x = 0 \).

2. If \( T^{-1} \) exists then it is linear.

3. If \( \dim D(T) = n < \infty \) and \( T^{-1} \) exists then \( \dim R(T) = \dim D(T) \).

Proof.

1. Suppose \( Tx = 0 \Rightarrow x = 0 \). Then
\[
Tx_1 = Tx_2 \Rightarrow T(x_1 - x_2) = 0 \Rightarrow x_1 = x_2
\]
So \( T \) is injective \( T^{-1} \) exists.
Conversely, if \( T^{-1} \) exists then \( Tx_1 = Tx_2 \Rightarrow x_1 = x_2 \). Therefore
\[
Tx = T0 = 0 \Rightarrow x = 0.
\]

2. We assume that \( T^{-1} \) exists and show that it is linear in a straightforward fashion.

3. It follows from the earlier result applied to both \( T \) and \( T^{-1} \).

\[ \square \]

Theorem 2.25 (Inverse of Product (Composition)). Let \( T : X \rightarrow Y \) and \( S : Y \rightarrow Z \) be bijections. Then \( (ST)^{-1} : Z \rightarrow X \) exists and

\[
(ST)^{-1} = T^{-1}S^{-1}.
\]

2.4 Bounded and Continuous Linear Operators

Definition (Bounded Linear Operator). Let \( X, Y \) be normed spaces and \( T : D(T) \xrightarrow{\text{into}} Y, D(T) \subset X \). Then \( T \) is said to be bounded if \( \exists k > 0 \) s.t.

\[
\|Tx\| \leq k \|x\|, \forall x \in D(T).
\]

Definition (Induced Operator Norm). Let \( X, Y \) be normed spaces and \( T : D(T) \xrightarrow{\text{into}} Y, D(T) \subset X \). Then the norms on \( X, Y \) induce a norm for operators

\[
\|T\| = \sup_{x \in D(T)} \frac{\|Tx\|}{\|x\|}.
\]

Equivalently,

\[
\|T\| = \sup_{\|x\| = 1} \|Tx\|.
\]
The preceding definition implies that
\[ \| Tx \| \leq \| T \| \| x \|, \forall x. \]

**Example 2.26** (Differentiation is Unbounded). Let \( T : C[0,1] \to C[0,1], \) be defined by \( Tx(t) = x'(t) \). Then \( T \) is linear but unbounded. Indeed, let \( x_n(t) = t^n \). Then \( \| x_n \| = 1 \) and \( \| Tx_n \| = n \), i.e., \( T \) is not bounded.

**Example 2.27** (Integration is Bounded). Let \( T : C[0,1] \to C[0,1], y \in C[0,1]^2, y = Tx \) via
\[ y(t) = \int_0^1 k(t, \tau)x(\tau)d\tau. \]
Since \( k \) is continuous, \( \exists k_0 > 0 \) s.t.
\[ |k(t, \tau)| \leq k_0. \]
So then
\[ \| y \| = \| Tx \| \leq \max_{t \in [0,1]} \int_0^1 |k(t, \tau)| \| x(t) \| d\tau \leq k_0 \| x \|. \]

**Theorem 2.28** (Finite Dimension). Let \( X \) be normed and \( \dim X = n < \infty \). Then every linear operator on \( X \) is bounded.

**Proof.** Let \( e_1, \ldots, e_n \) be a basis for \( X \) and let \( x \in X \) be arbitrary and represented as
\[ x = \sum_{i=1}^n \alpha_i e_i. \]
Then
\[ \| Tx \| = \left\| \sum_{i=1}^n \alpha_i Te_i \right\| \leq k \sum_{i=1}^n |\alpha_i|, \]
where \( k = \max_{i=1,\ldots,n} \| Te_i \| \). Also,
\[ \sum_{i=1}^n |\alpha_i| \leq \frac{1}{c} \| x \|. \]
It follows that \( T \) is bounded.

Next, we illustrate a general method for computing bounds on operators.
Example 2.29 (Calculating Bounds on Operators). Let
\[ x = \sum_{i=1}^{n} \xi_i e_i. \]

Then
\[ \|Tx\| = \left\| \sum_{i=1}^{n} \xi_i Te_i \right\| \leq \sum_{i=1}^{n} |\xi_i| \|Te_i\| \leq \max_{k=1,\ldots,n} \|Te_k\| \sum_{i=1}^{n} |\xi_i|. \]

Then using Theorem 2.9 we can conclude
\[ \frac{1}{c} \|x\| = \frac{1}{c} \left\| \sum_{i=1}^{n} \xi_i e_i \right\| \geq \sum_{i=1}^{n} |\xi_i| \]

which implies
\[ \|Tx\| \leq \gamma \|x\| \text{ where } \gamma = \frac{1}{c} \max_{k=1,\ldots,n} \|Te_k\|. \]

Definition (Continuity for Operators). Let \( T : \mathcal{D}(T) \rightarrow Y \). Then \( T \) is said to be continuous at \( x_0 \in \mathcal{D}(T) \) if
\[ \forall \varepsilon > 0 \ \exists \ \delta > 0 \ s.t. \ \|x - x_0\| < \delta \Rightarrow \|Tx - Tx_0\| < \varepsilon. \]

\( T \) is said to be continuous if it is continuous at every \( x_0 \in \mathcal{D}(T) \).

Theorem 2.30 (Continuity and Boundedness). Let \( X, Y \) be vector spaces \( \mathcal{D}(T) \subset X \), and \( T : \mathcal{D}(T) \rightarrow Y \) be linear.

1. \( T \) is continuous iff \( T \) is bounded
2. \( T \) is continuous at a single point \( \Rightarrow T \) is continuous everywhere

Proof.

1. The case \( T = 0 \) is trivial. Assume \( T \neq 0 \), then \( \|T\| \neq 0 \). Suppose \( T \) is bounded and let \( x_0 \in \mathcal{D}(T), \varepsilon > 0 \) be arbitrary. Then because \( T \) is linear, \( \forall x \in \mathcal{D}(T) \) s.t.
\[ \|x - x_0\| < \delta = \frac{\varepsilon}{\|T\|} \]

we have that 
\[ \|T \cdot x - T \cdot x_0\| = \|T( \cdot x - x_0)\| \leq \|T\| \cdot \|x - x_0\| < \|T\| \cdot \delta = \varepsilon. \]

Therefore \( T \) is continuous.

Conversely, suppose \( T \) is continuous at \( x_0 \in \mathcal{D}(T) \). Then \( \forall \varepsilon > 0 \ \exists \ \delta > 0 \ \text{s.t.} \)
\[ \|x - x_0\| \leq \delta \Rightarrow \|T \cdot x - T \cdot x_0\| \leq \varepsilon. \]

So choose any \( y \in \mathcal{D}(T), \ y \neq 0 \) and set
\[ x = x_0 + \delta \frac{y}{\|y\|}. \]

Then
\[ x - x_0 = \delta \frac{y}{\|y\|} \Rightarrow \|x - x_0\| = \delta, \]
so
\[
\begin{align*}
\|T \cdot x - T \cdot x_0\| &= \|T( \cdot x - x_0)\| \\
&= \left\| T \left( \delta \frac{y}{\|y\|} \right) \right\| \\
&= \frac{\delta}{\|y\|} \cdot \|T \cdot y\| \\
&\leq \varepsilon.
\end{align*}
\]

Therefore we have
\[ \|T \cdot y\| \leq c \|y\| \text{ where } c = \frac{\varepsilon}{\delta}. \]

2. Above we only used continuity at a single point to show boundedness. Boundedness implies continuity.

\[ \square \]

**Corollary 2.31.** Let \( T : \mathcal{D}(T) \xrightarrow{\text{into}} Y \) be a bounded, linear operator. Then

1. \( (x_n) \subset \mathcal{D}(T), x \in \mathcal{D}(T) \) and \( x_n \to x \Rightarrow T \cdot x_n \to T \cdot x \)

2. \( \mathcal{N}(T) \) is closed.

**Proof.**

1. \( \|T \cdot x_n - T \cdot x\| = \|T( \cdot x_n - x)\| \leq \|T\| \cdot \|x_n - x\| \to 0 \)

2. \( \forall x \in \mathcal{N}(T) \ \exists (x_n) \subset \mathcal{N}(T) \ \text{s.t.} \)
\[ x_n \to x \Rightarrow T \cdot x_n \to T \cdot x. \]

But \( x_n \in \mathcal{N}(T) \Rightarrow T \cdot x_n = 0, \ \forall n \Rightarrow T \cdot x = 0 \Rightarrow x \in \mathcal{N}(T). \) Therefore \( \mathcal{N}(T) \) is closed.

\[ \square \]

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Compare to the notion of subadditivity (triangle inequality).

**Remark 2.32** (Operator Norm is Submultiplicative).

1. \[ \|TS\| \leq \|T\| \|S\| \]
2. \[ \|T^n\| \leq \|T\|^n \]

**Remark 2.33** (Equality of Operators). Two operators $T, S$ are said to be equal and we write $T = S$ if \[ \mathcal{D}(T) = \mathcal{D}(S) \text{ and } Tx = Sx, \forall x \in \mathcal{D}(T). \]

**Definition** (Restriction). Let $T : \mathcal{D}(T) \xrightarrow{\text{into}} Y$ and let $B \subset \mathcal{D}(T)$. We define the restriction of $T$ to $B$, denoted $T|_B : B \xrightarrow{\text{into}} Y$, as \[ T|_B x = Tx, \forall x \in B. \]

**Definition** (Extension). Let $T : \mathcal{D}(T) \xrightarrow{\text{into}} Y$ and $\mathcal{D}(T) \subset M$. An extension of $T$ is \( \hat{T} : M \xrightarrow{\text{into}} Y \) s.t. 
\[ \hat{T}|_{\mathcal{D}(T)} = T \]
\[ \hat{T}x = Tx, \forall x \in \mathcal{D}(T). \]

**Theorem 2.34** (Bounded Linear Extension). Let $\mathcal{D}(T) \subset X$, $X$ a normed space, $Y$ a Banach space, and $T : \mathcal{D}(T) \xrightarrow{\text{into}} Y$ be linear. Then there is a unique bounded, linear extension $\hat{T}$ of $T$ s.t. \[ \|\hat{T}\| = \|T\|. \]

**Proof.** Let $x \in \overline{\mathcal{D}(T)}$. Then $\exists (x_n) \in \mathcal{D}(T)$ s.t. $x_n \to x$. Since $T$ is linear and bounded we have 
\[ \|Tx_n - Tx\| = \|T(x_n - x)\| \leq \|T\| \|x_n - x\|, \]
so $(Tx_n)$ is Cauchy. Since $Y$ is Banach, $(Tx_n)$ converges, say 
\[ Tx_n \to y \in Y. \]

In this way, we define 
\[ \hat{T}x = y. \]
Because limits are unique, $\hat{T}$ is uniquely defined $\forall x \in \overline{\mathcal{D}(T)}$. Of course $\hat{T}$ inherits linearity and 
\[ \hat{T}x = Tx, \forall x \in \mathcal{D}(T) \]
so $\hat{T}$ is a genuine extension.
Finally, we need to show $\hat{T}$ is bounded. Note
\[ \|Tx_n\| \leq \|T\| \|x_n\| \]
and recall
\[ x \mapsto \|x\| \]
is continuous. Therefore $Tx_n \to y = \hat{T}x$ and
\[ \|\hat{T}x\| \leq \|x\| \|x\| . \]
Therefore $\hat{T}$ is bounded and $\|\hat{T}\| \leq \|T\|$. By definition, $\|\hat{T}\| \geq \|T\|$ so
\[ \|\hat{T}\| = \|T\|. \]

2.5 Linear Functionals

Definition (Linear Functional). A linear functional $f$ is a linear operator with $D(f) \subset X$ where $X$ is vector space and $R(f) \subset K$ where $K$ is the scalar field corresponding to $X$ ($\mathbb{R}$ or $\mathbb{C}$). Then
\[ f: D(f) \rightarrow K. \]

Definition (Bounded Linear Functional). Let $f: D(f) \rightarrow K$ be a linear functional. Then $f$ is said to be bounded if $\exists c > 0$ s.t.
\[ |f(x)| \leq c \|x\| , \forall x \in D(f) . \]

Definition (Norm of a Linear Functional). Let $f$ be a linear functional. Then
\[ \|f\| = \sup_{x \in D(f), x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{x \in D(f), \|x\| = 1} |f(x)| , \]
so
\[ |f(x)| \leq \|f\| \|x\| . \]

Theorem 2.35 (Continuity and Boundedness). Let $f$ be a linear functional. $f$ is continuous iff $f$ is bounded.
**Example 2.36** (Dot Product). Let $a \in \mathbb{R}, a \neq 0$ be fixed and let $f : \mathbb{R}^n \to \mathbb{R}$ be defined as

$$f(x) = x \cdot a.$$ 

Then $f$ is linear functional and

$$|f(x)| \leq \|x\| \|a\|.$$ 

On the other hand, if we take $x = a$

$$|f(x)| = \|a\|^2 \Rightarrow \|f\| \geq \|a\|.$$ 

**Example 2.37** (Integral). Define $f : C[a, b] \to \mathbb{R}$ by

$$f(x) = \int_a^b x(t)dt.$$ 

Then

$$|f(x)| \leq (b-a) \|x\|.$$ 

Further, if $x = 1$

$$|f(x)| = b-a \Rightarrow \|f\| = b-a.$$ 

**Example 2.38** (Point Evaluation). Let $t_0 \in [a, b]$ be fixed and define $f : C[a, b] \to \mathbb{R}$ by

$$f(x) = x(t_0).$$ 

Choosing the sup-norm, we see

$$|f(x)| = |x(t_0)| \leq \|x\|,$$

and since $f(1) = 1 = \|1\|$, we have

$$\|f\| = 1.$$ 

**Example 2.39** (Point Evaluation in $L^2$). Let $f : L^2[a, b] \to \mathbb{R}$ by

$$f(x) = x(a).$$ 

Choose the $L^2$ norm, and let $x \in L^2[a, b]$ be such that $x(a) = 1$ but then $x$ rapidly decreases to 0. Then $f(x) = 1$ but

$$\|x\| = \left( \int_a^b |x(t)|^2 \, dt \right)^{\frac{1}{2}}$$

and for any $\varepsilon > 0$ we can choose an $x$ such that $\|x\| < \varepsilon$. Therefore there is no such $c > 0$ s.t.

$$|f(x)| = 1 \leq c \|x\|.$$ 

So point evaluation in this space with this norm is unbounded.
**Example 2.40** (Inner Product in $\ell^2$). Fix $a \in \ell^2$ and define $f : \ell^2 \to \mathbb{R}$ by

$$f(x) = \sum_{i=1}^{\infty} x_i a_i = \langle x, a \rangle.$$ 

Then

$$|f(x)| \leq \sum_{i=1}^{\infty} |x_i a_i| \leq \sqrt{\sum_{i=1}^{\infty} x_i^2} \sqrt{\sum_{i=1}^{\infty} a_i^2} = \|x\| \|a\|.$$ 

**Definition** (Algebraic Dual). Let $X$ be a vector space. Then we denote $X^*$ as the space of linear functionals on $X$. $X^{**}$ represents the dual of $X^*$, et cetera.

<table>
<thead>
<tr>
<th>Space</th>
<th>General Element</th>
<th>Value at a Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>$x$</td>
<td>n/a</td>
</tr>
<tr>
<td>$X^*$</td>
<td>$f$</td>
<td>$f(x)$</td>
</tr>
<tr>
<td>$X^{**}$</td>
<td>$g$</td>
<td>$g(f)$</td>
</tr>
</tbody>
</table>

**Definition** (Isomorphism = Bijective Isometry). Let $X, Y$ be vector spaces over the same scalar field $\mathbb{K}$. Then an isomorphism $T : X \to Y$ is a bijective mapping which preserves the two algebraic operations of vector spaces, i.e.

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty.$$ 

So $T$ is a bijective linear operator and we say $X, Y$ are isomorphic.

If in addition $X, Y$ are normed spaces, then $T$ must also preserve the norms

$$\|x\| = \|Tx\|.$$ 

We can obtain $g \in X^{**}$ picking a fixed $x \in X$ and setting

$$g(f) = g_x(f) = f(x), \forall f \in X^*.$$ 

This $g$ is linear, and $x \in X \Rightarrow g_x \in X^{**}.$

**Definition** (Canonical Mapping/Embedding). Define $C : X \to X^{**}$ by

$$x \mapsto g_x.$$ 

Then $C$ is injective and linear. Therefore $C$ is an isomorphism of $X$ and $\mathcal{R}(C) \subset X^{**}.$
An interesting and important problem is when $\mathcal{R}(C) = X^{**}$, i.e., $X, X^{**}$ are isomorphic.

**Definition** (Embeddable). Let $X, Y$ be vector spaces. $X$ is said to be embeddable in $Y$ if it is isomorphic with a subspace of $Y$.

From Definition 2.5 we can see $X$ is always embeddable in $X^{**}$. $C$ is also called the canonical embedding of $X$ into $X^{**}$.

**Definition** (Algebraic Reflexive). If the canonical mapping $C$ is surjective (and hence bijective) then

$$\mathcal{R}(C) = X^{**}$$

and $X$ is said to be algebraically reflexive.

**Problem 13.** If $Y$ is a subspace of a vector space $X$ and $f$ is a linear functional on $X$ such that $f(Y)$ is not the whole scalar field $\mathbb{K}$, show $f(y) = 0$, $\forall y \in Y$.

**Solution.** Suppose that $f(y) = b \neq 0$ for some $y \in Y$. For arbitrary $a \in R$ we have $\frac{a}{b}y \in Y$ and $f\left(\frac{a}{b}y\right) = a$. It follows that $f(Y) = R$; a contradiction. $\square$

**Problem 14.** Let $f \neq 0$ be a linear functional on a vector space $X$, and let $x_0 \in X - N(f)$ be fixed. Show that every $x \in X$ has a unique representation

$$x = \alpha x_0 + y,$$

some $y \in N(f)$.

**Solution.** Let $f \neq 0$ and consider $x_0 \in X - N(f)$. Then $f(x_0) = c \neq 0$. Let $a = f(x) = f\left(\frac{a}{c}x_0\right)$. It follows that $x - \frac{a}{c}x_0 \in N(f)$. We can write

$$x = \frac{a}{c}x_0 + y,$$

where $y = x - \frac{a}{c}x_0 \in N(f)$. $\square$

### 2.6 Finite Dimensional Case

Let $X, Y$ be finite dimensional vector spaces over the same field $\mathbb{K}$, and let $T : X \to Y$ be a linear operator. Let $E = \{e_1, \ldots, e_n\}, B = \{b_1, \ldots, b_n\}$ be bases for $X, Y$ respectively. Then $x \in X$ and $y = Tx \in Y$ and $Te_k \in Y$, $\forall k$ have the unique representations

$$x = \sum_{k=1}^{n} \xi_k e_k$$

$$Te_k = \sum_{i=1}^{n} \tau_{ik} b_i$$

$$y = \sum_{i=1}^{n} \eta_i b_i.$$
Then calculate
\[ y = Tx \]
\[ = T \left( \sum_{k=1}^{n} \xi_k e_k \right) \]
\[ = \sum_{k=1}^{n} \xi_k T e_k \]
which implies
\[ y = \sum_{i=1}^{n} \eta_i b_i = \sum_{k=1}^{n} \xi_k \left( \sum_{i=1}^{n} \tau_{ik} b_i \right) \]
\[ = \sum_{i=1}^{n} \left( \sum_{k=1}^{n} \tau_{ik} \xi_k \right) b_i \]
yielding
\[ \eta_i = \sum_{k=1}^{n} \tau_{ik} \xi_k. \]
So if we label
\[ T_{EB} = (\tau_{ik}), \bar{x} = (\xi_k), \bar{y} = (\eta_i) \]
then
\[ \bar{y} = T_{EB} \bar{x}. \]
We say \( T_{EB} \) represents \( T \) with respect to the bases \( E, B \).

