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Parameter Identification in Classes of Neutral Differential Equations with State-Dependent Delays

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Abstract

We consider a parameter identification algorithm and establish its theoretical convergence on initial value problems governed by neutral functional differential equations with state-dependent delays. In the discretization process we use an Euler-type approximation method based on equations with piecewise constant arguments. Numerical examples are included.

Keywords: Parameter identification, numerical approximation, neutral equations, state-dependent delays, Euler-method, equations with piecewise constant arguments.

1 Introduction

In this paper, making use of a general framework for parameter identification in distributed parameter systems (see e.g., [1], [2], [3], [17], and the references therein), we study convergence properties of numerical schemes producing approximate solutions of parameter estimation problems for a class of neutral functional differential equations (NFDEs) with state-dependent delays. Following the work in [14] (state-dependent delay equations), and in [11]–[13] (NFDEs with constant and time-dependent delays), in this paper we consider NFDEs of the form:

$$\frac{d}{dt}\Big(x(t)+q(t)x(t-\tau(t,x(t)))\Big) = f\Big(t,x(t),x(t-\sigma(t,x(t)))\Big),$$

and establish theoretical convergence of an Euler-type approximation scheme, based on equations with piecewise constant arguments (EPCAs), for approximate solutions of corresponding parameter identification problems.

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The remaining part of the paper is organized as follows:

In Section 2 we recall our existence and uniqueness results from [15], and introduce a simple EPCA-based numerical approximation scheme. Section 3 contains a brief description of the general identification method we follow. In Section 4 we define a modification of our approximation scheme, which is more appropriate for identification purposes, and prove the key steps of the general identification method, namely, the convergence of the approximate problem under a certain double limiting process, and the continuous dependence of the solution of the discretized initial value problem (IVP) on parameters. Section 5 contains a few numerical examples.

Note that EPCAs were used first in [8] to obtain numerical approximation schemes and to prove the convergence of the approximation method for linear delay and neutral differential equations with constant delays, and later in [9] for nonlinear delay equations with state-dependent delays. Finally, we note that existence and uniqueness questions for other classes of NFDEs with state-dependent delays have been studied in [6], [7], [10], and [16].

2 Existence and Uniqueness of Solutions

Consider the vector NFDE

$$\frac{d}{dt}\Big(x(t)+q(t)x(t-\tau(t,x(t)))\Big)=f\Big(t,x(t),x(t-\sigma(t,x(t)))\Big),\qquad t\in[0,T] \tag{2.1}$$

with initial condition

$$x(t)=\varphi(t),\qquad t\in[-r,0]. \tag{2.2}$$

We make the following assumptions:

(H1) $f\in C([0,T]\times\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$ is locally Lipschitz-continuous in its second and third arguments, i.e., for every $M\geq 0$ there exists $L_1=L_1(M)\geq 0$ such that

$$|f(t,x,y)-f(t,\bar{x},\bar{y})|\leq L_1\Big(|x-\bar{x}|+|y-\bar{y}|\Big),$$

for $t\in[0,T]$, $x,\bar{x},y,\bar{y}\in\mathbb{R}^n$, $|x|$, $|\bar{x}|$, $|y|$, $|\bar{y}|\leq M$,

(H2) $q\in C([0,T];\mathbb{R})$ is Lipschitz-continuous, i.e., there exists $L_2\geq 0$ such that

$$|q(t)-q(\bar{t})|\leq L_2|t-\bar{t}|,\qquad \text{for } t\in[0,T],$$

(H3) $\tau,\sigma\in C([0,T]\times\mathbb{R}^n;\mathbb{R})$ are such that

(i) there exist $r>0$ and $r_0>0$ such that

$$-r\leq t-\tau(t,x)\leq t-r_0\qquad\text{and}\qquad -r\leq t-\sigma(t,x)\leq t,\qquad \text{for } t\in[0,T],\ x\in\mathbb{R}^n,$$

(ii) τ is locally Lipschitz-continuous in its first and second arguments, i.e., for every $M\geq 0$ there exists constants $L_3=L_3(M)\geq 0$ and $L_4=L_4(M)\geq 0$ such that

$$|\tau(t,x)-\tau(\bar{t},\bar{x})|\leq L_3|t-\bar{t}|+L_4|x-\bar{x}|,$$

for $t,\bar{t}\in[0,T]$, $x,\bar{x}\in\mathbb{R}^n$, $|x|$, $|\bar{x}|\leq M$,

(iii) σ is locally Lipschitz-continuous in its second argument, i.e., for every $M \geq 0$ there exists $L_5 = L_5(M) \geq 0$ such that

$$|\sigma(t,x) - \sigma(t,\bar{x})| \leq L_5|x - \bar{x}|$$

for $t \in [0,T]$, $x, \bar{x} \in \mathbb{R}^n$, $|x|$, $|\bar{x}| \leq M$,

(H4) $\varphi \in C([-r,0];\, \mathbb{R}^n)$ is Lipschitz-continuous on $[-r,0]$, i.e., there exists $L_6 \geq 0$ such that

$$|\varphi(t) - \varphi(\bar{t})| \leq L_6|t - \bar{t}|, \qquad \text{for } t \in [0,T].$$

Here, and throughout this paper, $|\cdot|$ denotes a vector norm on $\mathbb{R}^n$.

For $h > 0$ we introduce the "greatest integer function with respect to h", $[t]_h \equiv [t/h]h$, where $[\cdot]$ is the greatest integer function. It is a piecewise constant, right continuous function satisfying

$$t - h < [t]_h \leq t. \tag{2.3}$$

Following the ideas of [8] and [13], we discretize (2.1) by changing the time variable t to the piecewise constant function, $[t]_h$. Consider

$$\begin{aligned} &\frac{d}{dt}\Big(y_h(t) + q([t]_h)y_h(t - [\tau([t]_h, y_h([t]_h - h))]_h)\Big) \\ &\qquad = f\Big([t]_h, y_h([t]_h), y_h([t]_h - [\sigma([t]_h, y_h([t]_h))]_h)\Big) \end{aligned} \tag{2.4}$$

for $t \in [0,T]$, with the initial condition

$$y_h(t) = \varphi(t), \qquad t \in [-r,0]. \tag{2.5}$$

It is easy to see that IVP (2.4)-(2.5) has a unique solution on [0,T] for $0 < h < r_0$, and the values of the solution at mesh points can be computed by a simple recursive formula using past values of the solution at mesh points only. (For details, see [15].)

We recall the following results concerning the local existence and uniqueness of solutions of IVP (2.1)-(2.2) from [15].

Lemma 2.1 *Assume (H1)–(H4). Let $h_0 \equiv r_0/2$. Then there exist constants $M_1 > 0$ and $\alpha = \alpha(M_1)$, $M_2 = M_2(M_1)$ such that $0 < \alpha \leq r_0/2$, $|y_h(t)| \leq M_1$ and $|\dot{y}_h(t)| \leq M_2$ for $t \in [-r,\alpha]$, $0 < h \leq h_0$.*

Lemma 2.2 *Assume (H1)–(H4). Let h_0, α, M_1 and M_2 be defined by Lemma 2.1, $L_4 = L_4(M_1)$ be the constant from (H3) (ii), and assume that $|q|_C L_4 L_6 < 1$. Then there exists a constant $M_3 \geq 0$ such that $|a_h(k) - b_h(k)| \leq M_3 h$ for $0 < h \leq h_0$, $k = 1, 2, \ldots, [\alpha/h]$.*

Theorem 2.3 *Assume (H1)–(H4). Let h_0, α, M_1, M_2 and M_3 be defined by Lemma 2.1 and 2.2, $L_4 = L_4(M_1)$ be the constant from (H3) (ii), and assume that $|q|_C L_4 L_6 < 1$. Then IVP (2.1)-(2.2) has a Lipschitz-continuous solution, $x(t)$, on $[-r,\alpha]$.*

Theorem 2.4 *Assume (H1)–(H4), and let $x(t)$ be a Lipschitz-continuous solution of IVP (2.1)-(2.2) on $[-r,\alpha]$. Let $M_1^* \equiv \max\{|x(t)| : t \in [-r,\alpha]\} + \varepsilon$ for some $\varepsilon > 0$, $L_4 = L_4(M_1^*)$ be the corresponding constant from (H3) (ii), and $M_2^* \equiv \operatorname{ess\,sup}\{|\dot{x}(t)| : t \in [-r,\alpha]\}$. If $|q|_C L_4 M_2^* < 1$, then $x(t)$ is the unique solution of IVP (2.1)-(2.2) on $[-r,\alpha]$, and*

$$\lim_{h\to 0+} \sup_{0\le t\le \alpha} |x(t) - y_h(t)| = 0,$$

where y_h is the solution of IVP (2.4)-(2.5).