**Remark 2.41** (Finite Dimensional Linear Operators are Matrices). *The above argument shows that every f.d.l.o. can be represented as a matrix.*

### 2.6.1 Linear Functionals

Let \( X \) be a vector space, \( n = \text{dim} \ X \), and let \( \{e_1, \ldots, e_n\} \) be a basis for \( X \). Label
\[ \alpha_i = f(e_i). \]
Then
\[ f(x) = f \left( \sum_{i=1}^{n} \xi_i e_i \right) = \sum_{i=1}^{n} \xi_i \alpha_i. \]
\( (f \) is uniquely determined by its values \( \alpha_i \) at the \( n \) basis vectors of \( X \)).
Definition (Dual Basis). Define \( f_k \) by
\[
f_k(e_i) = \delta_{ik}, \; \forall \; k = 1, \ldots, n.
\]
Then
\[
\{ f_1, \ldots, f_n \} \leftrightarrow \{ e_1, \ldots, e_n \}.
\]

Theorem 2.42 (Dimension of \( X^* \)). Let \( X \) be a vector space, \( n = \dim X \), and let \( E = \{ e_1, \ldots, e_n \} \) be a basis for \( X \). Then \( \{ f_1, \ldots, f_k \} \) is a basis for \( X^* \) and \( \dim X^* = n \). So then if \( f \in X^* \) then we can represent
\[
f = \sum_{i=1}^{n} \alpha_i f_i.
\]

Proof. See Theorem 2.41.

Lemma 2.43 (Zero Vector). Let \( X \) be a f.d.v. If \( x_0 \in X \) has the property that \( f(x_0) = 0 \), \( \forall \; f \in X^* \) then \( x_0 = 0 \).

Proof. Let \( \{ e_1, \ldots, e_n \} \) be a basis for \( X \) and let
\[
x_0 = \sum_{i=1}^{n} \xi_0_i e_i.
\]
Then
\[
f(x_0) = \sum_{i=1}^{n} \xi_0_i \alpha_i.
\]
By assumption \( f(x_0) = 0 \), \( \forall \; f \in X^* \Rightarrow \xi_0_i = 0, \; \forall \; i = 1, \ldots, n. \)

Theorem 2.44 (Algebraic Reflexivity). A f.d.v. \( X \) is algebraically reflexive, i.e., \( X \) is isomorphic to \( X^{**} \).

Proof. By construction \( C \) is linear. Then
\[
Cx_0 = 0 \Rightarrow (Cx_0)(f) = g_{x_0}(f) = f(x_0) = 0, \; \forall \; f \in X^*
\]
and therefor \( x_0 = 0 \) by the previous lemma. Therefore \( C \) has an inverse \( C^{-1} : \mathcal{R}(C) \rightarrow X \). Therefore \( \dim \mathcal{R}(C) = \dim X \). Therefor \( C \) is an isomorphism and \( X \) is algebraically reflexive.

Problem 15. Find a basis for the null space of functional \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \) given by
\[
f(x) = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \alpha_3 \xi_3
\]
where \( \alpha_1 \neq 0 \).

Solution. Let \( (\alpha_1, \alpha_2, \alpha_3) \) with \( \alpha_1 \neq 0 \) be a fixed point in \( \mathbb{R}^3 \) and let \( f(x) = \sum a_i x_i \). The vectors \( \left(-\frac{\alpha_2}{\alpha_1}, 1, 0\right) \) and \( \left(-\frac{\alpha_3}{\alpha_2}, 0, 1\right) \) form a basis for \( N(f) \).
2.7 Dual Space

**Definition** \((B(X, Y))\). Let \(X, Y\) be vector spaces. Then we define \(B(X, Y)\) as the set of all bounded linear operators from \(X\) to \(Y\).

**Remark 2.45** \((B(X, Y)\) is a Vector Space). \((\alpha S + \beta T)x = \alpha Sx + \beta Tx\)

**Theorem 2.46** \((B(X, Y)\) is a Normed Space). Let \(X, Y\) be normed spaces. The vector space \(B(X, Y)\) is also a normed space with norm

\[
\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|.
\]

**Theorem 2.47** \((B(X, Y)\) is a Banach Space). If in the assumptions of the previous proof \(Y\) is Banach, then \(B(X, Y)\) is also a Banach space.

**Proof.** Let \((T_n)\) be an arbitrary Cauchy sequence in \(B(X, Y)\). Then

\[
\forall \varepsilon > 0 \exists N \text{ s.t. } \|T_n - T_m\| < \varepsilon, \forall m, n > N.
\]

Therefore \(\forall x \in X, \text{ and } m, n > N\)

\[
\|T_n x - T_m x\| = \|(T_n - T_m)x\| \\
\leq \|T_n - T_m\| \|x\| \\
< \varepsilon \|x\|.
\]

So fix \(x \in X\). We see \(\forall x \in X\) that \((T_n x) \subset Y\) is Cauchy. \(Y\) is complete so \(\exists y \in Y\) s.t. \(T_n x \to y\). We need to construct a limit for \((T_n)\). Define \(T : X \to Y\) by

\[Tx = y.\]

By construction, \(T\) is linear

\[
\lim_{n \to \infty} T_n (\alpha x + \beta y) = \lim_{n \to \infty} (\alpha T_n x + \beta T_n x) \\
= \alpha \lim_{n \to \infty} T_n x + \beta \lim_{n \to \infty} T_n x \\
= \alpha Tx + \beta Tx.
\]

The following

\[
\|T_n x - Tx\| = \left\|T_n x - \lim_{m \to \infty} T_m x\right\| \\
= \lim_{m \to \infty} \|T_n x - T_m x\| \\
\leq \varepsilon \|x\|
\]
implies that $T_n - T$ is bounded. Since $T_n \in B(X, Y) \Rightarrow T_n$ is bounded and 
\[T = T_n - (T_n - T)\]

$T$ is also bounded. Therefore $T \in B(X, Y)$.

Finally,
\[
\frac{\|T_n x - Tx\|}{\|x\|} \leq \varepsilon
\]

implies
\[
\|T_n - T\| \leq \varepsilon \Rightarrow \|T_n - T\| \rightarrow 0.
\]

\[\square\]

**Definition** (Normed Dual Space $X'$). Let $X$ be a n.s. Then the set of all bounded linear functionals on $X$ constitutes a n.s. with norm
\[
\|f\| = \sup_{x \in X, x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{x \in X, \|x\|=1} |f(x)|.
\]

This is the normed dual of $X$.

**Theorem 2.48** ($X'$ is a Banach Space). Let $X$ be a n.s. and let $X'$ be the dual of $X$. Then $X'$ is a Banach space.

**Proof.** See Theorem 2.47. \[\square\]

**Example 2.49** ($\mathbb{R}^n$). The dual of $\mathbb{R}^n$ is $\mathbb{R}^n$.

**Proof.** We claim $\mathbb{R}^{n*} = \mathbb{R}^n$ and $\forall f \in \mathbb{R}^{n*}$ we have 
\[
f(x) = \sum_{k=1}^{n} \xi_k \alpha_k, \alpha_k = f(e_k).
\]

So then
\[
|f(x)| \leq \sum_{k=1}^{n} |\xi_k \alpha_k| \leq \|x\| \sqrt{\sum_{k=1}^{n} \alpha_k^2},
\]

which implies
\[
\|f\| \leq \sqrt{\sum_{k=1}^{n} \alpha_k^2}.
\]

If we choose $x = (\alpha_k)$ then
\[
|f(x)| = \sqrt{\sum_{k=1}^{n} \alpha_k^2} \Rightarrow \|f\| = \sqrt{\sum_{k=1}^{n} \alpha_k^2}.
\]
So take the onto mapping from $\mathbb{R}^{n'}$ to $\mathbb{R}^n$

$$f \mapsto \alpha = (\alpha_k) = (f(e_k))$$

which is norm-preserving, linear, and a bijection (an isomorphism), which completes the proof. □

Problem 16.

1. Show $\ell^1' = \ell^{\infty}$

2. Show $\ell^p' = \ell^q$ where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$

Solution.

1. A Schauder basis for $\ell^1$ is $(e_k)$, where $e_k = (\delta_{ki})$. Then every $x \in \ell^1$ has a unique representation

$$x = \sum_{k=1}^{\infty} \xi_k e_k.$$  

We consider any $f \in \ell^1'$, where $\ell^1'$ is the dual space of $\ell^1$. Since $f$ is linear and bounded,

$$f(x) = \sum_{k=1}^{\infty} \xi_k \gamma_k , \text{ where } \gamma_k = f(e_k).$$

It is easy to see that $(\gamma_k) \in \ell^{\infty}$.

On the other hand, for every $b = (\beta_k) \in \ell^{\infty}$,

$$g(x) = \sum_{k=1}^{\infty} \xi_k / \beta_k , \text{ where } x = (\xi_k) \in \ell^1$$

is a bounded linear functional on $\ell^1$, i.e., $g \in \ell^1'$.

It is also easy see that $\|f\| = \|c\|_{\infty}$, where $c = (\gamma_k) \in \ell^{\infty}$. It follows that the bijective linear mapping of $\ell^1'$ onto $\ell^{\infty}$ defined by $f \mapsto c$ is an isomorphism.

2. Take the same Schauder basis and representation for $x$ and $f$ as in 1. Let $q$ be the conjugate of $p$ and consider $x_n = (\xi_k^{(n)})$ with

$$\xi_k^{(n)} = \frac{|\gamma_k|^q}{\gamma_k} , \text{ if } k \leq n \text{ and } \gamma_k \neq 0,$$

and

$$\xi_k^{(n)} = 0 \text{ if } k > n \text{ or } \gamma_k = 0.$$
Then we have

\[ f(x_n) = \sum_{k=1}^{\infty} \xi_k^{(n)} \gamma_k = \sum_{k=1}^{n} |\gamma_k|^q \leq \|f\| \left( \sum_{k=1}^{n} |\gamma_k|^q \right)^{\frac{1}{p}}. \]

Using \( 1 - \frac{1}{p} = \frac{1}{q} \) we get

\( \left( \sum_{k=1}^{n} |\gamma_k|^q \right)^{\frac{1}{q}} \leq \|f\|. \)

Since \( n \) is arbitrary, letting \( n \to \infty \), we obtain

\( \left( \sum_{k=1}^{\infty} |\gamma_k|^q \right)^{\frac{1}{q}} \leq \|f\|, \)

and therefore \( (\gamma_k) \in \ell^q \).

Conversely, for any \( b = (\beta_k) \in \ell^q \) we may define \( g \) on \( \ell^p \) by

\[ g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k, \text{ where } x = (\xi_k) \in \ell^p. \]

Then \( g \) is linear, and boundedness follows from the Holder inequality. Hence \( g \in \ell^{p'} \).

From the Holder inequality we get

\[ |f(x)| \leq \|x\| \left( \sum_{k=1}^{\infty} |\gamma_k|^q \right)^{\frac{1}{p}}. \]

It follows that \( \|f\| = \|c\|_q \), where \( c = (\gamma_k) \in \ell^q \) and \( \gamma_k = f(e_k) \). The mapping of \( \ell^{p'} \) onto \( \ell^q \) defined by \( f \mapsto c \) is linear, bijective, and norm-preserving, so it is an isomorphism.

\[ \square \]
3 Hilbert Spaces

**Definition (Inner Product).** Let $X$ be vector space with scalar field $\mathbb{K}$. $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$ is called an inner product if

1. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
2. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
3. $\langle x, y \rangle = \langle y, x \rangle$
4. $\langle x, x \rangle \geq 0$, $\langle x, x \rangle = 0$ iff $x = 0$

**Definition (Hilbert Space).** A Hilbert space is a complete inner product space.

**Remark 3.1.**

1. A vector space $X$ with inner product $\langle \cdot, \cdot \rangle$ has the induced norm and metric:
   \[ \|x\| = \langle x, x \rangle \]
   \[ d(x, y) = \|x - y\|. \]

2. If $X$ is real then $\langle x, y \rangle = \langle y, x \rangle$.

3. From the definition of inner product, we obtain
   \[ \langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \]
   \[ \langle x, \alpha y \rangle = \overline{\alpha} \langle x, y \rangle \]
   \[ \langle x, \alpha y + \beta z \rangle = \overline{\alpha} \langle x, y \rangle + \overline{\beta} \langle x, z \rangle \]

   So the inner product is linear in the first argument and conjugate linear in the second argument.

   \[ \|x + y\|^2 + \|x - y\|^2 = 2 \left( \|x\|^2 + \|y\|^2 \right) \]  \hspace{1cm} (5)

   Any norm which is induced from an inner product must satisfy this equality.

5. Orthogonality. $x, y \in X$ are said to be **orthogonal** if $\langle x, y \rangle = 0$ and we write $x \perp y$.

   For subsets $A, B \subset X$ we say $A \perp B$ if
   \[ a \perp b \quad \forall \ a \in A, b \in B. \]
Example 3.2.
1. \( \mathbb{R}^n \) with inner product \( \langle x, y \rangle = x^T y \).
2. \( \mathbb{C}^n \) with inner product \( \langle x, y \rangle = x^T \overline{y} \).
3. Real \( L^2[a, b] \) with
   \[ \langle x, y \rangle = \int_a^b x(t)y(t)dt. \]
4. Complex \( L^2[a, b] \) with
   \[ \langle x, y \rangle = \int_a^b x(t)\overline{y(t)}dt. \]
5. \( \ell^2 \) with inner product
   \[ \langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}. \]

Problem 17.
1. Show \( \ell^p \) with \( p \neq 2 \) is not a Hilbert space.
2. Show \( C[a, b] \) is not a Hilbert space.

Solution.
1. Consider \( x = (1, 1, 0, \ldots, 0, \ldots) \) and \( y = (1, -1, 0, \ldots, 0, \ldots) \). Then \( x, y \in \ell^p \) and \( \|x\|_p = \|y\|_p = 2^{\frac{1}{p}}, \) and \( \|x+y\|_p = \|x-y\|_p = 2 \). It follows that if \( p \neq 2 \), then the parallelogram equality is not satisfied.
2. Let \( x(t) = \frac{t-a}{b-a} \) and \( y(t) = 1 - \frac{t-a}{b-a} \). Then we have \( \|x\| = \|y\| = \|x+y\| = \|x-y\| = 1 \) and the parallelogram equality is not satisfied.

Problem 18. Show that \( \|x\| = \sqrt{\langle x, x \rangle} \) defines a norm.

Solution. It follows from the definition of the inner product that \( \|x\| \geq 0 \) and \( \|x\| = 0 \) iff \( x = 0 \). Also, we have using the definition of the inner product that
   \[ \|\alpha x\| = (\langle \alpha x, \alpha x \rangle)^{\frac{1}{2}} = (\alpha \overline{\alpha} \langle x, x \rangle)^{\frac{1}{2}} = (|\alpha|^2 \langle x, x \rangle)^{\frac{1}{2}} = |\alpha|\|x\|. \]

Concerning the triangle inequality we get
   \[ \|x + y\|^2 = \langle x + y, x + y \rangle = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle = \|x\|^2 + 2Re \langle x, y \rangle + \|y\|^2 \]
   \[ \leq \|x\|^2 + 2\|x\||y\| + \|y\|^2 = (\|x\| + \|y\|)^2, \]
   from which the result follows.
Lemma 3.3 (Continuity of Inner Product). If in an inner product space $y_n \to y$ and $x_n \to x$ then 
$$\langle x_n, y_n \rangle \to \langle x, y \rangle.$$ 

Theorem 3.4 (Completion). For any inner product space $X$ there is a Hilbert space $H$ and isomorphism $A$ from $X$ onto a dense subspace $W \subset H$. The space $H$ is unique up to isomorphisms.

Theorem 3.5 (Subspace). Let $Y \subset H$ with $H$ a Hilbert space. Then
1. $Y$ is complete iff $Y$ is closed in $H$.
2. If $Y$ is finite dimensional then $Y$ is complete.
3. If $H$ is separable then $Y$ is separable.

Definition (Convex Subset). $M \subset X$ is said to be convex if
$$ax + (1 - \alpha)y \in M, \forall \alpha \in [0, 1], a, b \in M.$$ 

Theorem 3.6. Let $X$ be an inner product space and $M \neq \emptyset$ a convex subset which is complete. Then $\forall x \in X \exists! y \in M$ s.t.
$$\delta = \inf_{\hat{y} \in M} \|x - \hat{y}\| = \|x - y\|.$$ 

Proof.
1. Existence. By definition of inf,
$$\exists (y_n) \text{ s.t. } \delta_n \to \delta, \text{ where } \delta_n = \|x - y_n\|.$$ 

We will show $(y_n)$ is Cauchy. Write $v_n = y_n - x$. Then $\|v_n\| = \delta_n$ and
$$\|v_n + v_m\| = \|y_n + y_m - 2x\|
= 2\left\| \frac{1}{2}(y_n + y_m) - x \right\| \geq 2\delta$$ 

Furthermore, $y_n - y_m = v_n - v_m, \Rightarrow$
$$\|y_n - y_m\|^2 = \|v_n - v_m\|^2
= - \|v_n + v_m\|^2 + 2(\|v_n\|^2 + \|v_m\|)
\leq -(2\delta)^2 + 2(\delta_n^2 + \delta_m^2).$$ 

Since $M$ is complete, $y_n \to y \in M$. Then $\|x - y\| \geq \delta$. Also, we have
$$\|x - y\| \leq \|x - y_n\| + \|y_n - y\|
= \delta_n + \|y_n - y\|
\to \delta + 0$$ 

therefore $\|x - y\| = \delta.$
2. Uniqueness. Suppose that two such elements exist, \( y_1, y_2 \in M \) and from above,

\[ \|x - y_1\| = \delta = \|x - y_2\|. \]

By (5) we have

\[
\begin{align*}
\|y_1 - y_2\|^2 &= \|(y_1 - x) + (x - y_2)\|^2 \\
&= 2\|y_1 - x\|^2 + 2\|y_2 - x\|^2 - \|(y_1 - x) + (y_2 - x)\|^2 \\
&= 2\delta^2 + 2\delta^2 - 2\left\|\frac{1}{2}(y_1 + y_2) - x\right\|^2 \\
&\leq 0
\end{align*}
\]

since

\[ 2^2\left\|\frac{1}{2}(y_1 + y_2) - x\right\|^2 \geq 4\delta^2. \]

Therefore \( y_1 = y_2 \).

\[ \square \]

**Lemma 3.7 (Orthogonality).** With the setup of the previous theorem, \( z = x - y \) is orthogonal to \( M \).

**Proof.** Suppose \( z \perp M \) were false, then \( \exists w \in M \) s.t.

\[ \langle z, w \rangle = \beta \neq 0. \]

and so \( w \neq 0 \). For any \( \alpha \),

\[
\begin{align*}
\|z - \alpha w\|^2 &= \langle z - \alpha w, z - \alpha w \rangle \\
&= \langle z, z \rangle - \alpha \langle z, w \rangle - \alpha \left[ \langle w, z \rangle - \bar{\alpha} \langle w, w \rangle \right] \\
&= \|z\|^2 - \alpha \beta - \alpha \left[ \beta - \bar{\alpha} \langle w, w \rangle \right].
\end{align*}
\]

If we choose

\[ \bar{\alpha} = \frac{\bar{\beta}}{\langle w, w \rangle}, \]

and continue the equality above,

\[
\begin{align*}
\|z - \alpha w\|^2 &= \delta^2 - \frac{\|\beta\|^2}{\langle w, w \rangle} - \alpha [0] \\
&< \delta^2
\end{align*}
\]

contradicting the minimality of \( y, \delta \). Therefore we must have \( z \perp M \).

\[ \square \]
Definition (Direct Sum). A vector space $X$ is said to be direct sum of subspaces $Y$ and $Z$, written
\[ X = Y \oplus Z, \]
if $\forall x \in X \exists! y \in Y, z \in Z$ s.t.
\[ x = y + z. \]

Definition (Orthogonal Complement). Let $Y$ be a closed subspace of $X$. Then the orthogonal complement of $Y$ in $X$ is
\[ Y^\perp = \{ z \in X \text{ s.t. } z \perp Y \}. \]

Theorem 3.8 (Direct Sum). Let $Y$ be a closed subspace of $H$. Then
\[ H = Y \oplus Y^\perp. \]

Proof.
1. Existence. $H$ is complete and $Y$ is closed implies $Y$ is complete. Further, we know $Y$ is convex. Therefore $\forall x \in H \exists y \in Y$ s.t.
\[ x = y + z, \text{ where } z \in Y^\perp. \]

2. Uniqueness. Assume $x = y + z = y_1 + z_1$. Then
\[ y - y_1 \in Y \]
\[ z - z_1 \in Y^\perp. \]
But $Y \cap Y^\perp = \{0\} \Rightarrow y - y_1 = z - z_1 = 0.$

Definition (Orthogonal Projection). Let $Y$ be a closed subspace of $H$, so $H = Y \oplus Y^\perp$, and let
\[ x = y + z \]
\[ y \in Y \]
\[ z \in Y^\perp. \]
Then the orthogonal projection onto $Y$ is $P : H \to Y$ given by
\[ Px = y. \]
Clearly $P$ is bounded. Further $P$ is idempotent, that is, $P^2 = P$, which we call idempotent.
Definition (Annihilator). Let $X$ be an inner product space and let $M$ be a nonempty subset of $X$. Then the annihilator of $M$ is

$$M^\perp = \{ x \in X \text{ s.t. } x \perp M \} = \{ x \in X \text{ s.t. } \langle x,v \rangle = 0 \ \forall \ v \in M \}.$$

Theorem 3.9. Let $M, X$ be as the definition above and denote $(M^\perp)^\perp = M^{\perp\perp}$. Then

1. $M^\perp$ is a vector space.
2. $M^\perp$ is closed.
3. $M \subset M^{\perp\perp}$.