We remark that the conditions $|q|_C L_4 L_6 < 1$ and $|q|_C L_4 M_2^* < 1$ in Theorem 2.3 and 2.4 are essential. Examples were given in [15] to show that without these conditions IVP (2.1)-(2.2) may not have a solution or unique solution, and the scheme (2.4)-(2.5) may not be convergent.

Next we recall Lemma 3.2 from [8]. The proof of Theorem 2.4 in [15] was based on a slightly modified version of this inequality. We shall use this "Gronwall type" inequality in Section 4 as well.

Lemma 2.5 *Let $a > 0$, $b \ge 0$, $\alpha \ge 0$, $\beta > 0$, $\gamma \equiv \max\{\alpha,\beta\}$, and $g : [0,T] \to [0,\infty)$ be continuous and nondecreasing. Let $u : [-\gamma,T] \to [0,\infty)$ be continuous, and satisfy the inequality*

$$u(t) \le g(t) + bu(t-\beta) + a\int_0^t u(s-\alpha)\,ds, \qquad t \in [0,T].$$

Then $u(t) \le d(t)e^{ct}$ for $t \in [0,T]$, where c is the unique positive solution of $cbe^{-c\beta} + ae^{-c\alpha} = c$, and

$$d(t) \equiv \max\left\{\frac{g(t)}{1-be^{-c\beta}},\ \max_{-\gamma\le s\le 0} e^{-cs}u(s)\right\}, \qquad t \in [0,T].$$

3 A General Identification Method

In this section we briefly recall a general method frequently used to identify parameters in various classes of differential equations (see, e.g., [1], [3], [17], and also [11]–[14]). We present the method for our IVP (2.1)-(2.2).

Assume that (2.1)-(2.2) contains some "unknown" parameters, γ. In this paper we concentrate on identifying the initial function, φ, and the parameters, q and τ in the neutral term of the equation, i.e., $\gamma = (\varphi, q, \tau)$. Other parameters on the right-hand side of the equation can be treated similarly (see [14]).

We assume that some information is available via measurements $(X_0, X_1, \ldots, X_l)$ of the solution, $x(t)$, at discrete time values $(t_0, t_1, \ldots, t_l)$. The goal is to find the parameter value, which minimizes the least squares fit-to-data criterion

$$J(\gamma) = \sum_{i=0}^{l} |x(t_i;\gamma) - X_i|^2,$$

where γ belongs to an admissible set Δ contained in the parameter space Γ. (Denote this problem by $\mathcal{P}$). The general method consists of the following steps:

Step 1) First take finite dimensional approximations of the parameters, γ^N, (i.e., $\gamma^N \in \Delta^N \subset \Gamma^N \subset \Gamma$, $\dim \Gamma^N < \infty$, $\gamma^N \to \gamma$ as $N \to \infty$).

Step 2) Consider a sequence of approximate IVPs corresponding to a discretization of IVP (2.1)-(2.2) for some fixed parameter $\gamma^N \in \Gamma^N$ with solutions $y^M(\cdot;\gamma^N)$ satisfying $y^M(t,\gamma^N) \to x(t,\gamma)$ as $N, M \to \infty$, uniformly on compact time intervals, and $\gamma^N \in \Delta^N$.

Step 3) Define the least square minimization problems ($\mathcal{P}^{N,M}$): for each $N, M = 1, 2, \ldots$, i.e., find $\gamma^{N,M} \in \Delta^N \subset \Gamma^N$, which minimizes the least squares fit-to-data criterion

$$J^{N,M}(\gamma^N) = \sum_{i=0}^{l} |y^M(t_i;\gamma^N) - X_i|^2, \qquad \gamma^N \in \Delta^N.$$

Often Δ^N is the projection of Δ to Γ^N, and we restrict our discussion to this case.

Step 4) Assuming that Δ is a compact subset of Γ, and the approximate solution, $y^M(t;\gamma^N)$, depends continuously on the parameter, γ^N, we get, that $J^{N,M}(\cdot)$ is continuous for each M, N. Hence the finite dimensional minimization problems, $\mathcal{P}^{N,M}$, have a solution, $\bar{\gamma}^{N,M}$. Since $\bar{\gamma}^{N,M} \in \Delta$, the sequence $\bar{\gamma}^{N,M}$ $(N, M = 1, 2, \ldots)$ has a convergent subsequence, say $\bar{\gamma}^{N_j,M_j}$, with limit $\bar{\gamma} \in \Gamma$.

Step 5) It follows from Step 2 that $J^{N_j,M_j}(\bar{\gamma}^{N_j,M_j}) \to J(\bar{\gamma})$ as $j \to \infty$. Let $\gamma \in \Delta$ be fixed, and let $\gamma^N \to \gamma$ satisfying Step 1. Then, in particular, $\gamma^{N_j} \to \gamma$ as $j \to \infty$. Using that $\bar{\gamma}^{N_j,M_j}$ is a solution of $\mathcal{P}^{N_j,M_j}$, Step 2 implies

$$J(\bar{\gamma}) = \lim_{j\to\infty} J^{N_j,M_j}(\bar{\gamma}^{N_j,M_j}) \leq \lim_{j\to\infty} J^{N_j,M_j}(\gamma^{N_j}) = J(\gamma),$$

therefore $\bar{\gamma}$ is the solution of the minimization problem $\mathcal{P}$.

In practice we take "large enough" N and M, and use the solution of $\mathcal{P}^{N,M}$ as an approximate solution of $\mathcal{P}$. Note that Step 4 and 5 yield that the limit of any convergent subsequence of $\gamma^{N,M}$ is a solution of $\mathcal{P}$ (with the same cost). It is possible that the minimizer of $J(\gamma)$ is not unique (see Example 5.4 below). Identifyability of parameters, i.e., the uniqueness of the parameter minimizing the cost function $J(\gamma)$, is an important research topic. However, except for some comments in Section 4, we do not address this question here. For related works we refer to [2] and [18].

In our examples, we will use linear spline approximation to discretize the parameters φ, q and τ, in the case of nonconstant functions, in Step 1. In the next section we introduce a set of approximate IVPs corresponding to IVP (2.1)-(2.2) we use in Step 2, and show uniform convergence of the scheme, and continuous dependence of the approximate solution on parameters, as required in Step 2 and Step 4, respectively.

4 Main Results

In this section we assume that $T > 0$ is such that IVP (2.1)-(2.2) has a unique solution on $[0,T]$. Our goal is to identify the parameters $\gamma = (\varphi, q, \tau)$. We introduce the parameter space $\Gamma \equiv C([-r,0];\mathbb{R}^n) \times C([0,T];\mathbb{R}) \times BC([0,T]\times\mathbb{R}^n;\mathbb{R}^+)$, where $BC([0,T]\times\mathbb{R}^n;\mathbb{R}^+)$ denotes the Banach-space of bounded and continuous functions from $[0,T]\times\mathbb{R}^n$ to $[0,\infty)$ with the norm $|\tau|_C \equiv \sup\{\tau(t,x) \,:\, t \in [0,T],\ x \in \mathbb{R}^n\}$. We denote the usual supremum norms on $C([-r,0];\mathbb{R}^n)$ and $C([0,T];\mathbb{R})$ by $|\varphi|_C$ and $|q|_C$, respectively. (We will always use the symbols φ, q and τ to denote elements of these spaces, so there should be no confusion with this simplified notation of the norms.) The norm on Γ is defined by $|(\varphi,q,\tau)|_\Gamma \equiv |\varphi|_C+|q|_C+|\tau|_C$.