Theorem 3.10. If $M$ is a closed subspace of a Hilbert space $H$ then

$$M^{\perp\perp} = M.$$

Lemma 3.11 (Dense Set). Let $M \neq \emptyset$ be a subset of a Hilbert space $H$. Then

$$\text{span}(M) = H \iff M^\perp = \{0\}.$$

Proof.

1. $\Rightarrow$. Assume $x \in M^\perp, V = \text{span}(M)$ is dense in $H$. Then $x \in \overline{V} = H \Rightarrow \exists (x_n) \subset V \text{ s.t. } x_n \to x$. Now $x \in M^\perp$ and $M^\perp \perp V \Rightarrow \langle x_n, x \rangle = 0$. But $\langle \cdot, \cdot \rangle$ is continuous $\Rightarrow \langle x_n, x \rangle \to \langle x, x \rangle = 0 \Rightarrow x = 0 \Rightarrow M^\perp = \{0\}$, since $x \in M^\perp$ was arbitrary.

2. $\Leftarrow$. Suppose $M^\perp = \{0\}$ and write $V = \text{span}(M)$. Then

$$x \perp V \Rightarrow x \perp M$$

$$\Rightarrow x \in M^\perp$$

$$\Rightarrow x = 0$$

$$\Rightarrow V^\perp = \{0\}$$

$$\Rightarrow \overline{V} = H.$$

Definition (Orthonormal Set/Sequence). $M$ is said to be an orthogonal set if $\forall x, y \in M$

$$x \neq y \Rightarrow \langle x, y \rangle = 0.$$

$M$ is said to be orthonormal if $\forall x, y \in M$

$$\langle x, y \rangle = \delta_{xy} = \begin{cases} 0 & x \neq y \\ 1 & x = y \end{cases}.$$

If $M$ is countable and orthogonal (resp. orthonormal) then we can write $M = (x_n)$ and we say $(x_n)$ is an orthogonal (resp. orthonormal) sequence.
Remark 3.12 (Pythagorean Relation, Linear Independence). Let \( M = \{x_1, \ldots, x_n\} \) be an orthogonal set. Then

1. 
\[
\left\| \sum_{i=1}^{n} x_i \right\|^2 = \sum_{i=1}^{n} \|x_i\|^2 .
\]

2. \( M \) is linearly independent.

Example 3.13 (Orthonormal Sequences).

1. \( \{(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1),(0,\ldots,0,1)\} \subset \mathbb{R} \).

2. \( (e_n) \subset \ell^2 \) where \( e_n = \delta_{ni} \).

3. Let \( X = C[0,2\pi] \) with \( \langle x, y \rangle = \int_0^{2\pi} x(t)y(t)dt \). Then the functions

\[(\sin nt), n = 1, 2, \ldots, (\cos nt), n = 0, 1, 2, \ldots \]

form an orthogonal sequence.

Remark 3.14 (Unique Representation with Orthonormal Sequences). Let \( X \) be an inner product space and let \( (e_k) \) be an orthonormal sequence in \( X \), and suppose \( x \in \text{span}(\{e_1, \ldots, e_n\}) \) where \( n \) is fixed. Then we can represent

\[ x = \sum_{k=1}^{n} \alpha_k e_k \Rightarrow \langle x, e_i \rangle = \left\langle \left[ \sum_{k=1}^{n} \alpha_k e_k \right], e_i \right\rangle = \alpha_i \]

\[ \Rightarrow x = \sum_{k=1}^{n} \langle x, e_k \rangle e_k . \]

Now let \( x \in X \) be arbitrary and take \( y \in \text{span}(\{e_1, \ldots, e_n\}) \), where

\[ y = \sum_{k=1}^{n} \langle x, e_k \rangle e_k . \]

Define \( z \) by setting \( x = y + z \Rightarrow z \perp y \) because

\[
\langle z, y \rangle = \langle x - y, y \rangle = \langle x, y \rangle - \langle y, y \rangle = \left[ \sum \langle x, e_k \rangle e_k \right] - \|y\|^2
\]

\[ = \sum \langle x, \langle x, e_k \rangle e_k \rangle - \|y\|^2 \]

\[ = \sum \langle x, e_k \rangle \langle x, e_k \rangle - \|y\|^2 \]

\[ = 0 . \]
Then

\[ \|x\|^2 \Rightarrow \|y\|^2 + \|z\|^2 \]

\[ \|z\|^2 \Rightarrow \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 \geq 0 \]

\[ \sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2 \]

**Remark 3.15 (Bessel’s Inequality).** Let \( X \) be an inner product space and let \((e_k)\) be an orthonormal sequence in \( X \). Then the last inequality in the above remark is called **Bessel’s Inequality**:

\[ \sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2. \]

**Definition (Fourier Coefficients).** The sequence \((\langle x, e_k \rangle)\) is called the Fourier coefficients of \( x \) w.r.t. \((e_k)\).

**Problem 19 (Fourier Coefficients are Minimizers).** Let \( \{e_1, \ldots, e_n\} \) be an orthonormal set in an inner product space \( X \) (\( n \) is fixed). Let \( x \in X \) be an arbitrary, fixed element and let \( y = \sum_{k=1}^{n} \beta_k e_k \). Then \( \|x - y\| \) depends on \( \beta_1, \ldots, \beta_n \). Show by direct calculation that \( \|x - y\| \) is minimized iff \( \beta_i = \langle x, e_i \rangle \), \( \forall \; i = 1, \ldots, n \).

**Solution.** Let \( \gamma_i = \langle x, e_i \rangle \), and \( y = \sum \beta_i e_i \). Then

\[ \|x - y\|^2 = \left\langle x - \sum \beta_i e_i, x - \sum \beta_i e_i \right\rangle = \|x\|^2 - \sum \beta_i \gamma_i - \sum |\beta_i|^2 \]

and this is minimum for given \( x \) and \( e_i \satisfies \beta_i = \gamma_i \).

**Problem 20 (Gramm-Schmidt).** Orthonormalize the first three terms of the sequence \((1, t, t^2, t^3, \ldots)\) on the interval \([-1, 1]\) where

\[ \langle x, y \rangle = \int_{-1}^{1} x(t) y(t) dt. \]

**Solution.** Let \( f_1(t) = 1 \), then \( e_1(t) = \frac{f_1(t)}{\|f_1(t)\|} = \frac{1}{\sqrt{2}} \). Let \( f_2(t) = t \). We have that \( \langle f_2(\cdot), e_1(\cdot) \rangle = 0 \), so we just need to normalize \( f_2(t) \) to get \( e_2(t) = \sqrt{\frac{3}{2}} t \). Let \( f_3(t) = t^2 \). Easy calculation shows that \( \langle f_3(\cdot), e_1(\cdot) \rangle = \sqrt{\frac{3}{2}} \) and \( \langle f_3(\cdot), e_2(\cdot) \rangle = 0 \). Then \( f_3(t) = \langle f_3(\cdot), e_2(\cdot) \rangle e_2(t) = t^2 - \frac{4}{3} \). Normalizing this quantity we get \( e_3(t) = \sqrt{\frac{3}{8}} (3t^2 - 1) \).
Theorem 3.16 (Convergence). Let $H$ be a Hilbert space and let $(e_n) \subset H$ be an orthonormal sequence. Consider

$$
\sum_{k=1}^{\infty} \alpha_k e_k.
$$

(6)

1. The series (6) converges in the induced norm of $H$ iff

$$
\sum_{k=1}^{\infty} |\alpha_k|^2 < \infty.
$$

2. If (6) converges to $x$, then $\alpha_k = \langle x, e_k \rangle$ and

$$
x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k.
$$

3. For any $x \in H$ the series (6) with $\alpha_k = \langle x, e_k \rangle$ converges (in the norm of $H$).

Proof. Let

$$
s_n = \sum_{k=1}^{n} \alpha_k e_k
$$

and

$$
\sigma_n = \sum_{k=1}^{n} |\alpha_k|^2.
$$

1. For $n > m$

$$
\|s_n - s_m\|^2 = \left\| \sum_{k=m+1}^{n} \alpha_k e_k \right\|^2
$$

$$
= \sum_{k=m+1}^{n} |\alpha_k|^2
$$

$$
= \sigma_n - \sigma_m.
$$

Hence $(s_n)$ is Cauchy in $H$ iff $(\sigma_n)$ is Cauchy in $\mathbb{R}$.

2. Note $\langle s_n, e_i \rangle = \alpha_i$, $\forall \ i = 1, \ldots, k \leq n$. By assumption $s_n \to x$, so

$$
\Rightarrow \alpha_i = \langle s_n, e_i \rangle \to \langle x, e_i \rangle , \text{ for } i \leq k
$$

$$
\Rightarrow \alpha_i = \langle x, e_i \rangle \ \forall \ i = 1, 2, \ldots
$$

3. This follows from Bessel’s inequality and 1.
Definition (Total Set). Let $X$ be an inner product space and let $M \subset X$.

1. $\text{span}(M) = X \Rightarrow M$ is a total set.
2. If $M$ is an orthonormal set then $M$ is a total orthonormal set.

Remark 3.17.

1. A total orthonormal family in $X$ is sometimes called an orthonormal basis for $X$. Note this is not equivalent to an algebraic basis unless $X$ is finite dimensional.

2. In every nontrivial Hilbert space $H \neq \{0\}$ there is a total orthonormal set.

Definition (Hilbert Dimension). The Hilbert dimension of $X$ is the cardinality of the smallest orthonormal set, i.e., if $\Lambda = \{M \text{ s.t. } \text{span}(M) = H\}$ then the Hilbert dimension is $\inf_{M \in \Lambda} |M|$. 

Problem 21. Let $X$ be an inner product space and let $M \subset X$. Then

1. If $M$ is total in $X$ then $x \perp M \Rightarrow x = 0$ \hspace{1cm} (7)
2. If $X$ is complete, then (7) $\Rightarrow M$ is total in $X$.

Solution. 1. Let $H$ be the completion of $X$. Then, $X$ regarded as a subspace of $H$, is dense in $H$. By assumption, $M$ is total in $X$, so $\text{span}M$ is dense in $X$, and dense in $H$. It follows that the orthogonal complement of $M$ in $H$ is $\{0\}$.

2. If $x$ is a Hilbert space and $M^\perp = \{0\}$, then the Dense Set Lemma implies that $M$ is total in $X$.

Problem 22. Show that an orthonormal set $M$ in a Hilbert space $H$ is total iff

$$\sum_k |\langle x, e_k \rangle|^2 = \|x\|^2, \forall \ x \in M.$$ 

This is called Parseval’s equality.

Solution. If $M$ is not total, by Problem 21 there is a nonzero $x \perp M$ in $H$. Since $x \perp M$ we have 0 on the left-hand side of Parseval’s Equality which is not equal to $\|x\|$. Hence if Parseval’s Equality holds for all $x \in H$, then $M$ must be total in $H$.

Conversely, assume $M$ to be total in $H$. Consider any $x \in H$ and its nonzero Fourier coefficients arranged in a sequence, i.e., $\langle x, e_1 \rangle, \langle x, e_2 \rangle, \ldots$. Define $y = \sum \langle x, e_k \rangle e_k$. It follows that $x - y \in M^\perp$. Since $M$ is total, then $x = y$. 

Theorem 3.18 (Separable Hilbert Space). Let $H$ be a Hilbert space.

1. If $H$ is separable then every orthonormal set is countable.

2. If $H$ contains an orthonormal sequence which is total then $H$ is separable.

3.1 Representation of Functionals on Hilbert Spaces

Theorem 3.19 (Riesz). Let $H$ be a Hilbert space. Every bounded, linear functional on $H$ can be represented in terms of the inner product on $H$, i.e.,

$$f(x) = (x, z)$$

where $z$ is uniquely determined by $f$ and

$$\|z\| = \|f\|.$$

Proof.

1. Existence of $z$. If $f = 0$ take $z = 0$. Otherwise assume $f \neq 0$. Then $\mathcal{N}(f) \neq H$ and $\mathcal{N}(f) \neq H \Rightarrow \mathcal{N}(f)^\perp \neq \{0\}$. Let $w \in \mathcal{N}(f)^\perp$ s.t. $w \neq 0$ and set

$$v = f(x)w - f(w)x, \ x \in H.$$  

Then

$$\Rightarrow f(v) = f(x)f(w) - f(w)f(x) = 0$$  

$$\Rightarrow v \in \mathcal{N}(f).$$

Since $w \perp \mathcal{N}(f)$ we have

$$0 = (v, w)$$  

$$= (f(x)w - f(w)x, w)$$  

$$= f(x)\langle w, w \rangle - f(w)\langle x, w \rangle$$  

$$= f(x)\|w\|^2 - f(w)\langle x, w \rangle.$$  

Then

$$\Rightarrow f(x) = \frac{f(w)}{\|w\|^2} \langle x, w \rangle$$  

$$\Rightarrow f(x) = \left\langle x, \frac{f(w)}{\|w\|^2} w \right\rangle.$$  

Then

$$z = \left(\frac{f(w)}{\|w\|^2}\right) w.$$
2. Uniqueness of $z$. Suppose $f(x) = \langle x, z_1 \rangle = \langle x, z_2 \rangle$. Then

$$\Rightarrow (x, z_1 - z_2) = 0, \; \forall \; x \in H.$$ Choose $x = z_1 - z_2$. Then

$$\Rightarrow \langle z_1 - z_2, z_1 - z_2 \rangle = 0$$
$$\Rightarrow z_1 = z_2.$$

3. $\|f\| = \|z\|$. If $f = 0$ then $z = 0$ and $\|f\| = \|z\| = 0$. For $f \neq 0$, $z \neq 0$. Note

$$f(z) = \langle z, z \rangle$$
$$= \|z\|^2, \text{ and}$$
$$\|z\|^2 \leq \|f\| \|z\|$$
$$\Rightarrow \|z\| \leq \|f\|.$$ Also

$$|f(x)| = |\langle x, z \rangle|$$
$$\leq \|x\| \|z\|$$
$$\Rightarrow \|f\| \leq \|z\|.$$ This yields

$$\|f\| = \|z\|.$$

Lemma 3.20 (Equality). Let $X$ be an inner product space. Then

$$\langle x, w \rangle = \langle y, w \rangle \; \forall \; w \in X \Rightarrow x = y.$$ In particular,

$$\langle x, w \rangle = 0 \; \forall \; w \in X \Rightarrow x = 0.$$

Definition (Sesquilinear Form). Let $X, Y$ be vector spaces over the same scalar field $K$ ($\mathbb{R}$ or $\mathbb{C}$). A sesquilinear form (or sesquilinear functional) $h$ on $X \times Y$ is a mapping

$$h : X \times Y \to \mathbb{K}$$

such that

$$\forall \; x, x_1, x_2 \in X \; \text{and} \; y, y_1, y_2 \in Y \; \text{and} \; \alpha, \beta \in \mathbb{K}$$
$$1. \; h(x_1 + x_2, y) = h(x_1, y) + h(x_2, y)$$

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2. \( h(x, y_1 + y_2) = h(x, y_1) + h(x, y_2) \)

3. \( h(\alpha x, y) = \alpha h(x, y) \)

4. \( h(x, \beta y) = \overline{\beta} h(x, y) \)

**Remark 3.21.**

1. If \( X, Y \) are real (\( \mathbb{K} = \mathbb{R} \)), then
   \[
   h(x, \beta y) = \beta h(x, y)
   \]
   and \( h \) is said to be **bilinear**.

2. If \( X, Y \) are vector spaces and \( \exists c \in \mathbb{R} \) s.t.
   \[
   |h(x, y)| \leq c \|x\| \|y\| , \forall x, y
   \]
   then \( h \) is bounded and
   \[
   \|h\| = \sup_{x \neq 0, y \neq 0} \frac{|h(x, y)|}{\|x\| \|y\|}
   \]
   \[
   = \sup_{\|x\|=1, \|y\|=1} |h(x, y)|
   \]
   \[
   \Rightarrow |h(x, y)| \leq \|h\| \|x\| \|y\| .
   \]

**Theorem 3.22** (Riesz Representation). Let \( H_1, H_2 \) be Hilbert spaces and

\[
h : H_1 \times H_2 \rightarrow \mathbb{K}
\]

a bounded sesquilinear form. Then \( h \) has a representation

\[
h(x, y) = \langle Sx, y \rangle
\]

where \( S : H_1 \rightarrow H_2 \) is a bounded, linear operator. \( S \) is uniquely determined by \( h \) and

\[
\|S\| = \|h\| .
\]

**Proof.**

1. Existence of \( S \). Consider \( \overline{h(x, y)} \) which is linear in \( y \). If we fix an \( x \), then \( \overline{h(x, y)} \) is a bounded, linear functional and we can use the Riesz theorem to find \( z \) and represent

   \[
   \overline{h(x, y)} = \langle y, z \rangle ,
   \]
hence
\[ h(x, y) = \langle z, y \rangle. \quad (8) \]

Note that \( z \) is unique, but it depends on \( x \in H_1 \). This means that (8) with variable \( x \) defines an operator \( S : H_1 \to H_2 \) given by
\[ z = Sx, \]
and we write
\[ h(x, y) = \langle z, y \rangle = \langle Sx, y \rangle. \]

We must show \( S \) is linear. Observe
\[
\langle S(\alpha x_1 + \beta x_2), y \rangle = h(\alpha x_1 + \beta x_2, y)
= \alpha h(x_1, y) + \beta h(x_2, y)
= \alpha \langle Sx_1, y \rangle + \beta \langle Sx_2, y \rangle, \quad \forall y \in H_2
\Rightarrow S(\alpha x_1 + \beta x_2) = \alpha Sx_1 + \beta Sx_2.
\]

2. Uniqueness of \( S \). If \( h(x, y) = \langle Sx, y \rangle = \langle Tx, y \rangle \), then \( S = T \) by the equality lemma.

3. Boundedness of \( S \). Note first
\[
\|h\| = \sup_{x \neq 0 \atop y \neq 0} \left| \frac{\langle Sx, y \rangle}{\|x\|\|y\|} \right|
\geq \sup_{x \neq 0 \atop Sx \neq 0} \left| \frac{\langle Sx, Sx \rangle}{\|x\|\|Sx\|} \right|
= \sup_{x \neq 0} \left| \frac{\|Sx\|}{\|x\|} \right|
= \|S\|
\Rightarrow \|h\| \geq \|S\|.
\]

So \( S \) is bounded. Also
\[
\|h\| = \sup_{x \neq 0 \atop y \neq 0} \left| \frac{\langle Sx, y \rangle}{\|x\|\|y\|} \right|
\leq \sup_{x \neq 0 \atop y \neq 0} \left| \frac{\|Sx\|\|y\|}{\|x\|\|y\|} \right|
\leq \sup_{x \neq 0} \left| \frac{\|Sx\|}{\|x\|} \right|
= \|S\|
\Rightarrow \|h\| \leq \|S\|. \]
This yields

\[ \|S\| = \|h\|. \]

**Problem 23.** Let \( H \) be a Hilbert space. Show that \( H' \) (the dual of \( H \)) is a Hilbert space with inner product \( \langle \cdot, \cdot \rangle_1 \) defined by

\[ \langle f_z, f_v \rangle_1 = \langle z, v \rangle = \langle v, z \rangle, \]

where

\[ f_z(x) = \langle x, z \rangle \]
\[ f_v(x) = \langle x, v \rangle \]

**Solution.** It is easy to verify that \( \langle \cdot, \cdot \rangle_1 \) is an inner product on \( H' \). Since \( \|f_z\| = \|z\| = ((\langle z, z \rangle)_1)^{1/2} = ((\langle f_z, f_z \rangle)_1)^{1/2} \) the norm on \( H' \) is induced by the inner product \( \langle \cdot, \cdot \rangle_1 \). We know that the normed dual is always a Banach space. It follows that \( H' \) is complete, and therefore a Hilbert space.

**Problem 24.** Show that any Hilbert space \( H \) is isomorphic with its second dual, i.e.,

\[ H \cong (H')' = H''. \]

**Solution.** Let \( T : H \rightarrow h'' \) by \( z \rightarrow F_z \), where \( F_z : H' \rightarrow K \) is defined by \( F_z(f) = \langle f, f_z \rangle_1 \), where \( \langle \cdot, \cdot \rangle_1 \) and \( f_z \) are the same as in Problem 23. By repeated applications of the Riesz representation theorem we have that \( F(f) = \langle f, g \rangle_1 \) for some \( g \in h' \) and \( g = f_z \) for some \( z \in H \). It follows that \( T \) is surjective.

Suppose that \( z, w \in H \) and \( F_z = F_w \). Then \( \langle f, f_z \rangle_1 = \langle f, f_w \rangle_1 \) for all \( f \in H' \) and we get that \( f_z = f_w \) and \( z = w \). Therefore, \( T \) is injective.

It is easy to see that \( T \) is linear.

Also, by the Riesz representation theorem we have that \( \|F_z\| = \|f_z\| = \|z\| \), i.e., \( T \) preserves norms, and inner products. It follows that \( T \) is an isomorphism.

### 3.2 Hilbert Adjoint

**Definition** (Hilbert Adjoint \( T^* \)). Let \( H_1, H_2 \) be Hilbert spaces and let \( T : H_1 \rightarrow H_2 \) be a bounded, linear operator. Then the adjoint is \( T^* : H_2 \rightarrow H_1 \) s.t.

\[ \langle Tx, y \rangle = \langle x, T^* y \rangle, \quad \forall \ x \in H_1 \text{ and } x \in H_2. \]

**Theorem 3.23** (Existence of the Hilbert Adjoint).

1. \( T^* \) exists.
2. $T^*$ is unique.

3. $\|T^*\| = \|T\|$. 

Proof. Observe that $h(y, x) = \langle y, Tx \rangle$ is sesquilinear form on $H_2 \times H_1$.

$$\Rightarrow |h(x, y)| \leq \|y\| \|Tx\|$$
$$\leq \|T\| \|x\| \|y\|$$
$$\Rightarrow \|h\| \leq \|T\|.$$ 

Also

$$\|h\| = \sup_{x \neq 0, y \neq 0} \frac{|\langle y, Tx \rangle|}{\|y\| \|x\|}$$
$$\geq \sup_{x \neq 0, Tx \neq 0} \frac{|\langle Tx, Tx \rangle|}{\|Tx\| \|x\|}$$
$$= \|T\|$$
$$\Rightarrow \|h\| = \|T\|.$$ 

Therefore $h$ is a bounded sesquilinear form. Using a Riesz representation, $h(x, y) = \langle T^* y, x \rangle$ where $T^*$ exists, is unique with norm

$$\|T^*\| = \|h\| = \|T\|.$$ 

Also

$$\Rightarrow \langle y, Tx \rangle = \langle T^* y, x \rangle$$
$$\Rightarrow \langle Tx, y \rangle = \langle x, T^* y \rangle.$$ 

\[ \square \]

**Lemma 3.24 (Zero Operator).** Let $X, Y$ be inner product spaces and $Q : X \to Y$ be bounded, linear operators. Then

1. $Q = 0$ iff $\langle Qx, y \rangle = 0 \ \forall \ x \in X \text{ and } y \in Y$

2. Let $X$ be complex and $Q : X \to X$. Then

$$\langle Qx, x \rangle = 0 \ \forall \ x \in X \Rightarrow Q = 0.$$ 

Proof.
1. \(\Rightarrow\).

\[
Q = 0 \Rightarrow Qx = 0 \forall x \in X
\]

\[
\Rightarrow \langle Qx, y \rangle = (0, y) = 0 \langle w, y \rangle = 0.
\]

\(\Leftarrow\).

\[
\langle Qx, y \rangle = 0 \forall x, y \Rightarrow Qx = 0 \forall x, y
\]

\[
\Rightarrow Q = 0.
\]

2. Let \(x, y \in X\), then \(v = \alpha x + y \in X\). Then

\[
\langle Qv, v \rangle = \langle Qx, x \rangle = \langle Qy, y \rangle = 0
\]

\[
\Rightarrow 0 = \langle Q(\alpha x + y), \alpha x + y \rangle
\]

\[
= |\alpha|^2 \langle Qx, x \rangle + \langle Qy, y \rangle + \alpha \langle Qx, y \rangle + \overline{\alpha} \langle Qy, x \rangle
\]

\[
= \alpha \langle Qx, y \rangle + \overline{\alpha} \langle Qy, x \rangle.
\]

Now, take \(\alpha = 1\) then \(\alpha = i\) to obtain the two relations

\[
\langle Qx, y \rangle + \langle Qy, x \rangle = 0
\]

\[
\langle Qx, y \rangle - \langle Qy, x \rangle = 0.
\]

Then \(\langle Qx, y \rangle = 0 \Rightarrow Q = 0\) by 1. \(\square\)

**Remark 3.25.** In the previous lemma, 2 is not necessarily true if \(X\) is real.

**Theorem 3.26** (Properties of The Hilber Adjoint). Let \(H_1, H_2\) be Hilbert spaces. Let \(S, T : H_1 \to H_2\) be bounded, linear operators and let \(\alpha\) be a scalar. Then

1. \(\langle T^*y, x \rangle = \langle y, Tx \rangle\)
2. \((S + T)^* = S^* + T^*\)
3. \((\alpha T)^* = \overline{\alpha}T^*\)
4. \((T^*)^* = T\)
5. \(\|T^*T\| = \|TT^*\| = \|T\|^2\)
6. \(T^*T = 0\) iff \(T = 0\)
7. \((ST)^* = T^*S^*\), assuming \(H_1 = H_2\).

**Definition** (Self-Adjoint, Unitary, Normal). Let \(H\) be a Hilbert space and \(T : H \rightarrow H\).

1. \(T\) is self-adjoint (or Hermitian) if \(T^* = T\)
2. \(T\) is unitary if \(T\) is bijection and \(T^* = T^{-1}\)
3. \(T\) is normal if \(TT^* = T^*T\)

**Remark 3.27.**

1. If \(T\) is self-adjoint then \(\langle Tx, y \rangle = \langle x, Ty \rangle\).
2. If \(T\) is self-adjoint or normal then \(T\) is normal.

**Example 3.28.** Consider \(\mathbb{C}^n\) with \(\langle x, y \rangle = x^T \overline{y}\). Let \(T : \mathbb{C}^n \rightarrow \mathbb{C}^n\). If we specify a basis for \(\mathbb{C}^n\), we can represent \(T, T^*\) by matrices \(A, B\). Then

\[
\langle Tx, y \rangle = (Ax)^T \overline{y} = x^T A^T \overline{y} \\
\langle x, T^* y \rangle = x^T B \overline{y}.
\]

Therefore

\[
\Rightarrow A^T = \overline{B} \\
\Rightarrow B = \overline{A}^T.
\]

If \(T\) is self-adjoint then \(A = \overline{A}^T\).

**Theorem 3.29** (Self-Adjointness). Let \(H\) be a Hilbert space and let \(T : H \rightarrow H\) be a bounded, linear operator. Then

1. If \(T\) is self-adjoint then \(\langle Tx, x \rangle\) is real for all \(x \in H\).
2. If \(H\) is complex and \(\langle Tx, X \rangle\) is real for all \(x \in H\) then \(T\) is self-adjoint.

**Proof.**

1. If \(T\) is self-adjoint, then for all \(x \in X\) we have

\[
\overline{\langle Tx, x \rangle} = \langle x, Tx \rangle = \langle Tx, x \rangle.
\]
2. If \( \langle Tx, x \rangle \) is real for all \( x \in X \), then
\[
\langle Tx, x \rangle = \langle T^*x, x \rangle = \langle x, T^*x \rangle = \langle T^*x, x \rangle.
\]
Hence
\[
0 = \langle Tx, x \rangle - \langle T^*x, x \rangle
\]
\[
= \langle (T - T^*)x, x \rangle
\]
\[
\Rightarrow T = T^*
\]
by the zero operator lemma, since \( H \) is complex.

\[\square\]

**Theorem 3.30.** Let \( H \) be a Hilbert space and let \((T_n)\) be a sequence of bounded, linear, self-adjoint operators \( T_n : H \to H \). Suppose that \( T_n \to T \) in norm, that is \( \|T_n - T\| \to 0 \), where \( \|\cdot\| \) is the norm on the space \( B(H, H) \). Then \( T \) is a bounded, linear, self-adjoint operator on \( H \).

**Proof.** We show \( T^* = T \).
\[
\|T - T^*\| \leq \|T - T_n\| + \|T_n - T_n^*\| + \|T_n^* - T^*\|
\]
\[
= \|T - T_n\| + 0 + \|T_n^* - T^*\|
\]
\[
= 2 \|T - T_n\|
\]
\[
\to 0.
\]

\[\square\]

**Theorem 3.31 (Unitary Operators).** Let \( H \) be a Hilbert space and let \( U, V : H \to H \) be unitary. Then

1. \( U \) is isometric, i.e., \( \|Ux\| = \|x\| \) \( \forall \ x \in H \)
2. \( \|U\| = 1 \) provided \( H \neq \{0\} \)
3. \( U^{-1} = U^* \) is unitary
4. \( UV \) is unitary
5. \( U \) is normal.
6. If \( T \) is a bounded, linear operator on \( H \) and \( H \) is complex, then \( T \) is unitary iff \( T \) is isometric and surjective.

**Proof.**

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1. \[ \|Ux\|^2 = \langle Ux, Ux \rangle = \langle x, U^*Ux \rangle = \langle x, Ix \rangle = \|x\|^2 \]

2. Follows from 1.

3. Since $U$ is bijective, so is $U^{-1}$ and
\[ (U^{-1})^* = U^{**} = U = (U^{-1})^{-1}. \]

4. $UV$ is bijective and
\[ (UV)^* = V^*U^* = V^{-1}U^{-1} = (UV)^{-1}. \]

5. $U^{-1} = U^*$ and $UU^{-1} = U^{-1}U = I$.

6. $\Rightarrow$. Suppose that $T$ is isometric and surjective. Isometry implies injectivity, so $T$ is bijective. Need to show $T^* = T^{-1}$. By isometry,
\[ \langle T^*Tx, x \rangle = \langle Tx, Tx \rangle = \langle x, x \rangle = \langle Ix, x \rangle \]
\[ \Rightarrow \langle (T^*T - I)x, x \rangle = 0 \]
\[ \Rightarrow T^*T = I. \]

Also,
\[ TT^* = TT^*(TT^{-1}) = T(T^*T)T^{-1} = I. \]

Therefore $T^* = T^{-1}$.

$\Leftarrow$. Conversely, $T$ is isometric by 1 and surjective by definition.

\[ \square \]
4 Fundamental Theorems

Definition (Partially Ordered Set). Let $M$ be a set with a relation $\leq$. This relation is said to be a partial order if

1. $a \leq a \forall a \in M$
2. $a \leq b$ and $b \leq a \Rightarrow a = b$
3. $a \leq b$ and $b \leq c \Rightarrow a \leq c$.

We say $M$ is a partially ordered set.

Definition (Upper Bound). Let $M$ be partially ordered set. If $\exists u \in M$ s.t. $x \leq u \forall x \in M$, then $u$ is said to be a upper bound. Such a $u$ is not guaranteed to exist.

Definition (Maximal Element). Let $M$ be partially ordered set. If $\exists m \in M$ s.t. $m \leq x \Rightarrow m = x \forall x \in M$, then $m$ is said to be a maximal elemtn. Such a $m$ is not guaranteed to exist or be unique.

Example 4.1.

1. $\mathbb{R}$ with the usual $\leq$.
2. The power set $P(X)$ is partially ordered by set inclusion. $X$ is the unique maximal element.
3. $x = (x_i), y = (y_i) \in \mathbb{R}^n$ ordered by
   \[ x \leq y \Leftrightarrow x_i \leq y_i \forall i = 1, \ldots, n. \]
4. The positive integers ordered by
   \[ x \leq y \Leftrightarrow x | n. \]

Definition (Chain). Let $M$ be a partially ordered set. A subset $C \subset M$ is said to be a chain if $a, b \in C \Rightarrow a \leq c$ or $c \leq a$. I.e., all elements in $C$ are comparable.

The next statement is equivalent to the axiom of choice. It retain’s its name (lemma) for historical reasons.

Lemma 4.2 (Zorn’s Lemma). Let $M \neq \emptyset$ be a partially ordered set. Suppose that every chain $C \subset M$ has an upper bound. Then $M$ has at least one upper bound.

If we decide to live with the the axiom of choice, we can use Zorn’s Lemma to prove various important results.
Theorem 4.3 (Hemel Basis). Every vector space $X \neq \{0\}$ has a Hemel basis.

Proof. Let $M$ be the set of all linearly independent subsets of $X$. Note that $M$ is not empty:

$$X \neq \{0\} \Rightarrow \exists x \in X \text{ s.t. } x \neq 0 \Rightarrow \{x\} \in M.$$ 

Set inclusion gives a partial order on $M$. Let $C \subset M$ be a chain. Then

$$A = \bigcup_{D \in C} D$$

is an upper bound for $C$.

With this setup we invoke Zorn's lemma to force the existence of a maximal element $B$. Let $Y = \text{span}(B)$ and so $Y \subset X$. We want to show $Y = X$. Assume not and let $z \in X - Y$. Then $B \cup \{z\}$ is linearly independent, contradicting the maximality of $B$. \qed

Theorem 4.4 (Total Orthonormal Set). In every Hilbert space $H \neq \{0\}$, there exists a total orthonormal set.

Proof. Let $M$ be the set of all orthonormal subsets of $H$. $M$ is not empty since

$$X \neq \{0\} \Rightarrow \exists x \in X \text{ s.t. } x \neq 0 \Rightarrow \{\frac{x}{\|x\|}\} \in M.$$ 

Set inclusion gives a partial order on $M$. Let $C \subset M$ be a chain. Then

$$A = \bigcup_{D \in C} D$$

is an upper bound for $C$.

With this setup we invoke Zorn's lemma to force the existence of a maximal element $F$. We claim this $F$ is total. To this end, suppose not. Then $\exists z \in H \text{ s.t. } z \neq 0$ and $z \perp F$. Hence $F \cup \{\frac{z}{\|z\|}\}$ is orthonormal, contradicting the maximality of $F$. \qed

Definition (Sublinear Functional). Let $X$ be a v.s. and let $p : X \to \mathbb{R}$. $p$ is said to be a sublinear functional if

1. $p(x + y) \leq p(x) + p(y) \ \forall \ x, y \in X$
2. $p(\alpha x) = \alpha p(x) \ \forall \ x \in X \ \forall \ \alpha \in \mathbb{R}, \alpha \geq 0.$

Remark 4.5. The norm on a n.s. is a sublinear functional.

Theorem 4.6 (Hanh-Banach (Extensions of Linear Functionals)). Let $X$ be a real v.s. and let $p$ be a sublinear functional on $X$. Let $f$ be a linear functional defined on a subspace $Z \subset X$ s.t.

$$f(x) \leq p(x) \ \forall \ x \in Z.$$ 

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Then f has a linear extension \( \hat{f} : Z \to X \) s.t.

\[
\hat{f}(x) \leq p(x) \quad \forall \ x \in X \\
\hat{f}(x) = f(x) \quad \forall \ x \in Z.
\]

**Proof.**

1. Let \( E \) be the set of all linear extensions \( g \) of \( f \) which satisfy

\[
g(x) \leq p(x) \quad \forall \ x \in \mathcal{D}(g).
\]

Then \( f \in E \Rightarrow E \neq \emptyset \). We now define the following partial order on \( E \): \( g \leq h \) means \( h \) is a linear extension of \( g \), i.e.

(a) \( \mathcal{D}(g) \subset \mathcal{D}(h) \)
(b) \( h(x) = g(x) \quad \forall \ x \in \mathcal{D}(g). \)

Let \( C \) be a chain in \( M \). Then we define the maximal element \( \hat{g} \) as follows

(a) \( \mathcal{D}(\hat{g}) = \bigcup_{g \in C} \mathcal{D}(g) \)
(b) \( \hat{g}(x) = g(x) \quad \forall \ g \in C \) s.t. \( x \in \mathcal{D}(g). \)

Then \( g \leq \hat{g} \quad \forall \ g \in C \Rightarrow \hat{g} \) is an upper bound for \( C \). Since \( C \subset E \) was arbitrary, we invoke Zorn’s lemma to produce a maximal element \( \hat{f} \). Therefore \( \hat{f} \) is a linear extension of \( f \).

2. Now we want to show \( \mathcal{D}(\hat{f}) = X \). In the manner above, assume otherwise and let \( y_1 \in x - \mathcal{D}(\hat{f}) \) and consider \( Y_1 = \text{span} (\{1\} \mathcal{D}(\hat{f}) \cup \{y_1\}) \). If \( x \in Y_1 \) then we can write \( x = y + \alpha y_1, y \in \mathcal{D}(\hat{f}) \) (and this representation is unique). Then let us define a functional \( g_1 : Y_1 \to \mathbb{R} \) as

\[
g_1(y + \alpha y_1) = \hat{f}(y) + \alpha c
\]

where \( c \) is any real constant. Then \( g_1 \) is linear extension of \( f \). If we can show

\[
g_1(x) \leq p(x) \quad \forall \ x \in \mathcal{D}(g_1)
\]

then \( g_1 \in E \), which would contradict the maximality of \( \hat{f} \) and imply \( \mathcal{D}(\hat{f}) = X \).