Consider a sequence of parameters, $\gamma^N \equiv (\varphi^N, q^N, \tau^N) \in \Gamma$, such that $|\gamma-\gamma^N|_\Gamma \to 0$ as $N\to\infty$. For large enough N such that $|\tau-\tau^N|_C \le r_0/2$, assumption (H2) (i) implies that

$$r_0/2 \le \tau^N(t,x), \qquad \text{for } t\in[0,T],\ x\in\mathbb{R}^n. \tag{4.1}$$

Define

$$-r_N \equiv \min\Big\{\min\{t-\tau^N(t,x) \,:\, t\in[0,T],\ x\in\mathbb{R}^n\},\, -r\Big\}. \tag{4.2}$$

Note that (H2) (i) yields that $-r-|\tau-\tau^N|_C \le -r_N \le -r$.

Let h be a positive number. Using the approximation scheme (2.4)-(2.5), we associate the following NFDE with piecewise constant arguments to (2.1) with parameters φ^N, q^N, τ^N:

$$\begin{aligned}\frac{d}{dt}\Big(y_{h,N}(t) + q^N([t]_h)y_{h,N}(t-[\tau^N([t]_h, y_{h,N}([t]_h-h))]_h)\Big)&\\ = f\Big([t]_h, y_{h,N}([t]_h), y_{h,N}([t]_h-[\sigma([t]_h, y_{h,N}([t]_h))]_h)\Big)&\end{aligned} \tag{4.3}$$

for $t\in[0,T]$, with the initial condition

$$y_{h,N}(t) = \widetilde{\varphi^N}(t), \qquad t\in[-r_N,0], \tag{4.4}$$

where $\widetilde{\varphi^N}$ is the extension of φ^N to $[-r_N,0]$ defined by

$$\widetilde{\varphi^N}(t) \equiv \begin{cases} \varphi^N(t), & t\in[-r,0],\\ \varphi^N(-r), & t\in[-r_N,-r].\end{cases} \tag{4.5}$$

The subscript h and N of $y_{h,N}(t)$ emphasizes that $y_{h,N}(t)$ is the solution of IVP (4.3)-(4.4) corresponding to the discretization parameter h and parameter values φ^N, q^N and τ^N.

Lemma 2.5 implies that IVP (4.3)-(4.4) has a unique solution on $[-r,T]$, and one can modify the proof of Theorem 2.4 to show that

$$\lim_{\substack{h\to0+\\ N\to\infty}} \max_{0\le t\le T} |x(t)-y_{h,N}(t)| = 0.$$

Consequently, this scheme satisfies Step 2 of the general identification method. It was used successfully in some examples in [12] for identifying parameters in (state-independent) neutral equations, but in other cases (e.g., when we tried to identify τ) our secant-type numerical minimization routine failed. The problem is that the solution of (4.3)-(4.4), and therefore the corresponding cost function, $J^{N,h}$, is piecewise-constant with respect to τ^N because of taking the integer part $[\tau^N(\cdot,\cdot)]_h$ in (4.3). The advantage of using scheme (4.3)-(4.4) is that it is very simple to compute the solution at mesh points, but it has some serious disadvantages as well:

1. The approximate solution is not linear between mesh points, therefore it is not easy to evaluate it between mesh points. Moreover, the solution is not continuous at positive mesh points.

2. The approximate solution does not depend continuously on the parameters because of using integer parts $[\tau^N(\cdot,\cdot)]_h$ and $[\sigma(\cdot,\cdot)]_h$ in (4.3).

To overcome these problems, we modify the approximate scheme (see also [12], [13]):

$$\begin{aligned}
&\frac{d}{dt}\Big(y_{h,N}(t) + \vartheta_h\Big\{q^N(t)y_{h,N}(t-\tau^N(t,y_{h,N}(t-h)))\Big\}\Big) \\
&\quad = f\Big([t]_h, y_{h,N}([t]_h), y_{h,N}([t]_h - \sigma([t]_h, y_{h,N}([t]_h)))\Big), \qquad t \in [0,T], \qquad (4.6)
\end{aligned}$$

with the initial condition

$$y_{h,N}(t) = \widetilde{\varphi^N}(t), \qquad t \in [-r_N, 0], \qquad (4.7)$$

where $\widetilde{\varphi^N}$ is defined by (4.5). Here $\vartheta_h\Big\{q^N(t)y_{h,N}(t-\tau^N(t,y_{h,N}(t-h)))\Big\}$ denotes the linear interpolate of the function $t \mapsto q^N(t)y_{h,N}(t-\tau^N(t,y_{h,N}(t-h)))$ using mesh points kh, i.e.,

$$\begin{aligned}
&\vartheta_h\Big\{q^N(t)y_{h,N}(t-\tau^N(t,y_{h,N}(t-h)))\Big\} \\
&\quad \equiv q^N([t]_h)y_{h,N}([t]_h - \tau^N([t]_h, y_{h,N}([t]_h - h)))\frac{[t]_h + h - t}{h} \\
&\qquad + q^N([t]_h + h)y_{h,N}([t]_h + h - \tau^N([t]_h + h, y_{h,N}([t]_h)))\frac{t - [t]_h}{h}.
\end{aligned}$$

By a solution of IVP (4.6)-(4.7) we mean a function $y_{h,N} : [-r_N, T] \to \mathbb{R}^n$, which is defined on $[-r_N, 0]$ by (4.7), such that

(i) the function $t \mapsto y_{h,N}(t) + \vartheta_h\Big\{q^N(t)y_{h,N}(t-\tau^N(t,y_{h,N}(t-h)))\Big\}$ is continuous on $[0,T]$,

(ii) its derivative exists at each point $t \in [0,T)$, with the possible exception of the points kh $(k = 0,1,2,\ldots)$ where finite one-sided derivatives exist, and

(iii) the function $y_{h,N}$ satisfies (4.6) on each interval $[kh,(k+1)h) \cap [0,T]$ $(k = 0,1,2,\ldots)$.

It follows from the definition that (4.6) is equivalent to the integral equation

$$\begin{aligned}
&y_{h,N}(t) + \vartheta_h\Big\{q^N(t)y_{h,N}(t-\tau^N(t,y_{h,N}(t-h)))\Big\} \\
&\quad = \varphi^N(0) + q^N(0)\widetilde{\varphi^N}(-\tau^N(0,\varphi^N(-h))) \\
&\qquad + \int_0^t f\Big([s]_h, y_{h,N}([s]_h), y_{h,N}([s]_h - \sigma([s]_h, y_{h,N}([s]_h)))\Big)\,ds. \qquad (4.8)
\end{aligned}$$

We will show that the scheme (4.6)-(4.7) preserves the convergence properties of (2.4)-(2.5) given by Theorem 2.4, and in addition, the approximate solution is continuous, piecewise linear, and we avoid taking integer part of τ^N and σ in (4.6).