3. To this end, let us construct a suitable \( c \). Let \( y, z \in \mathcal{D}(\hat{f}) \). Then

\[
\hat{f}(y) - \hat{f}(z) = \hat{f}(y - z) \\
\leq p(y - z) \\
= p(y + y_1 - y_1 - z) \\
\leq p(y + y_1) + p(-y_1 - z)
\]

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and therefore
\[-p(-y_1 - z) - \hat{f}(z) \leq p(y + y_1) - \hat{f}(y).\]

Note

(a) $y_1$ is fixed
(b) $y$ does not appear on the RHS
(c) $z$ does not appear on the LHS.

Then we can take $\sup_{z \in \mathcal{D}(\hat{f})} \inf_{y \in \mathcal{D}(\hat{f})}$ and call these values $m_0, m_1$, respectively. Then $m_0 \leq m_1$. Let $c$ be any value such that $m_0 \leq c \leq m_1$. Then
\[-p(-y_1 - z) - \hat{f}(z) \leq c \forall z \in \mathcal{D}(\hat{f})
\]
\[c \leq p(y + y_1) - \hat{f}(y) \forall y \in \mathcal{D}(\hat{f}).\]

Now let $\alpha \in \mathbb{R}$ and consider the cases

(a) $\alpha < 0$.
\[-p\left(-y_1 - \frac{1}{\alpha} y\right) - \hat{f}\left(\frac{1}{\alpha} y\right) \leq c \]
\[\alpha p\left(-y_1 - \frac{1}{\alpha} y\right) + \hat{f}(y) \leq -\alpha c \]

\[\Rightarrow\]
\[g_1(x) = g(y + \alpha y_1) = \hat{f}(y) + \alpha c \leq -\alpha p\left(-y_1 - \frac{1}{\alpha} y\right) \]
\[= p(\alpha y_1 + y) = p(x).\]

(b) $\alpha = 0$. Then $x \in \mathcal{D}(\hat{f})$ so $g_1(x) = \hat{f}(x) \leq p(x)$.

(c) $\alpha > 0$.
\[c \leq p\left(\frac{1}{\alpha} y + y_1\right) - \hat{f}\left(\frac{1}{\alpha} y\right) \]
\[\alpha c \leq \alpha p\left(\frac{1}{\alpha} y + y_1\right) - \hat{f}(y) \]
\[= p(x) - \hat{f}(y).\]
\[ g_1(x) = \hat{f}(y) + \alpha c \leq p(x). \]

Then \( g_1 \in E \) and we have our contradiction. Therefore \( \hat{f} \) is our bounded linear extension.

\[ \square \]

**Problem 25.**

1. Show that the norm on a v.s. \( X \) is a sublinear functional on \( X \).
2. Show that a sublinear functional \( p \) satisfies \( p(0) = 0 \) and \( p(-x) \geq -p(x) \).

**Solution.**

1. Let \( p(x) = \|x\| \). Then \( p(x+y) = \|x+y\| \leq \|x\| + \|y\| = p(x) + p(y) \). Also, if \( \alpha \geq 0 \), then
   \[ p(\alpha x) = \|\alpha x\| = |\alpha|\|x\| = \alpha p(x) \]
   . It follows that \( p \) is a sublinear functional.

2. \( X \) is not empty, \( 0 \in X \). Let \( x \in X \), then \( p(0) = p(0x) = 0p(x) = 0 \). Also, 
   \[ 0 = p(0) = p(x-x) \leq p(x) + p(-x) \]. It follows that \( p(-x) \geq -p(x) \).

\[ \square \]

**Problem 26.** If \( p \) is a sublinear functional on a real v.s. \( X \), show that there exists a linear functional \( \hat{f} \) on \( X \) s.t.

\[ -p(-x) \leq \hat{f}(x) \leq p(x). \]

**Solution.** Let \( Z = \{ x \in X : x = \alpha x + 0, \alpha \in \mathbb{R} \} \) and define the linear functional \( f \) on \( Z \) by

\[ f(x) = \alpha p(x_0). \]

Then \( f(x) \leq p(x) \). By the Hahn-Banach Theorem there exists a \( \hat{f} \) bounded linear functional defined on \( X \) such that \( \hat{f}(x) \leq p(x) \). Clearly, \( -\hat{f}(x) = \hat{f}(-x) \leq p(x) \). It follows that \( -p(-x) \leq \hat{f}(x) \leq p(x) \).

\[ \square \]

**Theorem 4.7 (Hahn-Banach Generalized).** Let \( X \) be a v.s. over \( \mathbb{K} \) (\( \mathbb{R} \) or \( \mathbb{C} \)) and let \( p \) be a real-valued functional on \( X \) s.t.

1. \( p(x+y) \leq p(x) + p(y) \forall x, y \in X \)
2. \( p(\alpha x) = |\alpha| p(x) \forall \alpha \in K \).

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Let \( Z \subset X \) be a subspace and let \( f : Z \to K \) be a functional s.t.
\[
|f(x)| \leq p(x) \quad \forall \ x \in Z.
\]

Then \( f \) has a bounded linear extension \( \hat{f} \) s.t.

1. \( \mathcal{D}(\hat{f}) = X \)

2. \( \hat{f}(x) = f(x) \quad \forall \ x \in Z \)

3. \( |\hat{f}(x)| \leq p(x) \quad \forall \ x \in X. \)

Proof.

1. \( K = \mathbb{R} \). Then
\[
|f(x)| \leq p(x) \Rightarrow f(x) \leq p(x).
\]

Then by the previous theorem \( \exists \hat{f} \) s.t. \( \hat{f}(x) \leq p(x) \quad \forall \ x \in X \). Also,
\[
-\hat{f}(x) = \hat{f}(-x) \leq p(-x) = p(x)
\]

so
\[
|\hat{f}(x)| \leq p(x).
\]

2. \( K = \mathbb{C} \). \( X \) complex \( \Rightarrow Z \) complex \( \Rightarrow f \) complex \( \Rightarrow \)
\[
f = f_1 + if_2
\]

where \( f_1, f_2 \) are real-valued. We introduce \( X_r, Z_r \) the v.s.’s obtained by restricting \( K \) to \( \mathbb{R} \). Now note that \( f \) linear on \( Z \) and \( f_1, f_2 \) real-valued means \( f_1, f_2 \) are real-valued linear functionals on \( Z_r \). Also,
\[
f_1(x) \leq |f(x)| \Rightarrow f_1(x) \leq p(x) \quad \forall \ x \in Z_r.
\]

Then by the previous theorems \( \exists \hat{f}_1 \) extension of \( f_1 \) from \( Z_r \) to \( X_r \) s.t.
\[
\hat{f}_1(x) \leq p(x) \quad \forall \ x \in X_r.
\]

Going back to \( Z \), we observe \( \forall \ x \in Z \)
\[
if(x) = i[f_1(x) + if_2(x)]
= f_1(ix) + if_2(ix)
= -f_2(x) + if_1(x)
\Rightarrow f_2(x) = -f_1(ix)
\]
So then \(\forall x \in X\) we define
\[
\hat{f}(x) = \hat{f}_1(x) - i\hat{f}_1(ix).
\]
Note
\[
\hat{f}(x) = f(x) \quad \forall x \in Z.
\]
(a) We claim \(\hat{f}\) is a linear functional on \(X\).
\[
\hat{f}((a + ib)x) = \hat{f}_1(ax + ibx) - i\hat{f}_1(ix) = a\hat{f}_1(x) + b\hat{f}_1(ix) - i \left[a\hat{f}_1(ix) - b\hat{f}_1(x)\right] = (a + ib) \left[\hat{f}_1(x) - i\hat{f}_1(ix)\right] = (a + ib) \hat{f}(x)
\]
(b) We claim \(|\hat{f}(x)| \leq p(x) \quad \forall x \in X\). Recall that \(p(x) \geq 0\). Then
\[
\hat{f}(x) = 0 \Rightarrow |\hat{f}(x)| \leq p(x).
\]
Assume \(\hat{f}(x) \neq 0\). We then use the exponential form of \(\hat{f}\)
\[
\hat{f}(x) = |\hat{f}(x)| e^{i\theta}.
\]
Then
\[
|\hat{f}(x)| = \hat{f}(x)e^{-i\theta}
\]
\[
= \hat{f}(e^{-i\theta}x)
\]
\[
= \hat{f}_1(e^{-i\theta}x) + i\hat{f}_1(ie^{-i\theta}x)
\]
\[
|\hat{f}(x)| \in \mathbb{R} \Rightarrow
\]
\[
|\hat{f}(x)| = \hat{f}_1(e^{-i\theta}x)
\]
\[
\leq p(e^{-i\theta}x)
\]
\[
= |e^{-i\theta}| p(x)
\]
\[
= p(x)
\]
Then \(\hat{f}\) is the bounded, linear extension.

\[\square\]
Theorem 4.8 (Hahn-Banach (for Normed Spaces)). Let $X$ be a n.s. and let $Z \subset X$ be a subspace. Let $f : Z \to \mathbb{K}$ be a bounded, linear functional. Then there exists a bounded, linear functional $\hat{f}$ which is an extension from $Z$ to $X$ s.t.

$$\left\| \hat{f} \right\|_X = \| f \|_Z$$

where

$$\left\| \hat{f} \right\|_X = \sup_{x \in X} \left| \hat{f}(x) \right|$$

$$\| f \|_Z = \sup_{x \in Z} |f(x)| .$$

Proof. If $Z = \{0\}$ then $f = 0$ and $\hat{f} = 0$. Otherwise, let $Z \neq \{0\}$. Then $\forall x \in Z$ we have

$$|f(x)| \leq \| f \|_Z \| x \| .$$

Then define

$$p(x) \leq \| f \|_Z \| x \| \ \forall \ x \in X.$$ 

Notice

$$p(x + y) = \| f \|_Z \| x + y \|$$

$$\leq \| f \|_Z (\| x \| + \| y \|)$$

$$= p(x) + p(y)$$

$$p(\alpha x) = \| f \|_Z \| \alpha x \|$$

$$= |\alpha| \| f \|_Z \| x \|$$

$$= |\alpha| p(x).$$

Then we can use the Hanh-Banach (Generalized) theorem $\Rightarrow \exists \hat{f} : X \to \mathbb{K}$ s.t.

$$\left| \hat{f}(x) \right| \leq p(x) = \| f \|_Z \| x \| \ \forall \ x \in X.$$

And so

$$\left\| \hat{f} \right\|_X = \sup_{x \in X} \left| \hat{f}(x) \right| \leq \| f \|_Z .$$

Finally, since $\hat{f}$ is an extension of $f$

$$\left\| \hat{f} \right\|_X \leq \| f \|_Z .$$

Therefore,

$$\left\| \hat{f} \right\|_X = \| f \|_Z .$$
**Remark 4.9.** In the previous theorem if $X = H$ is a Hilbert space and $Z \subset H$ is a closed subspace then we can represent $f$ (for some fixed $z \in Z$)

\[
f(x) = \langle x, z \rangle \quad \forall \ x \in Z
\]

\[
\|f\| = \|z\|
\]

\[
\hat{f}(x) = \langle x, z \rangle \quad \forall \ x \in X
\]

**Theorem 4.10** (Bounded Linear Functionals). Let $X$ be a n.s. and let $x_0 \in X, x_0 \neq 0$. Then $\exists \hat{f} : X \to \mathbb{K}$ s.t.

\[
\|\hat{f}\| = 1
\]

\[
\hat{f}(x_0) = \|x_0\|
\]

**Proof.** Let $Z \subset X$ be the subspace defined as

\[
Z = \{x \in X | x = \alpha x_0\}.
\]

Then

\[
f(x) = f(\alpha x_0)
\]

\[
= \alpha \|x_0\|
\]

\[
|f(x)| = |f(\alpha x_0)|
\]

\[
= |\alpha| \|x_0\|
\]

\[
= \|\alpha x_0\|
\]

\[
= \|x\|
\]

\[
\Rightarrow \|f\| = 1.
\]

Therefore $f$ has a linear extension $\hat{f}$ from $Z$ to $X$ with norm

\[
\|\hat{f}\| = \|f\| = 1.
\]

By definition of $f$ and $\hat{f}$, $\hat{f}(x_0) = f(x_0) = \|x_0\|$. □

**Problem 27.** To illustrate the Hanh-Banach (for Normed Spaces) consider a functional $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

\[
f(x) = \alpha_1 x_1 + \alpha_2 x_2 \quad \forall \ x = (x_1, x_2) \in \mathbb{R}^2.
\]

Construct its linear extension to $\mathbb{R}^3$ and the corresponding norms.

**Solution.** Let $\hat{f}(x) = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \alpha_3 \xi_3$. Then $||\hat{f}||(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)^{\frac{1}{2}} \geq \|f\|$ and $||\hat{f}|| = \|f\|$ iff $\alpha_3 = 0$. □
Problem 28. Consider the n.s. $\mathbb{R}^2$ and let $x_0 \in \mathbb{R}^2, x_0 \neq 0$. Find a bounded, linear functional $\hat{f} : \mathbb{R}^2 \rightarrow \mathbb{R}$ s.t.

1. $\|\hat{f}\| = 1$

2. $\hat{f}(x_0) = \|x_0\|

Solution. let $\hat{f} = \langle x, \frac{x_0}{\|x_0\|} \rangle$. Then $\|\hat{f}\|\frac{x_0}{\|x_0\|} = 1$ and $\hat{f}(x_0) = \langle x_0, \frac{x_0}{\|x_0\|} \rangle = \|x_0\|$. \n
4.1 Bounded, Linear Functionals on $C[a, b]$

Definition (Variation). Let $f : [a, b] \rightarrow \mathbb{R}$. We define the variation of $f$ as

$$\text{Var } f = \sup \left\{ \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| \right\}$$

where

$$a = t_0 < t_1 < \cdots < t_n = b$$

is a partition of $[a, b]$.

Definition (Bounded Variation). We say $f$ is of bounded variation and write $f \in BV[a, b]$ if

$$\text{Var } f < \infty.$$ 

Note $BV[a, b]$ is a vector space. If we define the norm

$$\|f\| = |f(a)| + \text{Var } f, \ \forall \ f \in BV[a, b]$$

then $BV[a, b]$ is a normed space.

Definition (Riemann-Stieltjes Integral). Let $x \in C[a, b]$ and $f \in BV[a, b]$ and $P_n$ be a partition of $[a, b]$. Define

$$\eta_{P_n} = \max \{ t_1 - t_0, \ldots, t_n - t_{n-1} \}.$$ 

Consider $S$ defined by

$$S(P_n) = \sum_{i=1}^{n} x(t_i) [f(t_i) - f(t_{i-1})].$$

Then $\exists I \in \mathbb{R}$ s.t. $\forall \ \varepsilon > 0 \ \exists \ \delta > 0$ s.t.

$$\eta_{P_n} < \delta \Rightarrow |I - S(P_n)| \leq \varepsilon.$$ 

We call $I$ the Riemann-Stieltjes integral of $x$ with respect to $f$ and write

$$I = \int_{a}^{b} x(t) df(t).$$

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Remark 4.11.

1. If \( f(t) = t \) then we have the usual integral.

2. If \( f \) has an integrable derivative then
\[
\int_a^b x(t)df(t) = \int_a^b x(t)f'(t)dt.
\]

3. We have
\[
\left| \int_a^b x(t)df(t) \right| \leq \sup_{t \in [a,b]} |x(t)| \text{Var}(f).
\]

Theorem 4.12 (Reisz - B.L.F.’s on \( C[a,b] \)). Every b.l.f. \( g \) on \( C[a,b] \) can be represented by a Riemann-Stieltjes integral
\[
g(x) = \int_a^b x(t)df(t)
\]
and
\[
\|g\| = \text{Var}(f).
\]

Proof. By the Hanh-Banach theorem \( g \) has an extension from \( C[a,b] \) to \( B[a,b] \) the space of bounded functions with norm
\[
\|x\| = \sum_{t \in [a,b]} |x(t)|
\]
and
\[
\|\hat{g}\| = \|g\|.
\]

Let
\[
x_t(s) = \begin{cases} 1 & s \in [0,t] \\ 0 & \text{otherwise} \end{cases},
\]
then \( x_t \in B[a,b] \). Let \( f(a) = 0 \) and \( f(t) = \hat{g}(x_t) \) \( \forall \ t \in (a,b] \). We want to show \( f \) is of b.v. and \( \text{Var} \ F \leq \|g\| \).

Use exponential form for a complex quantity \( z \), \( z = |z|e(z) \) where
\[
e(z) = \begin{cases} 1 & z = 0 \\ e^{i\theta} & z \neq 0 \end{cases}.
\]

Note that if \( z \neq 0 \) then \( |z| = ze^{-i\theta} \Rightarrow |z| = z\overline{e(z)} \).
We shall use the notation
\[ \varepsilon_i = e(f(t_i) - f(t_{i-1})) \] and \( x_t = x_i \).

Then
\[
\sum_{i=1}^{n} [f(t_i) - f(t_{i-1})] = |\hat{g}(x_1)| + \sum_{i=2}^{n} [\hat{g}(x_i) - \hat{g}(x_{i-1})] \\
= \varepsilon_1 \hat{g}(x_1) + \sum_{i=1}^{n} \varepsilon_i [\hat{g}(x_i) - \hat{g}(x_{i-1})] \\
= \hat{g} \left( \varepsilon_1 x_1 + \sum_{i=2}^{n} \varepsilon_i [x_i - x_{i-1}] \right) \\
\leq \| \hat{g} \| \| \varepsilon_1 x_1 + \sum_{i=2}^{n} \varepsilon_i [x_i - x_{i-1}] \| \\
= \| g \| \cdot 1 \\
\Rightarrow \text{Var } f \leq \| g \| \\
\Rightarrow f \in BV[a,b]
\]

Now, given \( P \) a partition of \([a,b]\) define
\[ z_n = x(t_0)x_1 + \sum_{i=2}^{n} x(t_{i-1})[x_i - x_{i-1}] \]
then \( z_n \in BV[a,b] \). Now we compute
\[
\hat{g}(z_n) = x(t_0)\hat{g}(x_1) + \sum_{i=2}^{n} x(t_{i-1})[\hat{g}(x_i) - \hat{g}(x_{i-1})] \\
= x(t_0)f(x_1) + \sum_{i=2}^{n} x(t_{i-1})[f(x_i) - f(x_{i-1})] \\
= \sum_{i=1}^{n} x(t_{i-1})[f(x_i) - f(x_{i-1})]
\]
Then taking a sequence of partitions \( (P_n) \) s.t. \( \eta P_n \to 0 \), we obtain
\[
\int_{a}^{b} x(t)df(t).
\]
We need to show \( \hat{g}(z_n) \to \hat{g}(x) = g(x) \) where \( x \in C[a,b] \). Note \( z_n(a) = x(a) \cdot 1 \Rightarrow z_n(z) = x(a) = 0 \). Note \( t_{i-1} < t \leq t_i \Rightarrow |z_n(t) - x(t)| = |x(t_{i-1}) - x(t)| \).
Note
\[ \eta_{P_n} \to 0 \Rightarrow \|z_n - x\| \to 0 \]
since \( x \in C[a, b] \Rightarrow x \) is uniformly continuous (since \([a, b]\) is compact). Then the continuity of \( \hat{g} \) implies \( \hat{g}(z_n) \to \hat{g}(x) \) and \( \hat{g}(x) = f(x) \).

Now we want to show \( \text{Var } f = \|g\| \). Computing,
\[ |g(x)| \leq \max_{t \in [a,b]} x(t) \text{ Var } f = \|x\| \text{ Var } f \]
and so
\[ \|g\| \leq \text{ Var } f. \]
Also \( \text{Var } f \leq \|g\| \), so
\[ \|g\| = \text{ Var } f. \]

\[ \square \]

**Remark 4.13.** \( f \) is not unique in the above theorem. However, we can make \( f \) unique by requiring:

1. \( f(a) = 0 \)
2. \( f(t^+) = f(t) \) (continuity from the right).

### 4.2 Adjoint Operator

Given a b.l.o. \( T : X \to Y \) we are interested in constructing a new operator \( T^* : Y' \to X' \). We proceed as follows. Let \( g \in Y', x \in X \). Setting \( y = Tx \) we obtain a function \( f \) on \( X \) by defining
\[ f(x) = g(Tx). \]

\( f \) is linear because \( g, T \) are linear. \( f \) is bounded because
\[ |f(x)| = |g(Tx)| \leq \|g\| \|T\| \|x\| \]
\[ \Rightarrow \|f\| \leq \|g\| \|T\|. \]

Therefore \( f \in X' \). Then
\[ f(x) = g(Tx) \quad \text{(9)} \]
with variable \( g \in Y' \) defines an operator called the adjoint of \( T \),
\[ T^* : Y' \to X'. \]

**Definition** (Adjoint). Given a b.l.o. \( T : X \to Y \) the adjoint \( T^* : Y' \to X' \) is given by
\[ f(x) = (T^* g)(x) = g(Tx). \]
Theorem 4.14. \( \|T^\times\| = \|T\| \).

Proof. We know \( T^\times \) is linear and \( f = T^\times g \). Then

\[
\|T^\times g\| = \|f\| \leq \|g\| \|T\| \Rightarrow \|T^\times\| \leq \|T\|.
\]

Now we want to show \( \|T^\times\| \geq \|T\| \). For any \( x_0 \in X, x_0 \neq 0 \) \exists \( g_0 \in Y' \) s.t. \( \|g_0\| = 1 \) and \( g_0(Tx_0) = \|Tx_0\| \).

Hence

\[
g_0(Tx_0) = (T^\times g_0)(x_0) \]
\[
f_0 = T^\times g_0
\]
\[
\Rightarrow \]
\[
\|Tx_0\| = g(Tx_0) \]
\[
= f_0(x_0) \]
\[
\leq \|f_0\| \|x_0\| \]
\[
= \|T^\times g_0\| \|x_0\| \]
\[
\leq \|T^\times\| \|g_0\| \|x_0\|.
\]

Recall \( \|g_0\| = 1 \), then

\[
\|Tx_0\| \leq \|T^\times\| \|x_0\| \Rightarrow \|T^\times\| \geq \frac{\|Tx_0\|}{\|x_0\|}.
\]

But \( x_0 \neq 0 \) is arbitrary, and taking sup on the RHS yields

\[
\|T^\times\| \geq \|T\|.
\]

Example 4.15. Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \). Let \( E \) be a basis, and let \( x, y \in \mathbb{R}^n \), and let \( T_E \) be the representation of \( T \) with respect to \( E \).

\[
E = \{e_1, \ldots, e_n\}
\]
\[
x = (\xi_1, \ldots, \xi_n)
\]
\[
y = (\eta_1, \ldots, \eta_n)
\]
\[
T_E = (\tau_{ik})
\]
\[
y = T_E x
\]
\[
\eta_i = \sum_{k=1}^{n} \tau_{ik} \xi_k.
\]
Let $F = \{f_1, \ldots, f_n\}$ be the dual basis of $E$, i.e., a basis for $\mathbb{R}^n$. Then for $g \in \mathbb{R}^n$, we can represent

$$g = \sum_{i=1}^{n} \alpha_i f_i$$

$$f_i(y) = f_i \left( \sum_{i=1}^{n} \eta_i e_i \right) = \eta_i$$

$$\Rightarrow g(y) = g(T_E x) = \sum_{i=1}^{n} \alpha_i \eta_i = \sum_{i=1}^{n} \sum_{k=1}^{n} \alpha_i \tau_{ik} \xi_k$$

$$\Rightarrow g(T_E x) = \sum_{k=1}^{n} \beta_k \xi_k$$

where $\beta_k = \sum_{i=1}^{n} \tau_{ik} \alpha_i$.

$$\Rightarrow f(x) = g(T_E x) = \sum_{k=1}^{n} \beta_k \xi_k$$

$$\Rightarrow f = (T_E)^{\times} g$$

$$\beta_k = \sum_{i=1}^{n} \tau_{ik} \alpha_i.$$ 

Therefore, if $T$ is represented by $T_E$, then $T^\times$ is represented by the transpose of $T_E$.

**Remark 4.16** (Properties of Adjoint).

1. $(S + T)^{\times} = S^{\times} + T^{\times}$
2. $(\alpha T)^{\times} = \alpha T^{\times}$
3. $(ST)^{\times} = T^{\times} S^{\times}$
4. If $T \in B(X, Y)$ and $T^{-1}$ exists and $T^{-1} \in B(Y, X)$, then $(T^{\times})^{-1}$ also exists, $(T^{\times})^{-1} \in B(X', Y')$ and $(T^{\times})^{-1} = (T^{-1})^{\times}$.

**Remark 4.17** (Relation between $T^\times$ and $T^\ast$). Let $T : X \rightarrow Y, X = H_1, Y = H_2, T : H_1 \rightarrow H_2, T^\ast : H_2' \rightarrow H_1'$, $f \in H_1' g \in H_2'$

$$T^\times g = f$$

$$g(Tx) = f(x)$$
where $H_1, H_2$ are Hilbert spaces. Then let us represent (Riesz)
\[ f(x) = \langle x, x_0 \rangle, \quad x_0 \in H_1 \]
\[ g(y) = \langle y, y_0 \rangle, \quad y_0 \in H_2. \]

Then we can define $A_1 : H'_1 \to H_1$ by $A_1 f = x_0$ and $A_2 : H'_2 \to H_2$ by $A_2 g = y_0$. Then $A_1, A_2$ are bijective, isometric, and conjugate-linear. So we can define $T^* : H_2 \to H_1$ by
\[ T^* y_0 = A_1 T^* A_2^{-1} y_0 = x_0. \]

\[ \Rightarrow \]
\[ \langle Tx, y_0 \rangle = g(Tx) \\
= f(x) \\
= \langle x, x_0 \rangle \\
= \langle x, T^* y_0 \rangle \]

Remark 4.18.

1. $(\alpha T)^\times = \alpha T^\times$ but $(\alpha T)^* = \overline{\alpha} T^*$

2. In the finite dimensional setting $T^*$ is represented as the transpose and $T^*$ is represented by the conjugate transpose:
\[ T^\times = T^T \]
\[ T^* = \overline{T}^T. \]

Problem 29.

1. Show $(\alpha T)^* = \alpha T^\times$.

2. Show $(T^n)^\times = (T^\times)^n$.

Solution.

1. $((\alpha T)^x g)(x) = g((\alpha T)x) = g(\alpha Tx) = \alpha g(Tx) = \alpha (t^x g)(x) = ((\alpha T^x)g)(x)$.

2. We have $(ST)^x = T^x S^x$ because $((ST)^x g)(x) = g((ST)(x)) = g(S(T(x))) = (s^x g)(T(x)) = T^x((S^x g)(x)) = (T^x(S^x g))(x) = ((T^x S^x)g)(x)$. It follows that $(T^n)^x = (T^{n-1})^x T^x = \cdots = (t^x)^n$. \qed

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4.3 Reflexive Spaces

**Definition (Reflexive (Algebraic)).** A vector space \( X \) is algebraically reflexive if the canonical map \( C : X \to X^{**} \) is surjective (and thus bijective). Recall \( C \) is the mapping
\[
x \mapsto g_x \text{ where } g_x(f) = f(x), \forall f \in X^*.
\]

**Definition (Reflexive (Dual)).** A normed space \( X \) is called reflexive if \( X = X'' \), i.e., if \( \mathcal{R}(C) = X'' \).

**Problem 30.** Let \( X \) be a n.s. Define \( g_x : X' \to \mathbb{K} \) by fixing an \( x \in X \) and setting
\[
g_x(f) = f(x) \quad \forall f \in X'.
\]

1. Show \( \forall \) fixed \( x \in X, g_x \) is a b.l.f. on \( X' \) and
\[
\|g_x\| = \|x\|.
\]

2. \( C \) is an isomorphism \( X \xrightarrow{\text{onto}} \mathcal{R}(C) \).

3. If a n.s. \( X \) is reflexive (i.e., \( \mathcal{R}(C) = X'' \)) then \( X \) is complete.

**Solution.**

1. Let \( g_x(f) = f(x) \), where \( x \in X \) is fixed. Then \( |g_x(f)| = |f(x)| \leq \|f\| \|x\| \) and therefore \( g \) is bounded and \( \|g\| \leq \|x\| \). Let \( \hat{f} \in X' \) such that \( \|\hat{f}\| = 1 \) and \( \hat{f}(x) = \|x\| \).
   (We established the existence of such \( \hat{f} \).) Then \( \|g_x\| \geq \frac{|g_x(\hat{f})|}{\|\hat{f}\|} = \|x\| \). It follows that \( \|g_x\| = \|x\| \).

2. Consider \( C : x \to X'' \), where \( x \to g_x \). Then \( C \) is linear, because \( \forall f \in X' \) we have
\[
g_{\alpha x + \beta y}(f) = f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) = \alpha g_x(f) + \beta g_y(f).
\]
Then \( C(\alpha x + \beta y) = \alpha g_x + \beta g_y = \alpha C(x) + \beta C(y) \). Also, \( \|C(x)\| = \|g_x\| = \|x\| \), and then \( \|g_x - g_y\| = \|g_{x-y}\| = \|x-y\| \) and \( C \) is isometric. Moreover, if \( x \neq y \), then \( g_x \neq g_y \) and therefore \( C \) is injective. It follows that \( C \) is an isomorphism onto its range.

3. \( X'' \) is complete being the dual of \( X' \). By assumption \( X \) is reflexive, hence \( R(C) = X'' \). Part (ii) implies the completeness of \( X \) via isomorphism.

\[\square\]

**Theorem 4.19.** Every f.d.n.s. is reflexive.

**Example 4.20.**
1. \( L^p, \mathcal{L}^p[a, b], \forall 1 < p < \infty \) are reflexive.

2. \( C[a, b], \mathcal{L}^1[a, b], \ell^1 \) are not reflexive.

**Theorem 4.21** (Hilbert Spaces are Reflexive). *If \( H \) is a Hilbert space then \( H \) is reflexive.*

**Proof.** The canonical map \( C : H \to H'' \) given by \( x \mapsto g_x \) is injective. We want to show \( C \) is surjective, i.e.,

\[
\forall g \in H'' \exists x \in H \text{ s.t. } g = Cx.
\]

Define \( A : H' \to H \) by \( Af = z \) where \( z \) is the Riesz representation of \( f \):

\[
f(x) = \langle x, z \rangle.
\]

Then \( A \) is bijective, an isometry and conjugate-linear. We view \( H' \) as the Hilbert space with inner product

\[
\langle f_1, f_2 \rangle_1 = \langle Af_2, Af_1 \rangle.
\]

Let \( g \in H'' \) be arbitrary. Then

\[
g(f) = \langle f, f_0 \rangle_1 = \langle Af_0, Af \rangle.
\]

Writing \( f(x) = \langle x, z \rangle, z = Af, Af_0 = z \) we have

\[
\langle Af_0, Af \rangle = \langle x, z \rangle = f(x).
\]

\[\Rightarrow g(f) = f(x) \Rightarrow g = Cx. \text{ Then } H \text{ is reflexive.} \]

**Lemma 4.22** (Existence of a Functional). *Let \( Y \) be a n.s. and let \( Y \subset X \) be a proper, closed subspace. Let \( x_0 \in X - Y \) be arbitrary. Calculate

\[
\delta = \inf_{\hat{y} \in Y} \| \hat{y} - x_0 \|.
\]

Note \( Y \) closed \( \Rightarrow \delta > 0. \) Then \( \exists \hat{f} \in X' \) s.t.

\[
\| \hat{f} \| = 1 \text{ and } \hat{f}(y) = 0 \forall y \in Y \text{ and } \hat{f}(x_0) = \delta. \quad (10)
\]

**Proof.** Consider a subspace \( Z \subset X \) spanned by \( Y \) and \( x_0 \) and define a sublinear functional \( f \) on \( Z \) by

\[
f(z) = f(y + \alpha x_0) = \alpha \delta.
\]

Note \( \delta \neq 0 \Rightarrow f \neq 0. \) Then \( f \) satisfies (10). Now we use the Hanh-Banach theorem to extend \( f \) to \( X. \) In a slight abuse of notation we refer to the extension of \( f \) as \( f. \) Then \( f \) is bounded. It is clear that \( \alpha = 0 \Rightarrow f(z) = 0. \)
For the case $\alpha \neq 0$

\[
|f(z)| = |\alpha| \delta \\
= |\alpha| \inf_{\hat{y} \in Y} \|\hat{y} - x_0\| \\
\leq |\alpha| \|\alpha^{-1}y - x_0\| \\
= \|y + \alpha x_0\| \\
\Rightarrow |f(z)| \leq \|\hat{z}\| \\
\Rightarrow \|f\| \leq 1
\]

Now to show $\|f\| \geq 1$. \exists (y_n) \subset Y s.t. \|y_n - x_0\| \to \delta$. Let $z_n = y_n - x_0$, then $f(z_n) = -\delta$. Then

\[
\|f\| = \sup_{z \in Z, z \neq 0} \frac{|f(z)|}{\|z\|} \\
\geq \frac{|f(z_n)|}{\|z_n\|} \\
= \frac{\delta}{\|z_n\|} \\
\to 1,
\]

$\Rightarrow \|f\| = 1$. \hfill \square

**Theorem 4.23** (Separability). $X'$ separable \Rightarrow $X$ separable.

**Proof.** Assume $X'$ is separable. Then

\[
U' = \{f| \|f\| = 1\} \subset X'
\]
contains a countable, dense subset, say $(f_n) \subset U'$. Since $f_n \in U' \forall n$, 

\[
\|f_n\| = \sup_{\|x\|=1} |f(x)| = 1.
\]

By the definition of sup we can find a sequence $(x_n) \in X$ s.t. $\|x_n\| = 1 \forall n$ s.t.

\[
|f_n(x_n)| \geq \frac{1}{2}.
\]

Let $Y = \text{span}((x_n))$, then $Y$ is separable. We want to show $Y = X$ (because then $(x_n)$ is a countable dense subset of $X$).
Suppose \( Y \subseteq X \). Then \( \exists \hat{f} \in X' \) s.t. \( \|\hat{f}\| = 1, \hat{f}(y) = 0 \ \forall \ y \in Y \). Since \( (x_n) \subset Y \) we have \( \hat{f}(x_n) = 0 \ \forall \ n \). Then
\[
\frac{1}{2} \leq |f_n(x_n)|
= |f_n(x_n) - \hat{f}(x_n)|
= |(f_n - \hat{f})(x_n)|
\leq \|f_n - \hat{f}\| \|x_n\|
= \|f_n - \hat{f}\|.
\]
Then \( \|f_n - \hat{f}\| \geq \frac{1}{2} \) and \( \|\hat{f}\| = 1 \) contradicts that \( (f_n) \subset U' \) is dense.

**Corollary 4.24.** \( X \) separable and \( X' \) not separable \( \Rightarrow X \) not reflexive.

**Proof.** \( X \) reflexive \( \Rightarrow X = X'' \Rightarrow X'' \) separable \( \Rightarrow X' \) separable, which contradicts our assumptions.

**Example 4.25.** \( \ell^1 \) is not reflexive. This is because \( (\ell^1)' = \ell^\infty \) and \( \ell^\infty \) is not separable.

### 4.4 Baire Category Theorem

**Definition** (Category). Let \( X \) be a metric space and \( M \subset X \).

1. \( M \) is said to be **nowhere dense** in \( X \) if \( \overline{M} \) has no interior points.

2. \( M \) is said to be of **first category** or **meager** if \( X \) is the countable union of nowhere dense sets.

3. \( M \) is of the **second category** if it is not of first category.

**Theorem 4.26** (Baire’s Theorem). If \( X \neq \emptyset \) then \( X \) is of the second category (i.e., \( X \) is not meager).

**Proof.** Suppose \( X \neq \emptyset \) and \( X \) is meager. Then
\[
X = \bigcup_{k=1}^{\infty} M_k
\]
with \( M_k \) nowhere dense in \( X \ \forall \ k \). By assumption, \( \overline{M_1} \) does not contain a nonempty open set \( \Rightarrow \overline{M_1} \neq X, \Rightarrow \overline{M_1} = X - \overline{M_1} \) is not empty and is open. Pick \( p_1 \in \overline{M_1} \) and an open ball containing \( p_1 \), say \( B_1 = B(p_1, \varepsilon_1) \subset \overline{M_1} \), with \( \varepsilon_1 \leq \frac{1}{2} \).
Similarly, $\overline{M}_2$ does not contain a nonempty open set $\Rightarrow B(p_1, \frac{\epsilon_1}{2}) \subseteq \overline{M}_2$. Then $\overline{M}_2^c \cap B(p_1, \frac{\epsilon_1}{2}) \neq \emptyset$. Then

$$\exists B_2 = B(p_2, \epsilon_2) \subset \left( \overline{M}_2^c \cap B(p_1, \frac{\epsilon_1}{2}) \right) \text{ with } \epsilon_2 < \frac{\epsilon_1}{2}.$$  

Continuing inductively, we can find $B_k = B(p_k, \epsilon_k)$ with $\epsilon_k < \frac{1}{2^n}$ s.t.

$$B_k \cap M_k \neq \emptyset \quad B_{k+1} \subset B(p_k, \frac{\epsilon_k}{2}) \subset B_k.$$

Since $\epsilon < \frac{1}{2^n}$ we find $(p_n)$ is Cauchy. Then because $X$ is complete $\exists p \in X$ s.t. $p_n \to p$. Also $\forall n > m$

$$d(p_m, p) \leq d(p_m, p_n) + d(p_n, p) < \frac{\epsilon_m}{2} + d(p_n, p) \to \frac{\epsilon_m}{2}$$

and therefore $p \in B_m \forall m$. Since $B_m \subset \overline{M}_m^c$, we have $p \notin M_m \forall m$. Therefore $p \notin \bigcup M_m = X$, a contradiction. \qed

**Theorem 4.27 (Uniform Boundedness).** Let $(T_n) \subset B(X, Y)$ with $X$ a Banach space and $Y$ a normed space. Suppose $\forall x \in X \exists c_x$ s.t.

$$\|T_n x\| \leq c_x \forall n.$$  

Then $(\|T_n\|)$ is bounded.

**Proof.** For each $k$ define

$$A_k = \{ x \in X \mid \|T_n x\| \leq k \forall n \}.$$

Then we can see $A_k$ is closed:

1. Let $x \in \overline{A_k}$. Then $\exists (x_i) \subseteq A_k$ s.t. $x_i \to x$.
2. For fixed $n$, $\|T_n x_i\| \leq k \Rightarrow \|T_n x\| \leq k$ by continuity of $\|\cdot\|$.
3. Then $x \in A_k$.

By assumption $\forall x \in X \exists k$ s.t. $x \in A_k$, so

$$X = \bigcup_{k=1}^{\infty} A_k.$$  

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Then since $X$ is complete we invoke Baire to obtain $A_{k_0}$, a set which contains an open ball, say

$$B_0 = B(x_0,v) \subset A_{k_0}.$$  

Let $x \in X$ s.t. $x \neq 0$ be arbitrary. Set $z = x_0 + \gamma x, \gamma = \frac{v}{2\|x\|}$. Then

$$\|z - x_0\| < r \Rightarrow z \in B_0 \Rightarrow \|T_n z\| \leq k$$  

Also, $x_0 \in B_0 \Rightarrow \|T_n x_0\| \leq k_0$. Then

$$\|T_n x\| = \gamma^{-1} \|T_n (z - x_0)\|$$

$$\leq \gamma^{-1} (\|T_n z\| + \|T_n x_0\|)$$

$$\leq \frac{4}{v} \|x\| k_0.$$  

We conclude

$$\|T_n\| = \sup_{\|x\|=1} \|T_n x\| \leq \frac{4}{v} k_0 = c,$$

and $c$ is independent of $x$.