Lemma 4.1 *Let $(\varphi^N, q^N, \tau^N) \in \Gamma$, and assume (4.1). Then IVP (4.6)-(4.7) has a unique continuous solution on $[-r_N, T]$ for all $0 < h < r_0/2$. Moreover, the solution is linear on the intervals $[kh,(k+1)h]$.*

Proof Assumption (4.1), and inequality (2.3) imply that

$$[t]_h - \tau^N([t]_h, y_{h,N}([t]_h - h)) \le t - r_0/2, \qquad t \in [0,T], \tag{4.9}$$

and

$$[t]_h + h - \tau^N([t]_h + h, y_{h,N}([t]_h)) \le t + h - r_0/2 \le t - r_0/4, \qquad t \in [0,T], \quad 0 < h < r_0/4. \tag{4.10}$$

Therefore $\vartheta_h\Big\{q^N(t)y_{h,N}(t-\tau^N(t, y_{h,N}(t-h)))\Big\}$ in (4.8) always uses past values of the solution, hence the existence and uniqueness of the solution follows from (4.8) using the method of steps on the intervals $[kh,(k+1)h]$. The solution is continuous, since, clearly, $t \mapsto \vartheta_h\{q^N(t)y_{h,N}(t-\tau^N(t, y_{h,N}(t-h)))\}$ is a continuous function. Equation (4.6) yields for $t \in (kh,(k+1)h)$:

$$\begin{aligned}\dot y_{h,N}(t) &= -\frac{d}{dt}\vartheta_h\Big\{q^N(t)y_{h,N}(t-\tau^N(t,y_{h,N}(t-h)))\Big\}\\ &\quad + f\Big(kh, y_{h,N}(kh), y_{h,N}(kh-\sigma(kh,y_{h,N}(kh)))\Big).\end{aligned}$$

Therefore $\dot y_{h,N}(t)$ is constant on $(kh,(k+1)h)$, and the solution is linear on $[kh,(k+1)h]$. ■

Integrating both sides of (4.6) from kh to $(k+1)h$, and using the notation $a_{h,N}(k) \equiv y_{h,N}(kh)$, we get the recursive formula

$$\begin{aligned}a_{h,N}(k+1) &= a_{h,N}(k) + q^N(kh)y_{h,N}(kh - \tau^N(kh, a_{h,N}(k-1)))\\ &\quad - q^N((k+1)h)y_{h,N}((k+1)h - \tau^N((k+1)h, a_{h,N}(k))) \qquad (4.11)\\ &\quad + hf(kh, a_{h,N}(k), y_{h,N}(kh - \sigma(kh, a_{h,N}(k)))), \quad \text{for } k = 0,1,2,\dots,\\ a_{h,N}(k) &= \varphi^N(kh), \qquad -r \le kh \le 0, \quad k = 0,-1,-2,\dots, \qquad (4.12)\end{aligned}$$

which is simple to compute, using that $y_{h,N}(t)$ is linear between mesh points. Equation (4.11) immediately implies the explicit formula

$$\begin{aligned}a_{h,N}(k+1) &= \varphi^N(0) + q^N(0)\widetilde{\varphi^N}(-\tau^N(0,\varphi^N(0)))\\ &\quad - q^N((k+1)h)y_{h,N}((k+1)h - \tau^N((k+1)h, a_{h,N}(k)))\\ &\quad + h\sum_{i=0}^{k} f(ih, a_{h,N}(i), y_{h,N}(i - \sigma(ih, a_{h,N}(i)))).\end{aligned}$$

If h is small, this is better to use in numerical computations, since in (4.11) the difference $q^N(kh)y_{h,N}(kh-\tau^N(kh,a_{h,N}(k-1))) - q^N((k+1)h)y_{h,N}((k+1)h-\tau^N((k+1)h,a_{h,N}(k)))$ is close to 0, therefore some accuracy can be lost by using that formula.

Theorem 4.2 *Assume (H1)–(H4), and that IVP (2.1)-(2.2) has a unique Lipschitz-continuous solution, $x(t)$, on $[-r,T]$. Let $\gamma^N = (\varphi^N, q^N, \tau^N) \in \Gamma$ be such that $|\gamma - \gamma^N|_\Gamma \to 0$ as $N \to \infty$. Define $M_1 \equiv \max\{|x(t)| : t \in [-r,T]\} + \varepsilon$ and $M_2 \equiv \operatorname{ess\,sup}\{|\dot x(t)| : t \in [-r,T]\}$ for some $\varepsilon > 0$, and let $L_4 = L_4(M_1)$ be the constant from (H3) (ii). If*

$$|q|_C L_4 M_2 < 1, \tag{4.13}$$

then the solution, $y_{h,N}$, of IVP (4.6)-(4.7) converges uniformly on $[0,T]$ to the solution, x, of IVP (2.1)-(2.2) as $h \to 0^+$ and $N \to \infty$, i.e.,

$$\lim_{\substack{h\to 0^+\\ N\to\infty}} \max_{0\le t\le T} |x(t) - y_{h,N}(t)| = 0. \tag{4.14}$$

Proof We assume, without loss of generality, that ε is such that $(|q|_C + \varepsilon)L_4 M_2 < 1$, and N is large enough that $|q^N - q|_C < \varepsilon$, $|\varphi^N - \varphi|_C < \varepsilon$, and $|\tau^N - \tau|_C < r_0/2$. The last relation and (H2) (i) imply (4.1). Let $0 < \alpha_{h,N} \le T$ be the largest number such that $|y_{h,N}(t)| < M_1$ for $t \in [0, \alpha_{h,N})$. ($\alpha_{h,N}$ is well-defined since $|\varphi^N(0)| < M_1$ by our assumptions.) Using the integrated form of (2.1) and (4.8), and applying elementary estimates and (H1) with $L_1 = L_1(M_1)$, we get for $t \in [0, \alpha_{h,N}]$:

$$\begin{aligned}
&|x(t) - y_{h,N}(t)|\\
&\quad\le |\varphi(0) - \varphi^N(0)| + |q(0)\varphi(-\tau(0,\varphi(0))) - q^N(0)\widetilde{\varphi^N}(-\tau^N(0,\varphi^N(-h)))|\\
&\qquad + \Big|q(t)x(t-\tau(t,x(t))) - \vartheta_h\Big\{q^N(t)y_{h,N}(t-\tau^N(t,y_{h,N}(t-h)))\Big\}\Big|\\
&\qquad + \int_0^t \Big| f\Big(s, x(s), x(s-\sigma(s,x(s)))\Big)\\
&\qquad\qquad - f\Big([s]_h, y_{h,N}([s]_h), y_{h,N}([s]_h - \sigma([s]_h, y_{h,N}([s]_h)))\Big)\Big| ds\\
&\quad\le |\varphi(0) - \varphi^N(0)| + |q(0)\varphi(-\tau(0,\varphi(0))) - q^N(0)\widetilde{\varphi^N}(-\tau^N(0,\varphi^N(-h)))|\\
&\qquad + \Big|q(t)x(t-\tau(t,x(t))) - \vartheta_h\Big\{q^N(t)y_{h,N}(t-\tau^N(t,y_{h,N}(t-h)))\Big\}\Big|\\
&\qquad + \int_0^t \Big| f\Big(s, x(s), x(s-\sigma(s,x(s)))\Big) - f\Big([s]_h, x(s), x(s-\sigma(s,x(s)))\Big)\Big| ds \qquad (4.15)\\
&\qquad + L_1\int_0^t \Big(|x(s) - y_{h,N}([s]_h)| + \Big|x(s-\sigma(s,x(s))) - y_{h,N}([s]_h - \sigma([s]_h, y_{h,N}([s]_h)))\Big|\Big) ds.
\end{aligned}$$