Problem 31. Let $X$ be the normed space of all polynomials with norm

$$\|x\| = \max_i |\alpha_i|, \text{ where } x(t) = \alpha_0 t^k + \cdots + \alpha_0.$$  

Show $X$ is not complete.

Solution. Let $X$ be the normed space of all polynomials with norm $\|x\| = \max_i |\alpha_i|$ (the $\alpha_i$’s are the coefficients). Define the linear functional $f_n$ by $f_n(x) = \alpha_0 + \alpha_1 + \cdots + \alpha_n - 1$. Then $|f_n(x)| \leq n \|x\|$. Also for each fixed $x$ we have $|f_n(x)| \leq c_x$. On the other hand, for $x(t) = 1 + t + t^2 + \cdots + t^n$, we have $\|x\| = 1$ and $f_n(x) = n \|x\|$. Hence $\|f_n\| \geq \frac{|f_n(x)|}{\|x\|} = n$. It follows that the sequence $(\|f_n\|)$ is unbounded. The Uniform Boundedness Theorem implies that $X$ is not complete.

Problem 32. If $X,Y$ are Banach and $(T_n) \subset B(X,Y)$ is a sequence, show TFAE:

1. $(\|T_n\|)$ is bounded.
2. $(\|T_n x\|)$ is bounded $\forall x \in X$.
3. $(\|g(T_n x)\|)$ is bounded $\forall x \in X \forall g \in Y'$.

Solution. Recall that in a Banach space $X$ if a sequence $(x_n)$ is such that $(f(x_n))$ is bounded for all $f \in X'$, then $(\|x_n\|)$ is bounded and therefore 3 implies 2. Also, 2 implies 1 by the Uniform Boundedness Theorem. Finally, 1 implies 3 since $|g(T_n x)| \leq \|g\| \|T_n\| \|x\|$.
4.5 Strong and Weak Convergence

**Definition** (Strong and Weak Convergence). Let $X$ be a normed space and let $(x_n) \subset X$ be a sequence.

1. $(x_n)$ is **strongly convergent** if $\exists x \in X$ s.t.
   \[ \lim_{n \to \infty} \| x_n - x \| = 0. \]
   We denote this property with $\lim_{n \to \infty} x_n = x$ or $x_n \to x$.

2. $(x_n)$ is **weakly convergent** if $\exists x \in X$ s.t. $\forall f \in X'$
   \[ \lim_{n \to \infty} f(x_n) = f(x). \]
   We denote this property with $x_n \xrightarrow{w} x$.

**Lemma 4.28** (Weak Convergence). Let $x_n \xrightarrow{w} x$. Then

1. The weak limit $x$ is unique.
2. $x_{n_k} \xrightarrow{w} x \forall$ subsequences $(x_{n_k}) \subset (x_n)$.
3. $(\|x_n\|)$ is bounded.

**Proof.**

1. Suppose $x_n \xrightarrow{w} x$ and $x_n \xrightarrow{w} y$. Then $f(x_n) \to f(x)$ and $f(x_n) \to f(y) \Rightarrow f(x) = f(y)$. Then $0 = f(x) - f(y) = f(x - y) \forall f \in X'$. Then by a previous lemma $x = y$.

2. $(f(x_n)) \subset \mathbb{R} \Rightarrow$ all subsequences of $(f(x_n))$ converge and have the same limit.

3. $(f(x_n))$ converges $\Rightarrow |f(x_n)| \leq c_f \forall n$. Define $g_n \in X''$ by
   \[ g_n(f) = f(x_n). \]
   Then $|g_n(f)| = |f(x_n)| \leq c_f$, $\Rightarrow (\|g_n(f)\|)$ is bounded $\forall f \in X'$. Then $X'$ complete $\Rightarrow (\|g_n\|)$ bounded by the UBT. Now $\|g_n\| = \|x\|$ and the proof is complete.

**Theorem 4.29** (Strong and Weak Convergence). Let $X$ be a normed space and $(x_n) \subset X$.

1. $x_n \to x \Rightarrow x_n \xrightarrow{w} x$. The limits are the same.
2. \( x_n \overset{w}{\rightarrow} x \not\Rightarrow x_n \rightarrow x \) in general.

3. \( \dim X < \infty \Rightarrow x_n \overset{w}{\rightarrow} x \Rightarrow x_n \rightarrow x. \)

**Proof.**

1. \( \| f(x_n) - f(x) \| = \| f(x_n - x) \| \leq \| f \| \| x_n - x \|. \) Therefore \( x_n \rightarrow x \Rightarrow x_n \overset{w}{\rightarrow} x. \)

2. We show a counterexample. Let \( H \) be a Hilbert space and \( (e_n) \subset H \) and orthonormal sequence. Then \( \forall f \in H', f(x) = \langle x, z \rangle. \) In particular, \( f(e_n) = \langle e_n, z \rangle \) and by Bessel’s inequality

\[
\sum_{n=1}^{\infty} |\langle e_n, z \rangle|^2 \leq \| z \|^2.
\]

Therefore \( f(e_n) = \langle e_n, z \rangle \rightarrow 0 \) \( \forall f \in H' \Rightarrow e_n \overset{w}{\rightarrow} 0. \) But \( (e_n) \) does not converge strongly because for \( n \neq m \)

\[
\| e_m - e_n \|^2 = \langle e_m - e_n, e_m - e_n \rangle = 2.
\]

3. Suppose \( \dim X = k < \infty \) and \( x_n \overset{w}{\rightarrow} x. \) Let \( \{e_1, \ldots, e_k\} \) be a basis for \( X. \) Then we can represent

\[
x_n = \sum_{i=1}^{k} \alpha_i^{(n)} e_i
\]

\[
x = \sum_{i=1}^{k} \alpha_i e_i.
\]

By assumption \( f(x_n) \rightarrow f(x) \) \( \forall f \in X'. \) Take the dual basis \( \{f_1, \ldots, f_k\} \) defined by

\[
f_k(e_i) = \delta_{ki}.
\]

Then

\[
f_i(x_n) = \alpha_i^{(n)}
\]

\[
f_i(x) = \alpha_i.
\]

Hence

\[
f_i(x_n) \rightarrow f_i(x) \Rightarrow \alpha_i^{(n)} \rightarrow \alpha_i.
\]
Then
\[ \| x_n - x \| = \left\| \sum_{i=1}^{k} (\alpha_i^{(n)} - \alpha_i) e_i \right\| \]
\[ \leq \sum_{i=1}^{k} |\alpha_i^{(n)} - \alpha_i| \| e_i \| \]
\[ \to 0 \text{ as } n \to \infty. \]

Remark 4.30.
1. In \( \ell^1 \) strong and weak convergence are equivalent.
2. In a Hilbert space \( H \),
\[ x_n \stackrel{w}{\longrightarrow} x \iff \langle x_n, z \rangle \to \langle x, z \rangle \forall z \in H. \]

Lemma 4.31 (Weak Convergence). Let \( X \) be a normed space and let \( (x_n) \subset X \). Then \( x_n \stackrel{w}{\longrightarrow} x \) iff both:
1. \( (\| x_n \|) \) is bounded.
2. \( f(x_n) \to f(x) \forall f \in M \subset X' \forall M \text{ total } . \)

Proof.
1. “\( \Rightarrow \)”. Weak convergence implies 1,2 by previous results.
2. “\( \Leftarrow \)”. Suppose 1,2 hold. Let \( f \in X' \) be arbitrary and show \( f(x_n) \to f(x) \). By 1 we can find \( c > 0 \) s.t.
\[ \| x_n \| \leq c \]
\[ \| x \| \leq c. \]

Since \( M \subset X' \) is total, \( \forall f \in X' \exists (f_n) \subset \text{span}(M) \) s.t. \( f_n \to f \). Hence \( \forall \varepsilon > 0 \exists i \) s.t.
\[ \| f_i - f \| < \frac{\varepsilon}{3c}. \]
Moreover, since \( f_i \in \text{span}(M) \) \( (\text{by2}) \exists N \) s.t.
\[ |f_i(x_n) - f_i(x)| < \frac{\varepsilon}{3}, \forall n > N. \]
Then
\[|f(x_n) - f(x)| \leq |f(x_n) - f_i(x_n)| + |f_i(x_n) - f_i(x)| + |f_i(x) - f(x)|\]
\[< \|f - f_i\|\|x_n\| + \frac{\varepsilon}{3} + \|f_i - f\|\|x\|\]
\[< \frac{\varepsilon}{3c} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3c} = \varepsilon.\]

Therefore \(x_n \xrightarrow{w} x.\)

**Definition (Weak Cauchy, Weak Complete).**

1. In a normed space \(X\), a sequence \((x_n) \subset X\) is said to be **weak Cauchy** if \(\forall f \in X'\) the sequence \((f(x_n))\) is Cauchy.

2. A normed space \(X\) is said to be **weakly complete** if every weak Cauchy sequence converges weakly in \(X\).

**Problem 33.** Show \(X\) reflexive \(\Rightarrow\) \(X\) weakly complete.

**Solution.** Let \((x_n)\) be any weak Cauchy sequence in \(X\). Then \((f(x_n))\) converges for every \(f \in X'\). For \(x_n \in X\) there is a \(g_{x_n} \in X''\) such that \(f(x_n) = g_{x_n}(f)\). Hence \((g_{x_n}(f))\) converges, say, \(g_{x_n}(f) \to g(f)\). Weak Cauchyness of \((x_n)\) implies the boundedness of \((x_n)\) and then since \(\|g_{x_n}\| = \|x_n\|\) we have that \(g\) is bounded. Also, \(g\) is linear and therefore \(g \in X''\). Since \(X\) is reflexive, there is an \(x\) such that \(g(f) = f(x)\). Hence \(f(x_n) \to f(x)\). Since \(f \in X'\) was arbitrary, this shows that \((x_n)\) converges to \(x\) weakly. Since \((x_n)\) was any weak Cauchy sequence, \(X\) is weakly complete.

**Definition (Convergence of Operators).** Let \(X, Y\) be normed spaces, and let \((T_n) \subset B(X, Y)\) be a sequence. We say \((T_n)\) is:

1. **uniformly operator convergent** if \((T_n)\) converges in the norm of \(B(X, Y)\).

2. **strongly operator convergent** if \(\forall x \in X\) the sequence \((T_n x)\) converges strongly in \(Y\).

3. **weakly operator convergent** if \(\forall x \in X\) the sequence \((T_n x)\) converges weakly in \(Y\).

**Remark 4.32.** Uniform \(\Rightarrow\) strong \(\Rightarrow\) weak:

1. \(\|T_n - T\| \to 0 \Rightarrow \|(T_n - T)x\| \leq \|T_n - T\| \|x\| \to 0.\)

2. \(\|(T_n - T)x\| \to 0 \Rightarrow \|f(T_n - T)x\| \leq \|f\| \|(T_n - T)x\| \to 0.\)
Example 4.33 (Strong \( \not\Rightarrow \) Uniform). Let \( X = Y = \ell^2 \). Let \( T_n \) be “zero-overwrite” operator, i.e., let \((x_i) \in \ell^2 \)

\[
T_n(x_i) = (0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots).
\]

Then \( T_n(x_i) \to 0 \forall (x_i) \in \ell^2 \) so \((T_n)\) is strongly operator convergent to the zero operator. However given any \( n \), we can construct \( x' \in \ell^2 \) s.t. \( T_n x = x \), i.e., by making the first \( n \) terms of \( x' \) zero. Then

\[
\|T_n\| = \sup_{x \neq 0} \frac{\|T_n x\|}{\|x\|} \geq 1.
\]

So \((T_n)\) does not converge to \( 0 \in B(X,Y) \).

Example 4.34 (Weak \( \not\Rightarrow \) Strong). Let \( X = Y = \ell^2 \). Let \( T_n \) be “zero-shift” operator, i.e.,

\[
T_n(x_i) = (0, \ldots, 0, x_1, x_2, \ldots), \text{ n zeros}.
\]

Let \((z_i) \in \ell^2 \) be another element. Define \( f \in \ell^2' \) by

\[
f(x) = \langle x, z \rangle = \sum_{i=1}^{\infty} x_i z_i.
\]

Then

\[
f(T_n x) = \langle T_n x, z \rangle = \sum_{k=1}^{\infty} x_k z_{k+n}.
\]

Compute

\[
|f(T_n x)|^2 = |\langle T_n x, z \rangle|^2 \leq \sum_{k=1}^{\infty} |x_k|^2 \sum_{m=n+1}^{\infty} |z_m|^2 \to 0.
\]

Then \( f(T_n x) \to 0 = f(0x) \), i.e. \((T_n)\) converges weakly to zero. But consider the sequence \( x = (1,0,0,\ldots) \). Then

\[
\|T_n x - T_m x\| = \sqrt{2}, \text{ } n \neq m.
\]

So \((T_n)\) does not converge strongly to zero.

Definition (Strong, Weak and Weak* Convergence of Functionals). Let \( X \) be a normed space and let \((f_n) \in X'\) be a sequence.
1. \((f_n)\) is strongly convergent to \(f \in X'\) if \(\|f_n - f\| \to 0\) and we write \(f_n \to f\).

2. \((f_n)\) is weakly convergent to \(f \in X'\) if 
   \[g(f_n) \to g(f) \quad \forall \ g \in X''.\]

3. \((f_n)\) is weak* convergent to \(f \in X'\) if 
   \[f_n(x) \to f(x) \quad \forall \ x \in X.\]

   We write 
   \[f_n \overset{w^*}{\to} f.\]

**Remark 4.35.**

1. Weak \(\Rightarrow\) weak*.

2. Limit operators. Let \((T_n) \in B(X, Y)\) be a sequence.
   
   (a) If \(T_n \to T\) (uniform) then \(T \in B(X, Y)\).
   
   (b) If convergence is strong or weak, it is possible that the limit \(T\) is unbounded (not continuous) if \(X\) is not complete. **This is the why we use Banach spaces.**

**Example 4.36.** Let \(X\) be the subspace of \(\ell^2\) of sequences with finitely many nonzero terms. Then \(X\) is not complete. Define \(T_n\) via 
   \[T_n(x_i) = (x_1, 2x_2, \ldots, nx_n, x_{n+1}, x_{n+2}, \ldots).\]

Then \((T_n) \subset B(X, X)\) converges strong to the unbounded operator \(T\)
   \[T(x_i) = (ix_i).\]

**Lemma 4.37** (Strong Operator Convergence). Let \(X\) be a Banach space, \(Y\) a normed space, and \((T_n) \subset B(X, Y)\) a sequence. \((T_n)\) strongly operator convergent \(\Rightarrow\) \(T \in B(X, Y)\).

**Proof.** Note

1. \(T\) is linear.

2. \(T_n x \to Tx \quad \forall \ x \in X \Rightarrow T_n x\) is bounded \(\forall \ x \in X \Rightarrow \|T_n\| \leq c\), for some \(c \in \mathbb{R}\) by the uniform boundedness theorem.
Now,
\[ \|T_n x\| \leq \|T_n\| \|x\| \leq c \|x\| . \]
So \( \|T_n x\| \leq c \|x\| . \) Taking lim on the LHS yields
\[ \|T x\| \leq c \|x\| , \]
therefore \( T \in B(X,Y) . \)

**Theorem 4.38** (Strong Operator Convergence). Let \( X, Y \) be Banach spaces, and let \( (T_n) \subset B(X,Y) \) be a sequence. \( (T_n) \) is strongly operator convergent iff

1. \( (\|T_n\|) \) is bounded.

2. \( (T_n x) \) is Cauchy \( \forall \ x \in M \ \forall \ M \) total in \( X . \)

**Proof.**

1. \( \Rightarrow . \) If \( T_n x \to T x \ \forall \ x \in X \) then 1 follows by the UBT and 2 follows trivially.

2. \( \Leftarrow . \) Suppose \( \|T_n\| \leq c \ \forall \ n . \) Let \( x \in X \) be arbitrary and show \( (T_n x) \) converges strongly in \( Y . \)

Let \( \varepsilon > 0 \) be given. Since \( X = \text{span}(M) , \) then \( \exists y \in \text{span}(M) \text{ s.t.} \)
\[ \|x - y\| < \frac{\varepsilon}{3c} . \]

Also, \( (T_n y) \) Cauchy \( \Rightarrow \exists N \text{ s.t.} \)
\[ \|T_n y - T_m y\| < \frac{\varepsilon}{3} . \]

Then
\[ \|T_n x - T_m x\| \leq \|T_n x - T_n y\| + \|T_n y - T_m y\| + \|T_m y - T_m x\| \]
\[ < \frac{c \varepsilon}{3c} + \frac{\varepsilon}{3} + \frac{c \varepsilon}{3c} = \varepsilon . \]

Then \( (T_n) \) is Cauchy, and since \( Y \) is complete, \( (T_n x) \) converges in \( Y . \)

**Corollary 4.39.** Let \( X \) be a Banach space and \( (f_n) \subset X' \) a sequence which is weak* convergent to \( f , \)
\[ f_n \overset{w^*}{\to} f . \]

Then \( f \in X' \) iff

1. \( (\|f_n\|) \) is bounded.

2. \( (f_n x) \) is Cauchy \( \forall \ x \in M \ \forall \ M \) total in \( X . \)
4.6 The Open Mapping Theorem

**Definition** (Open Mapping). Let $X, Y$ be metric spaces, $T : D(T) \to Y$, $D(T) \subset X$. Then $T$ is said to be an open mapping if $B \subset D(T)$ open $\Rightarrow T(B) \subset Y$ open.

**Theorem 4.40** (Open mapping). Let $X, Y$ be Banach spaces and $T \in B(X,Y)$ s.t. $T : X \onto Y$. Then $T$ is an open mapping. Hence, if $T$ is injective then $T$ is bijective and so $T^{-1}$ is continuous and so $T^{-1} \in B(X,Y)$.

**Lemma 4.41** (Open Unit Ball). Let $X, Y$ be Banach spaces and $T \in B(X,Y)$ s.t. $T : X \onto Y$ and $B_0 = B(0,1) \subset X$. Then $T(B_0) \subset Y$ is open and $0 \in T(B_0)$.

*Proof.* Let $A \subset X$. Define

1. $\alpha A = \{x \in X | x = \alpha a, a \in A\}$
2. $A + w = \{x \in X | x = a + w, a \in A\}$

Consider $B_1 = B(0,1/2) \subset X$, and let $x \in X$ be arbitrary. Then choose $k$ s.t. $x \in kB_1$, e.g., $k > 2 \|x\|$. Then

$$X = \bigcup_{k=1}^{\infty} kB_1.$$ 

Since $T$ is surjective and linear,

$$Y = T(X) = T\left(\bigcup_{k=1}^{\infty} kB_1\right) = \bigcup_{k=1}^{\infty} kT(B_1) = \bigcup_{k=1}^{\infty} \overline{kT(B_1)}.$$ 

Then by the category theory, $Y$ complete $\Rightarrow Y$ not meager. Then at least one of $\overline{kT(B_1)}$ contains an open ball $\Rightarrow T(B_1)$ contains an open ball, say

$$B_* = B(y_0, \varepsilon) \subset \overline{T(B_1)}.$$ 

Then

$$B_* - y_0 = B(0, \varepsilon) \subset \overline{T(B_1)} - y_0.$$ 

With these sets constructed, we want to show

$$\overline{T(B_1)} - y_0 \subset \overline{T(B_0)}.$$ 

Let $y \in \overline{T(B_1)} - Y_0$. Then $y + y_0 \in \overline{T(B_1)}$. \hfill \Box
**Problem** 34. Let $X,Y$ be Banach spaces and $T \in B(X,Y)$ injective. Show that $T^{-1} : \mathcal{R}(T) \to X$ is bounded iff $\mathcal{R}(T)$ is closed in $Y$.