For convenience, we extend $x(t)$ to $t \in [-r_N, -r]$ by $x(t) = \varphi(-r)$, and denote the extended initial function by $\tilde\varphi(t)$. Clearly, $\tilde\varphi$ is Lipschitz-continuous with Lipschitz-constant L_6. Next we will estimate the terms on the right-hand side of (4.15) separately. Assumption (H4) and $|q - q^N|_C < \varepsilon$ yield

$$\begin{aligned}
&|q(0)\varphi(-\tau(0,\varphi(0))) - q^N(0)\widetilde{\varphi^N}(-\tau^N(0,\varphi^N(-h)))|\\
&\quad\le |q(0) - q^N(0)||\varphi(-\tau(0,\varphi(0)))| + |q^N(0)||\varphi(-\tau(0,\varphi(0))) - \widetilde{\varphi^N}(-\tau^N(0,\varphi^N(-h)))|\\
&\quad\le |q - q^N|_C|\varphi|_C + (|q|_C + \varepsilon)\Big(|\varphi(-\tau(0,\varphi(0))) - \widetilde{\varphi}(-\tau^N(0,\varphi^N(-h)))|\\
&\qquad\qquad + |\widetilde{\varphi}(-\tau^N(0,\varphi^N(-h))) - \widetilde{\varphi^N}(-\tau^N(0,\varphi^N(-h)))|\Big)\\
&\quad\le |q - q^N|_C|\varphi|_C + (|q|_C + \varepsilon)\Big(L_6|\tau(0,\varphi(0)) - \tau^N(0,\varphi^N(-h))| + |\varphi - \varphi^N|_C\Big). \qquad (4.16)
\end{aligned}$$

Let $x, y \in \mathbb{R}^n$, $|x|, |y| \le M_1$, then it follows from (H3) with $L_3 = L_3(M_1)$ and $L_4 = L_4(M_1)$

$$\begin{aligned}
|\tau(t,x) - \tau^N(u,y)| &\le |\tau(t,x) - \tau(u,x)| + |\tau(u,x) - \tau(u,y)| + |\tau(u,y) - \tau^N(u,y)|\\
&\le L_3|t-u| + L_4|x-y| + |\tau - \tau^N|_C. \qquad (4.17)
\end{aligned}$$

Consequently, we have that

$$\begin{aligned}|\tau(0,\varphi(0))-\tau^N(0,\varphi^N(-h))| &\leq L_4(|\varphi(0)-\varphi(-h)|+|\varphi(-h)-\varphi^N(-h)|)+|\tau-\tau^N|_C\\ &\leq L_4(L_6h+|\varphi-\varphi^N|_C)+|\tau-\tau^N|_C. \end{aligned}\tag{4.18}$$

Define the function $w_{h,N}(t)\equiv\max_{-r\leq s\leq t}|x(s)-y_{h,N}(s)|$, and let $\nu\equiv\nu(t,h)\equiv(t-[t]_h)/h$. Then $1-\nu=([t]_h+h-t)/h$, and the definition of ϑ_h implies

$$\begin{aligned}&\Big|q(t)x(t-\tau(t,x(t)))-\vartheta_h\Big\{q^N(t)y_{h,N}(t-\tau^N(t,y_{h,N}(t-h)))\Big\}\Big|\\ &\quad\leq \Big|q(t)x(t-\tau(t,x(t)))-\vartheta_h\Big\{q^N(t)x(t-\tau^N(t,y_{h,N}(t-h)))\Big\}\Big|\\ &\qquad+\Big|\vartheta_h\Big\{q^N(t)x(t-\tau^N(t,y_{h,N}(t-h)))\Big\}-\vartheta_h\Big\{q^N(t)y_{h,N}(t-\tau^N(t,y_{h,N}(t-h)))\Big\}\Big|\\ &\quad\leq |q(t)x(t-\tau(t,x(t)))-q^N([t]_h)x([t]_h-\tau^N([t]_h,y_{h,N}([t]_h-h)))|(1-\nu)\\ &\qquad+|q(t)x(t-\tau(t,x(t)))-q^N([t]_h+h)x([t]_h+h-\tau^N([t]_h+h,y_{h,N}([t]_h)))|\nu\\ &\qquad+|q^N([t]_h)|w_{h,N}([t]_h-\tau^N([t]_h,y_{h,N}([t]_h-h)))(1-\nu)\\ &\qquad+|q^N([t]_h+h)|w_{h,N}([t]_h+h-\tau^N([t]_h+h,y_{h,N}([t]_h)))\nu. \end{aligned}\tag{4.19}$$

Assumption (H2), the definition of M_2, (4.17), an estimate similar to (4.18), and the monotonicity of $w_{h,N}$ yield

$$\begin{aligned}&|q(t)x(t-\tau(t,x(t)))-q^N([t]_h)x([t]_h-\tau^N([t]_h,y_{h,N}([t]_h-h)))|\\ &\quad\leq \Big(|q(t)-q([t]_h)|+|q([t]_h)-q^N([t]_h)|\Big)|x(t-\tau(t,x(t)))|\\ &\qquad+|q^N([t]_h)||x(t-\tau(t,x(t)))-x([t]_h-\tau^N([t]_h,y_{h,N}([t]_h-h)))|\\ &\quad\leq (L_2h+|q-q^N|_C)M_1+(|q|_C+\varepsilon)M_2(h+|\tau(t,x(t))-\tau^N([t]_h,y_{h,N}([t]_h-h))|)\\ &\quad\leq (L_2h+|q-q^N|_C)M_1+(|q|_C+\varepsilon)M_2(h+L_3h+L_4(2M_2h+w_{h,N}(t))+|\tau-\tau^N|_C). \end{aligned}\tag{4.20}$$

In the same fashion we get

$$\begin{aligned}&|q(t)x(t-\tau(t,x(t)))-q^N([t]_h+h)x([t]_h+h-\tau^N([t]_h+h,y_{h,N}([t]_h)))|\\ &\quad\leq (L_2h+|q-q^N|_C)M_1+(|q|_C+\varepsilon)M_2(h+L_3h+L_4(2M_2h+w_{h,N}(t))+|\tau-\tau^N|_C). \end{aligned}\tag{4.21}$$

Therefore (4.19), (4.20) and (4.21), together with (4.9) and (4.10), imply

$$\begin{aligned}&\Big|q(t)x(t-\tau(t,x(t)))-\vartheta_h\Big\{q^N(t)y_{h,N}(t-\tau^N(t,y_{h,N}(t-h)))\Big\}\Big|\\ &\quad\leq (L_2h+|q-q^N|_C)M_1+(|q|_C+\varepsilon)M_2(h+L_3h+L_4(2M_2h)+|\tau-\tau^N|_C)\\ &\qquad+(|q|_C+\varepsilon)(L_4M_2w_{h,N}(t)+w_{h,N}(t-r_0/4)),\qquad 0<h<r_0/4. \end{aligned}\tag{4.22}$$

Finally, consider the terms in the last integral of (4.15). We have

$$\begin{aligned}|x(t)-y_{h,N}([t]_h)| &\leq |x(t)-x([t]_h)|+|x([t]_h)-y_{h,N}([t]_h)|\\ &\leq M_2h+w_{h,N}(t). \end{aligned}\tag{4.23}$$

Assumption (H2) (iii) with $L_5 = L_5(M_1)$, and (4.23) yield

$$\begin{aligned}
&|x(t-\sigma(t,x(t))) - y_{h,N}([t]_h - \sigma([t]_h, y_{h,N}([t])))| \\
&\quad\le |x(t-\sigma(t,x(t))) - x([t]_h - \sigma([t]_h, y_{h,N}([t])))| \\
&\qquad + |x([t]_h - \sigma([t]_h, y_{h,N}([t]))) - y_{h,N}([t]_h - \sigma([t]_h, y_{h,N}([t])))| \\
&\quad\le M_2(h + |\sigma(t,x(t)) - \sigma([t]_h, y_{h,N}([t]_h))|) + w_{h,N}([t]_h - \sigma([t]_h, y_{h,N}([t]))) \\
&\quad\le M_2(h + |\sigma(t,x(t)) - \sigma([t]_h, x(t))| + L_5|x(t) - y_{h,N}([t]_h)|) + w_{h,N}(t) \\
&\quad\le M_2(h + |\sigma(t,x(t)) - \sigma([t]_h, x(t))| + L_5M_2h + L_5w_{h,N}(t)) + w_{h,N}(t). \qquad (4.24)
\end{aligned}$$