**Solution.**

1. If $R(T)$ is closed in $Y$, it is complete, and boundedness follows from the Open Mapping Theorem.

2. Assume $T^{-1}$ to be bounded, $y \in \overline{R(T)} \subset Y$, $(y_n)$ in $R(T)$ such that $y_n \to y$, and $x_n = T^{-1}y_n$. Since $T^{-1}$ is continuous and $X$ is complete, $(x_n)$ converges, say, $x_n \to x$. Since $T$ is continuous, $y_n = Tx_n \to Tx$. Hence $y = Tx \in R(T)$, so that $R(T)$ is closed because $y \in \overline{R(T)}$ was arbitrary.

4.7 Closed Graph Theorem

**Definition** (Closed Linear Operator). Let $X,Y$ be normed spaces and $T : \mathcal{D}(T) \to Y$ a linear operator with $\mathcal{D}(T) \subset X$. $T$ is said to be **closed** if the graph of $T$, denoted $\Gamma_T$

$$\Gamma_T = \{(x,y) | x \in \mathcal{D}(T), y = Tx\}$$

is closed in $X \times Y$.

**Theorem 4.42** (Closed Graph Theorem). Let $X,Y$ be Banach spaces and let $T : \mathcal{D}(T) \to Y$ be a closed linear operator. If $\mathcal{D}(T)$ is closed in $X$ then $T$ is bounded.

**Proof.** Note that $X \times Y$ is normed space with norm

$$\|(x,y)\| = \|x\| + \|y\| \quad (11)$$

First we show $X \times Y$ is complete with norm (11). Let $z_n = (x_n, y_n)$ and $(z_n) \in X \times Y$ Cauchy. Let $\varepsilon > 0$ be given. $\exists$ $N$ s.t.

$$\|z_n - z_m\| = \|x_n - x_m\| + \|y_n - y_m\| < \varepsilon \ \forall \ n,m > N.$$  

Then $(x_n), (y_n)$ are Cauchy in $X,Y$ respectively. Since $X,Y$ are Banach, these sequences converge, say

$$x_n \to x \in X, y_n \to y \in Y.$$  

Then setting $z = (x,y)$,

$$z_n \to z \in X \times Y.$$  

Since $(z_n)$ was arbitrary, $X \times Y$ is complete.

Now, by assumption $\Gamma_T$ is closed in $X \times Y$ and $\mathcal{D}(T)$ is closed in $X$. Therefor $\Gamma_T, \mathcal{D}(T)$ are complete. Consider $P : \Gamma_T \to \mathcal{D}(T)$ given by the map

$$(x,Tx) \mapsto x.$$  

Then
1. $P$ is linear. Obvious.

2. $P$ is bounded.

\[
\|P(x,Tx)\| = \|x\| \\
\leq \|x\| + \|Tx\| \\
= \|(x,Tx)\|.
\]

3. $P$ is bijective with inverse $P^{-1} : \mathcal{D}(T) \to \Gamma T$ given by the map

\[x \mapsto (x,Tx).\]

Since $\Gamma T$ and $\mathcal{D}(T)$ are complete, $P^{-1}$ is bounded, say

\[\|(x,Tx)\| \leq b \|x\|.
\]

Then

\[
\|Tx\| \leq \|Tx\| + \|x\| \\
= \|(x,Tx)\| \\
\leq b \|x\|,
\]

\[\forall \ x \in \mathcal{D}(T).
\]

Then $T$ is bounded, as required. \(\square\)

**Theorem 4.43** (Closed Linear Operator). Let $X,Y$ be normed spaces, $T : \mathcal{D}(T) \to Y$ a linear operator, and $\mathcal{D}(T) \subset X$. Then $T$ is closed iff

1. Given $(x_n) \subset \mathcal{D}(T)$.

2. If $x_n \to x$ and $Tx_n \to y$ then

\[x \in \mathcal{D}(T) \text{ and } Tx = y.
\]

**Proof.** $z \in \overline{\Gamma T}$ iff \(\exists\) sequence $(z_n) \subset \Gamma T$

\[z_n = (x_n,Tx_n)
\]

s.t.

\[z_n \to z.
\]

Hence, $x_n \to x$ and $Tx_n \to y$. Then $z = (x,y) \in \Gamma T$ iff

\[x \in \mathcal{D}(T) \text{ and } y = Tx.
\]

\(\square\)
Example 4.44 (Differential Operator). Let \( X = C[0,1] \) and let \( T : \mathcal{D}(T) \to X \) be given by the map
\[
x \mapsto x'.
\]
Note \( \mathcal{D}(T) \) is the space of continuously differentiable functions. Then

1. \( T \) is bounded. Obvious.

2. \( T \) is closed. Suppose \( x_n \to x \) and \( Tx_n = x'_n \to y \). Then
\[
\int_0^1 y(\tau)d\tau = \int_0^1 \lim_{n \to \infty} x'_n(\tau)d\tau = \lim_{n \to \infty} \int_0^1 x'_n(\tau)d\tau = x(t) - x(0).
\]
\[
\Rightarrow \quad x(t) = x(0) + \int_0^1 y(\tau)d\tau.
\]
\[
\Rightarrow \quad x \in \mathcal{D}(T), \quad x' = y
\]

Remark 4.45. Boundedness does not imply closedness. To see this, let \( T : \mathcal{D}(T) \to \mathcal{D}(T) \subset X \) be the identity operator where \( \mathcal{D}(T) \) is proper, dense subset of \( X \). Then \( T \) is linear and bounded.

However \( T \) is not closed. Take \( x \in X - \mathcal{D}(T) \) and let \( (x_n) \subset \mathcal{D}(T) \) be s.t.
\[
x_n \to x.
\]
Then
\[
Tx_n = x_n \not\to x \notin \mathcal{D}(T).
\]

Lemma 4.46 (Closed Operator). Let \( X, Y \) be normed spaces, let \( T \in B(\mathcal{D}(T), Y) \) with \( \mathcal{D}(T) \subset X \).

1. If \( \mathcal{D}(T) \) is a closed subset of \( X \) then \( T \) is close.

2. If \( T \) is closed and \( Y \) is complete then \( \mathcal{D}(T) \) is a closed subset of \( X \).

Proof.

1. If \( (x_n) \subset \mathcal{D}(T) \) and \( x_n \to x \) and \( (Tx_n) \) converges then
(a) $x \in \overline{\mathcal{D}(T)} = \mathcal{D}(T)$ since $\mathcal{D}(T)$ is closed.

(b) $Tx_n \to Tx$ since $T$ is closed.

$\Rightarrow T$ is closed.

2. For $x \in \overline{\mathcal{D}(T)} \ni (x_n) \subset \mathcal{D}(T)$ s.t.

$$x_n \to x.$$ 

Since $T$ is bounded, 

$$\|Tx_n - Tx\| \leq \|T\| \|x_n - x_m\|.$$ 

Therefore $(Tx_n)$ is Cauchy, and since $Y$ is complete 

$$Tx_n \to y \in Y.$$ 

Since $T$ is closed we have $x \in \mathcal{D}(T)$ and $Tx = y$. Hence $\mathcal{D}(T)$ is closed because $y \in \overline{\mathcal{D}(T)}$ was arbitrary.

$\square$

**Problem 35.** Let $X$ and $Y$ be normed spaces and let $T : X \to Y$ be a closed, linear operator. Show the following.

1. The image $B$ of a compact subset $C \subset X$ is closed in $Y$.

2. The inverse image $A$ of compact subset $K \subset Y$ is closed in $X$.

**Solution.**

1. Consider any $a \in \overline{A}$. Let $a_n \to a$, where $a_n \in A$. Let $c_n \in C$ be such that $a_n = Tc_n$. Since $C$ is compact, $(c_n)$ has a subsequence $(c_{n_k})$ which converges, say, $c_{n_k} \to c \in C$. Also $Tc_{n_k} \to a$, and $Tc = a \in A$ because $T$ is closed by assumption.

2. Consider any $b \in \overline{B}$. Let $b_n \to b$, where $b_n \in B$. Let $k_n = Tb_n$. Since $K$ is compact, $(k_n)$ has a subsequence $(k_{n_i})$ which converges, say, $k_{n_i} \to k \in K$. Also $b_{n_i} \to b$, and $Tb = k \in K = T(B)$ by the closedness of $T$, so that $b \in B$ and $B$ is closed.

$\square$
5 Exam 1

Problem 1 (4 Points). Let $M \subset l^\infty$ be the subspace consisting of all sequences $x = (\xi_i)$ with at most finitely many nonzero terms. Is $M$ complete?

Solution. Let $M \subset l^\infty$, and $x = (1, 1/2, 1/3, \ldots) = (\xi_i)$. Consider the sequence $(x_n)$, where $x_n = (1, 1/2, \ldots, 1/n, 0, \ldots)$. Then $x_n \in M$ and $d(x_n, x) = 1/(n + 1)$, i.e., we have that $x_n \to x$, but $x \notin M$. It follows that $M$ is not closed. □

Problem 2 (4 Points). Does 

$$d(x, y) = \int_a^b |x(t) - y(t)| \, dt$$

define a metric or pseudometric on $X$ if $X$ is

1. the set of all real-valued continuous functions on $[a, b]$

2. the set of all real-valued Riemann integrable functions on $[a, b]$?

Solution. $d$ is a metric on $C[a, b]$, because if the integral is 0, then $|x(t) - y(t)| = 0$ for all $t \in [a, b]$. On the other hand $d$ is a pseudo-metric on $R[a, b]$, for example if $x(a) = 1$ and $x(t) = 0$ everywhere else on $[a, b]$, then the integral of $|x(t)|$ is 0. □

Problem 3 (4 Points). If $X$ is a compact metric space and $M \subset X$ is closed, show that $M$ is compact.

Solution. Let $(x_n) \subset M \subset X$. Since $X$ is compact $(x_n)$ has a convergent subsequence, say $(x_{n_k}) \subset M$, and $x_n \to x$. $M$ is closed, so we must have $x \in M$. It follows that $M$ is compact. □

Problem 4 (4 Points). Let $T : X \to Y$ be a linear operator and $\dim X = \dim Y = n < \infty$. Show that $R(T) = Y$ iff $T^{-1}$ exists.

Solution.

1. Suppose that $T^{-1}$ exists and let $\{e_1, e_2, \ldots, e_n\}$ a basis for $X$. We show that $\{Te_1, Te_2, \ldots, Te_n\}$ are linearly independent (and therefore $R(T) = Y$).

Suppose that $\alpha_1 Te_1 + \ldots + \alpha_n Te_n = T(\alpha_1 e_1 + \ldots + \alpha_n e_n) = 0$. $T$ is bijective by assumption, so we have that $\alpha_1 e_1 + \ldots + \alpha_n e_n = 0$, and then $\alpha_1 = \ldots = \alpha_n = 0$. It follows that $\{Te_1, \ldots, Te_n\}$ are linearly independent.

2. Suppose that $R(T) = Y$, and $(y_1, y_2, \ldots, y_n)$ is a basis for $Y$. Then there exist $x_1, x_2, \ldots, x_n \in X$ such that $Tx_i = y_i$, for $i = 1, 2, \ldots, n$. 93
We can show that \( \{x_1, x_2, ..., x_n\} \) are linearly independent using the same argument as above. It follows that \( \{x_1, x_2, ..., x_n\} \) is a basis for \( X \).

Suppose that \( Tx = 0 \). Then \( Tx = T(\alpha_1 x_1 + ... + \alpha_n x_n) = \alpha_1 y_1 + ... + \alpha_n y_n = 0 \), and \( \alpha_1 = ... = \alpha_n = 0 \). It follows that \( x = 0 \) and \( T \) is injective. Also, \( T \) is surjective by assumption. It follows that \( T^{-1} \) exists.

**Problem 5** (4 Points). Show that the operator \( T : l^\infty \rightarrow l^\infty \) defined by

\[
y = (\eta_i) = Tx, \quad \eta_i = \frac{\xi_i}{i}, \quad x(\xi_i),
\]

is linear and bounded. What can you say about \( T^{-1} \)?

**Solution.** Let \( T : l^\infty \rightarrow l^\infty \), \( x = (\xi_i) \in l^\infty \), \( Tx = (\xi_i/i) \). Then \( T \) is bounded, linear, injective, but not surjective, e.g., \( y = (1, 1, 1, ..) \in l^\infty \) is not in \( R(T) \). \( T^{-1} \) is only defined on \( R(T) \subset l^\infty \). \( T^{-1} \) is not bounded, because for \( x_n = (\xi_{ni}) \), where \( \xi_{ni} = 1 \) if \( i = n \) and 0 otherwise we get

\[
\frac{\|t^{-1}x_n\|}{\|x_n\|} = n.
\]
Problem 1 (4 Points). If \( x \) and \( y \) are different vectors in a finite dimensional vector space \( X \), show that there is a linear functional \( f \) on \( X \) such that \( f(x) \neq f(y) \).

Solution. Let \( x, y \in X, x \neq y, \dim X = n \).

1. Suppose that \( x \neq \alpha y \), for any \( \alpha \). Then \( x \) and \( y \) are linearly independent and we can construct a basis \( \{ x = e_1, y = e_2, e_3, ..., e_n \} \), and a corresponding dual basis \( \{ f_1, f_2, ..., f_n \} \), where \( f_i(e_j) = \delta_{ij} \). Then \( f_1(x) = 1 \neq f_1(y) \).

2. If \( x = \alpha y, \alpha \neq 1 \), then we take \( \{ x = e_1, e_2, e_3, ..., e_n \} \), and \( \{ f_1, f_2, ..., f_n \} \). Then \( f_1(x) = 1 \neq f_1(y) = \alpha \).

Problem 2 (4 Points). Let \( X \) and \( Y \) be normed spaces and \( T_n : X \to Y, n = 1, 2, 3, ..., \) be bounded linear operators. Show that convergence \( T_n \to T \) implies that for every \( \epsilon > 0 \) there is an \( N \) such that for all \( n > N \) and all \( x \) in any given closed ball we have \( \| T_n x - T x \| < \epsilon \).

Solution. Let \( B \) be a closed ball in \( X \). \( B \) is bounded, i.e., if \( x \in B \), then \( \| x \| < K \) for some \( K \). Let \( N \) be such that \( \| T_n - T \| < \frac{\epsilon}{K} \) for all \( n > N \). Then
\[
\| T_n x - T x \| = \| (T_n - T) x \| \leq \| T_n - T \| \| x \| < \epsilon.
\]

Problem 3 (4 Points). Show that \( y \perp x_n \) and \( x_n \to x \) together imply \( x \perp y \).

Solution. Suppose that \( y \perp x_n \) and \( x_n \to x \). Then
\[
\langle x, y \rangle = \lim_{n \to \infty} \langle x_n, y \rangle = 0.
\]

Problem 4 (4 Points). Let \( M \) be a total set in an inner product space \( X \). If \( \langle v, x \rangle = \langle w, x \rangle \) for all \( x \in M \), show that \( v = w \).

Solution. We have by assumption that \( \langle v - w, x \rangle = 0 \) for all \( x \in M \). \( M \) is total in \( X \), so \( v - w \in \text{span}M = X \). By the continuity of the inner product we have \( \langle v - w, v - w \rangle = \| v - w \|^2 = 0 \).

Problem 5 (4 Points). If \( S \) and \( T \) are bounded self-adjoint operators on a Hilbert space \( H \) and \( \alpha \) and \( \beta \) are real, show that \( L = \alpha S + \beta T \) is self-adjoint.

Solution. Let \( L = \alpha S + \beta T \). Then we have
\[
L^* = (\alpha S + \beta T)^* = (\alpha S)^* + (\beta T)^* = \alpha S^* + \beta T^* = \alpha S^* + \beta T^* = \alpha S + \beta T = L.
\]
7 Final Exam

Problem 1 (4 Points). What can you say about the reflexivity of $l^p$, $1 \leq p \leq \infty$?

Solution. $l^p$, For $1 < p < \infty$ is reflexive using the fact that the dual space of $l^p$ is $l^q$, where $\frac{1}{p} + \frac{1}{q} = 1$. On the other hand, $l^1$ and $l^\infty$ are nonreflexive spaces. For $l^1$ we can use the argument that a separable normed space with a nonseparable dual cannot be reflexive. Concerning $l^\infty$, we know that $c_0$, the space of convergent sequences with limit equal to zero is a subspace of $l^\infty$ and that the dual of $c_0$ is $l^1$. It follows that the dual of $l^\infty$ is larger than $l^1$, and therefore $l^\infty$ is not reflexive.

Problem 2 (4 Points). Let $X$ be a separable Banach space and $M \subset X'$ a bounded set. Show that every sequence of elements of $M$ contains a subsequence which is weak* convergent to an element of $X'$.

Solution. Any sequence $(f_n)$ in $M$ is bounded, say $\|f_n\| \leq r$. Since $X$ is separable, it contains a countable dense subset $V$, which we can arrange in a sequence $(x_m)$. Since

$$|f_n(x_m)| \leq \|f_n\||x_m| \leq r|x_m|,$$

we see that for fixed $m$ the sequence $(f_n(x_m))$ is bounded, so that it has a subsequence $A_1$ which converges at $x_1$, and $A_1$ has a subsequence $A_2$ which converges at $x_2$, ...etc.; hence $(f_{n_k}(x))$, where $f_{n_1} \in A_1, f_{n_2} \in A_2, ...$, is a subsequence which converges at every element of $V$. Since $V$ is dense in $X$ and $X$ is complete the statement follows.

Problem 3 (4 Points). Show that an open mapping need not map closed sets onto closed sets.

Solution. The mapping $T : R^2 \to R$ defined by $(x_1, x_2) \to (x_1)$ is open, it maps the closed set $\{(x_1, x_2)|x_1 x_2 = 1\} \subset R^2$ onto the set $R - \{0\}$ which is not closed in $R$.

Problem 4 (4 Points). Let $X$ and $Y$ be normed spaces and $X$ compact. If $T : X \to Y$ is a bijective closed linear operator, show that $T^{-1}$ is bounded.

Solution. $T^{-1}$ is closed, being the inverse of a closed operator. Hence $T^{-1} : Y \to X$, where $T^{-1}$ is closed and $X$ is compact, is bounded.

Problem 5 (4 Points). Let $f$ be an integrable function on the measure space $(X, \mathcal{B}, \mu)$. Show that given $\epsilon > 0$, there is a $\delta > 0$ such that for each measurable set $E$ with $\mu E < \delta$ we have

$$\left| \int_E f \right| < \epsilon.$$
Solution. Let $f$ be an integrable function on the measure space $(X, B, \mu)$. Let $f_n(x) = f(x)$ if $|f(x)| \leq n$, $f_n(x) = n$ if $f(x) \geq n$, $f_n(x) = -n$ if $f(x) \leq -n$. Then $|f - f_n| \to 0$ a.e. and $|f - f_n| \leq |f|$ for all $n$. By the dominated convergence theorem we have

$$\int |f - f_n| \to 0.$$ 

Let $\epsilon > 0$ be given. We can pick $N$ such that

$$\int |f - f_n| < \frac{\epsilon}{2} \text{ for all } N.$$ 

Let $\delta < \frac{\epsilon}{2N}$. If $E$ is measurable with $\mu E < \delta$, then

$$|\int_E f| = |\int_E (f_n + (f - f_n))| \leq \int_E |f_n| + \int_E |f - f_n| \leq \delta N + \int |f - f_n| \leq \epsilon.$$ 

\qed