Combining (4.15), (4.16), (4.18), (4.22), (4.23) and (4.24), we get for $t \in [0, \alpha_{h,N}]$ and $0 < h < r_0/4$:

$$\begin{aligned}
|x(t) - y_{h,N}(t)| \le\ & g_{h,N}(t) + (|q|_C + \varepsilon)L_4M_2w_{h,N}(t) \\
& + (|q|_C + \varepsilon)w_{h,N}(t - r_0/4) + L_1(2 + L_5M_2)\int_0^t w_{h,N}(s)\,ds, \qquad (4.25)
\end{aligned}$$

where

$$\begin{aligned}
g_{h,N}(t) \equiv\ & (1 + |q|_C + \varepsilon + (|q|_C + \varepsilon)L_4L_6)|\varphi - \varphi^N|_C + 2M_1|q - q^N|_C \\
& + (|q|_C + \varepsilon)(L_6 + M_2)|\tau - \tau^N|_C + L_2M_1h \\
& + (|q|_C + \varepsilon)(L_4(L_6)^2 + M_2 + L_3M_2 + 2L_4M_2)h + L_1(2M_2 + L_5M_2)Th \\
& + \int_0^t \Big| f\Big(s, x(s), x(s - \sigma(s,x(s)))\Big) - f\Big([s]_h, x(s), x(s - \sigma(s,x(s)))\Big)\Big|\,ds \\
& + L_1M_2\int_0^t |\sigma(s,x(s)) - \sigma([s]_h, x(s))|\,ds.
\end{aligned}$$

Note that $g_{h,N}(t)$ is defined on $[0,T]$. The monotonicity of $w_{h,N}(t)$, the inequality $w_{h,N}(t) \le |\varphi - \varphi^N|_C \le g_{h,N}(0)$ for $t \in [-r, 0]$, (4.13) and (4.22) imply

$$(1 - (|q|_C + \varepsilon)L_4M_2)w_{h,N}(t) \le g_{h,N}(t) + (|q|_C + \varepsilon)w_{h,N}(t - r_0/4) + L_1(2 + L_5M_2)\int_0^t w_{h,N}(s)\,ds$$

$t \in [0, \alpha_{h,N}]$, $0 < h < r_0/4$. Therefore an application of Lemma 2.5 yields

$$w_{h,N}(t) \le d_{h,N}(T)e^{\lambda T}, \qquad \text{for } t \in [0, \alpha_{h,N}], \quad 0 < h < r_0/4, \qquad (4.26)$$

where λ is the unique positive solution of

$$\lambda(|q|_C + \varepsilon)e^{-\lambda r_0/4} + L_1(2 + L_5M_2) = (1 - (|q|_C + \varepsilon)L_4M_2)\lambda,$$

and

$$d_{h,N}(t) \equiv \max\left\{ \frac{g_{h,N}(t)}{1 - (|q|_C + \varepsilon)L_4M_2 - (|q|_C + \varepsilon)e^{-\lambda r_0/4}},\ e^{\lambda r_0/4}|\varphi - \varphi^N|_C \right\}.$$

Since $[s]_h \to s$ as $h \to 0+$, the Lebesgue Dominant Convergence Theorem implies that the two intergals in $g_{h,N}(t)$ go to 0 as $h \to 0+$. Hence $g_{h,N}(t) \to 0$ uniformly in $t \in [0,T]$, as $h \to 0+$, $N \to \infty$, and therefore $\max_{0 \le s \le \alpha_{h,N}} |x(s) - y_{h,N}(s)| \to 0$ as $h \to 0+$, $N \to \infty$. Consequently $\alpha_{h,N} = T$ for small enough h and large enough N, and the statement of the theorem follows. ■

It is easy to obtain the following result for the rate of convergence in (4.14) from the definition of $g_{h,N}$ and (4.26):

Corollary 4.3 *Assume that the conditions of Theorem 4.2 are satisfied, and in addition, f and σ are locally Lipschitz-continuous in their first arguments. Then there exists a constant $K \geq 0$ such that*

$$|x(t) - y_{h,N}(t)| \leq K(|\varphi - \varphi^N|_C + |q - q^N|_C + |\tau - \tau^N|_C + h),$$

for $t \in [-r, T]$ and for small enough $h > 0$.

In the remaining part of this section we study continuous dependence of the solution of IVP (4.6)-(4.7) on the parameters φ^N, q^N and τ^N. To simplify the notation, we omit the upper index, N, and denote the solution of IVP (4.6)-(4.7) corresponding to $\gamma \equiv (\varphi, q, \tau) \in \Gamma$ and $\bar{\gamma} \equiv (\bar{\varphi}, \bar{q}, \bar{\tau}) \in \Gamma$ by $y_{h,\gamma}(t)$ and $y_{h,\bar{\gamma}}(t)$, respectively. Define $-r_\tau$ and $-r_{\bar{\tau}}$ by (4.2) for $\tau^N = \tau$ and $\tau^N = \bar{\tau}$, respectively. Since in our examples $\bar{\gamma} = (\bar{\varphi}, \bar{q}, \bar{\tau}) = (\varphi^N, q^N, \tau^N)$ are obtained by linear spline approximation from (φ, q, τ), we can assume that $\bar{\varphi}$ and $\bar{\tau}$ are Lipschitz-continuous functions with Lipschitz-constants equal to those of φ and τ, i.e., $\bar{\varphi}$ and $\bar{\tau}$ satisfy (H4) and (H3) (ii), respectively. We can assume that $\bar{\tau} = \tau^N$ satisfies (4.1) as well. Then, for small enough h, the solution, $y_{h,\gamma}(t)$, depends continuously on γ.

Theorem 4.4 *Assume that f and σ satisfy (H1) and (H3), respectively. Let $0 < h \leq r_0/6$, and $\bar{\gamma} = (\bar{\varphi}, \bar{q}, \bar{\tau}) \in \Gamma$ be such that $\bar{\tau}$ and $\bar{\varphi}$ satisfy (4.1), (H3) (ii) and (H4), respectively. Then*

$$\sup_{-r \leq t \leq T} |y_{h,\gamma}(t) - y_{h,\bar{\gamma}}(t)| \to 0, \qquad \text{as } \gamma \to \bar{\gamma}, \quad \gamma \in \Gamma. \tag{4.27}$$

Proof The proof is similar to that of Theorem 4.2, therefore we show only the main steps, and leave the details to the reader.

Fix $\varepsilon > 0$, and let $M_1^* \equiv \sup\{|y_{h,\gamma}(t)| \, : \, t \in [0, T]\} + \varepsilon$. We assume that $\gamma = (\varphi, q, \tau)$ is such that $|\varphi - \bar{\varphi}|_C < \varepsilon$ and $|\tau - \bar{\tau}|_C < r_0/4$. Then for such $\gamma \in \Gamma$, let $0 < \alpha_{h,\gamma} \leq T$ be the largest number such that $|y_{h,\gamma}(t)| < M_1^*$ for $t \in [0, \alpha_{h,\gamma})$. Applying (4.8) for γ and $\bar{\gamma}$, and taking the difference of the two equations, we get

$$\begin{aligned}
&|y_{h,\gamma}(t) - y_{h,\bar{\gamma}}(t)| \\
&\quad \leq |\varphi(0) - \bar{\varphi}(0)| + |q(0)\varphi(-\tau(0, \varphi(-h))) - \bar{q}(0)\bar{\varphi}(-\bar{\tau}(0, \bar{\varphi}(-h)))| \\
&\qquad + \Big|\vartheta_h\Big\{q(t)y_{h,\gamma}(t - \tau(t, y_{h,\gamma}(t-h)))\Big\} - \vartheta_h\Big\{\bar{q}(t)y_{h,\bar{\gamma}}(t - \bar{\tau}(t, y_{h,\bar{\gamma}}(t-h)))\Big\}\Big| \\
&\qquad + \int_0^t \Big| f\Big([s]_h, y_{h,\gamma}([s]_h), y_{h,\gamma}([s]_h - \sigma([s]_h, y_{h,\gamma}([s]_h)))\Big) \\
&\qquad\qquad - f\Big([s]_h, y_{h,\bar{\gamma}}([s]_h), y_{h,\bar{\gamma}}([s]_h - \sigma([s]_h, y_{h,\bar{\gamma}}([s]_h)))\Big)\Big| \, ds.
\end{aligned} \tag{4.28}$$

Linearity of ϑ_h, assumption (H1) with $L_1 = L_1(M_1^*)$, and (4.28) imply for $t \in [0, \alpha_{h,\gamma}]$:

$$|y_{h,\gamma}(t) - y_{h,\bar{\gamma}}(t)|$$

$$
\begin{aligned}
\leq\; & |\varphi-\bar\varphi|_C + |q(0)\varphi(-\tau(0,\varphi(-h))) - \bar q(0)\bar\varphi(-\bar\tau(0,\bar\varphi(-h)))| \\
& + \Big|\vartheta_h\Big\{(q(t)-\bar q(t))y_{h,\gamma}(t-\tau(t,y_{h,\gamma}(t-h)))\Big\}\Big| \\
& + \Big|\vartheta_h\Big\{\bar q(t)\Big(y_{h,\gamma}(t-\tau(t,y_{h,\gamma}(t-h))) - y_{h,\bar\gamma}(t-\tau(t,y_{h,\gamma}(t-h)))\Big)\Big\}\Big| \\
& + \Big|\vartheta_h\Big\{\bar q(t)\Big(y_{h,\bar\gamma}(t-\tau(t,y_{h,\gamma}(t-h))) - y_{h,\bar\gamma}(t-\bar\tau(t,y_{h,\bar\gamma}(t-h)))\Big)\Big\}\Big| \\
& + L_1\int_0^t \Big(|y_{h,\gamma}([s]_h) - y_{h,\bar\gamma}([s]_h)| \\
& \qquad + |y_{h,\gamma}([s]_h-\sigma([s]_h,y_{h,\gamma}([s]_h))) - y_{h,\bar\gamma}([s]_h-\sigma([s]_h,y_{h,\bar\gamma}([s]_h)))|\Big)ds. \quad (4.29)
\end{aligned}
$$

Introduce the notation $w_{h,\gamma}\equiv \max_{-r\leq u\leq t}|y_{h,\gamma}(u)-y_{h,\bar\gamma}(u)|$. The assumed inequalities $0<h<r_0/6$, $|\tau-\bar\tau|_C<r_0/6$ and (4.1) imply that

$$
\Big|\vartheta_h\Big\{\bar q(t)\Big(y_{h,\gamma}(t-\tau(t,y_{h,\gamma}(t-h))) - y_{h,\bar\gamma}(t-\tau(t,y_{h,\gamma}(t-h)))\Big)\Big\}\Big| \leq |\bar q|_C w_{h,\gamma}(t-r_0/6).
$$

Since $y_{h,\bar\gamma}(t)$ is equal to $\bar\varphi(t)$ for $t\in[-r,0]$, and it is piecewise linear for $t\in[0,T]$, and by our assumption, $\bar\varphi$ is Lipschitz-continuous, it follows that $M_2^*\equiv \operatorname{ess\,sup}\{|\dot y_{h,\bar\gamma}(t)| : t\in[-r,T]\}$ is finite. Let $L_4=L_4(M_1^*)$ be the Lipschitz-constant of $\bar\tau$ from (H3) (ii), then

$$
\Big|\vartheta_h\Big\{\bar q(t)\Big(y_{h,\bar\gamma}(t-\tau(t,y_{h,\gamma}(t-h))) - y_{h,\bar\gamma}(t-\bar\tau(t,y_{h,\bar\gamma}(t-h)))\Big)\Big\}\Big| \leq |\bar q|_C M_2^*(|\tau-\bar\tau|_C + L_4 w_{h,\gamma}([t]_h)).
$$

Therefore one can obtain from (4.29) the following estimates for $t\in[0,\alpha_{h,\gamma}]$:

$$
w_{h,\gamma}(t)\leq g_{h,\gamma} + |\bar q|_C L_4 M_2^* w_{h,\gamma}([t]_h) + |\bar q|_C w_{h,\gamma}(t-r_0/6) + L_1(2+L_5M_2^*)\int_0^t w_{h,\gamma}(s)\,ds, \tag{4.30}
$$

where

$$
g_{h,\gamma}\equiv |\varphi-\bar\varphi|_C + |q(0)\varphi(-\tau(0,\varphi(-h))) - \bar q(0)\bar\varphi(-\bar\tau(0,\bar\varphi(-h)))| + |q-\bar q|_C M_1^* + |\bar q|_C M_2^*|\tau-\bar\tau|_C.
$$

Clearly, $g_{h,\gamma}\to 0$ as $\gamma\to\bar\gamma$. Consider first the interval $[0,h]$. For $t\in[0,h]\cap[0,\alpha_{h,\gamma}]$, (4.30) is equivalent to

$$
w_{h,\gamma}(t)\leq g_{h,\gamma} + |\bar q|_C L_4 M_2^* w_{h,\gamma}(0) + |\bar q|_C w_{h,\gamma}(t-r_0/6) + L_1(2+L_5M_2^*)\int_0^t w_{h,\gamma}(s)\,ds. \tag{4.31}
$$

Let $\lambda>0$ be the unique positive solution of $\lambda|\bar q|_C e^{-\lambda r_0/6} + L_1(2+L_5M_2^*) = \lambda$. Then Lemma 2.5 yields that $w_{h,\gamma}(t)\leq d_\gamma^h e^{\lambda T}$ for $t\in[0,h]\cap[0,\alpha_{h,\gamma}]$, where

$$
d_\gamma^h \equiv \max\left\{\frac{g_{h,\gamma}+|\bar q|_C L_4 M_2^* w_{h,\gamma}(0)}{1-|\bar q|_C e^{-\lambda r_0/6}},\ e^{\lambda r_0/6}|\varphi-\bar\varphi|_C\right\}.
$$

Since $w_{h,\gamma}(0)=|\varphi-\bar\varphi|_C$, $d_\gamma^h\to 0$ as $\gamma\to\bar\gamma$, we have $h<\alpha_{h,\gamma}$ if γ is close enough to $\bar\gamma$, and $w_{h,\gamma}(h)\to 0$ as $\gamma\to\bar\gamma$.

We now consider the interval $[h,2h]$. For $t\in[0,2h]\cap[0,\alpha_{h,\gamma}]$, (4.30) is equivalent to

$$
w_{h,\gamma}(t)\leq g_{h,\gamma} + |\bar q|_C L_4 M_2^* w_{h,\gamma}(h) + |\bar q|_C w_{h,\gamma}(t-r_0/6) + L_1(2+L_5M_2^*)\int_0^t w_{h,\gamma}(s)\,ds. \tag{4.32}
$$

Lemma 2.5 implies that $w_{h,\gamma}(t) \le d_\gamma^{2h} e^{\lambda T}$ for $t \in [0, 2h] \cap [0, \alpha_{h,\gamma}]$, where

$$d_\gamma^{2h} \equiv \max\left\{ \frac{g_{h,\gamma} + |\bar{q}|_C L_4 M_2^* w_{h,\gamma}(h)}{1 - |\bar{q}|_C e^{-\lambda r_0/6}},\ e^{\lambda r_0/6} |\varphi - \bar{\varphi}|_C \right\}.$$

Therefore, as before, we obtain that $\alpha_{h,\gamma} > 2h$, and $w_{h,\gamma}(2h) \to 0$ as $\gamma \to \bar{\gamma}$. By extending the interval step-by-step, we get that $\alpha_{h,\gamma} = T$, and $w_{h,\gamma}(T) \to 0$ as $\gamma \to \bar{\gamma}$, which proves the theorem. ∎

We studied the single delay equation, (2.1), for simplicity of the presentation. Our results have a straightforward generalization to the multiple delay case, i.e., to NFDEs of the form

$$\frac{d}{dt}\Big(x(t) + \sum_{i=1}^{m} q_i(t) x(t - \tau_i(t, x(t))) \Big) = f\Big(t, x(t), x(t - \sigma_1(t, x(t))), \dots, x(t - \sigma_l(t, x(t))) \Big).$$

5 Numerical Examples

In this section we present some numerical examples to illustrate our identification method. The general method is the following: consider an IVP with unknown parameters. If the parameters are infinite dimensional, use linear spline approximation of the parameters. Then, for a fixed small $h > 0$, consider IVP (4.6)-(4.7), and solve the corresponding finite dimensional least-square minimization problem, $\mathcal{P}_{N,h}$ (see Step 3 in Section 3). If h is small and N is large, use the solution of $\mathcal{P}_{N,h}$ as an approximate solution of the identification method.

To solve $\mathcal{P}_{N,h}$, we used a nonlinear least square minimization code, based on a secant method with Dennis-Gay-Welsch update, combined with a trust region technique. See Section 10.3 in [5] for detailed description of this method.

Example 5.1 Consider the state-dependent NFDE

$$\begin{aligned} &\frac{d}{dt}\Big(x(t) + q(t) x(t - \tau(t, x(t))) \Big) \\ &\qquad = 0.0003 t x(t) - 0.0255 x(t - |x(t)|) + (0.5088t - 1.4895) x(t) + 2.99t, \quad t \ge 0, \\ x(t) &\qquad = t^2, \qquad t \in [-50, 0], \end{aligned}$$

where

$$\tau(t, x) = \min\Big\{ 0.5 + 0.5t + 0.01|x|,\ 50 \Big\}. \tag{5.1}$$

It is easy to check that $x(t) = t^2$ is a solution of this IVP (for $0.5 + 0.5t + 0.01t^2 \le 50$, e.g., on $[0, 49]$) corresponding to the parameter $q(t) = 0.5t^2 - t - 0.5$. In this example we identify q on $[0, 3]$.

One of the difficulties of working with state-dependent equations is that the exact initial interval, i.e., $-r = \min\{t - \tau(t, x(t)) \,:\, 0 \le t \le T\}$, depends on the actual solution. In our example $t - \tau(t, x(t)) = 0.5t - 0.5 - 0.01|x(t)|$ (for $\tau(t, x(t))| \le 50$). For the true solution, $x(t) = t^2$, we have that $t - \tau(t, x(t))$ is monotone increasing on $[0, 3]$, its minimum on this interval is -0.5, and it is positive at, e.g., $t = 1.5$. Therefore the solution uses all values of the initial function on $[-0.5, 0]$, but not the function values for $t < -0.5$. The true solution satisfies $M_2^* = \operatorname{ess\,sup}\{|\dot{x}(t)| : t \in [-0.5, 3]\} = 6$. Clearly, $L_4 = 0.01$, and the true parameter, q, satisfies $|q|_C = |q|_{C[0,3]} = 1$. Therefore, for the true parameter, $|q|_C L_4 M_2^* < 1$, and the conditions of Theorem 4.2 are satisfied.

We generate measurements, X_i, using the true solution, $x(t) = t^2$, and $t_i = 0.2i$, $i = 0, \ldots, 15$. We use N-dimensional linear splines with equidistant mesh points to approximate q on $[0, 3]$. Consider the minimization problem

$$\min J^{N,h}(q^N) = \sum_{i=0}^{15} |y_{h,N}(t_i) - X_i|^2. \tag{5.2}$$

We can use (4.13) to obtain an a priori estimate of $|q^N|_C$. Using $L_4 = 0.01$, $M_2^* = 6$, (4.13) implies that the possible parameters for which we expect convergence of the numerical method satisfy $|q|_C < 100/6$. Therefore, we can assume, e.g., that $|q^N|_C \le 16$, and solve (5.2) subject to this constraint. Table 1 and 2 contain the value of the cost function and the maximal error of the numerical solution, respectively, for different h and N. Figure 1 shows the approximate solution for $N = 3$, 5 and 7. In these runs we used the initial guess $q^N(t) = 0$. We observe good recovery of the coefficient, q.

Table 1: $J^{N,h}(\bar{q}^N)$

h	N=3	N=5	N=7	N=9	N=11	N=13
0.10000	5.0596e-03	1.9198e-03	9.8530e-04	1.7070e-04	1.6238e-05	9.2322e-05
0.01000	3.5235e-03	1.0431e-03	1.7979e-04	8.4079e-05	8.1028e-07	2.2984e-06
0.00100	3.5753e-03	1.0140e-03	1.6455e-04	7.9288e-05	2.8211e-07	5.0901e-07
0.00010	3.5818e-03	1.0117e-03	1.6349e-04	7.8894e-05	2.4407e-07	3.9908e-07

Table 2: Maximal error, i.e., $\max_{i=0,1,\ldots,N-1} |\bar{q}^N(\xi_i) - q(\xi_i)|$, $\quad \xi_i = 3i/(N-1)$

h	N=3	N=5	N=7	N=9	N=11	N=13
0.10000	1.0632e+00	5.3507e-01	6.8827e-01	1.9085e+00	1.8369e+00	1.1990e+00
0.01000	6.6414e-01	1.3126e-01	1.5645e-01	3.5141e-01	2.2176e-01	1.7703e-01
0.00100	6.2379e-01	1.8571e-01	6.6345e-02	1.5572e-01	8.2484e-02	6.8315e-02
0.00010	6.1638e-01	1.9258e-01	4.6795e-02	1.4026e-01	6.9176e-02	8.2794e-02

Example 5.2 Consider again the previous equation

$$\begin{aligned}
&\frac{d}{dt}\Big(x(t) + (0.5t^2 - t - 0.5)x(t - \tau(t, x(t)))\Big) \\
&\qquad = \quad 0.0003tx(t) - 0.0255x(t - |x(t)|) + (0.5088t - 1.4895)x(t) + 2.99t, \quad t \ge 0, \\
&\quad x(t) \quad = \quad \varphi(t), \qquad t \in [-r, 0],
\end{aligned}$$

where τ is defined by (5.1). We have that $x(t) = t^2$ is the solution of this IVP (for $\tau(t, x(t)) \le 50$) corresponding to the initial function $\varphi(t) = t^2$. We generate measurements at $t_i = 0.05i$, $i = 0, \ldots, 30$ using the function $x(t) = t^2$.

The knowledge of the exact initial interval is more essential in this example than in the previous one. It follows from (5.1) that the "true" initial interval is a subset of $[-50, 0]$. But one does not want to define the approximate initial functions, φ^N, on a larger interval than necessary, because that would introduce unnecessary variables to the minimization problem. Moreover, the solution, and therefore the cost function, $J^{N,h}$ would be independent of those variables. One possible approach is the following: define $-r^* \equiv \min\{t_i - \tau(t_i, X_i)\ :\ i = 0, \ldots, 30\}$, and make the assumption that $r = r^*$. (Which is, in fact, true for our solution, $x(t) = t^2$.) Since $\max\{t_i - \tau(t_i, X_i)\ :\ i = 0, \ldots, 30\} > 0$, we can see that all values of the initial function between $t = -0.5$ and $t = 0$ are used in the equation. If, during the run, initial function values for $t < -r^*$ are requested, then one could restart the process by selecting a value $-r < -r^*$, or, if possible, taking more measurements, and recomputing $-r^*$ to get a better guess of $-r$.

Our goal is then to identify φ on the interval $[-0.5, 0]$. Let $\bar{\tau}^N$ be the minimizer of

$$\min J^{N,h}(\tau^N) = \sum_{i=0}^{30} |y_{h,N}(t_i) - X_i|^2,$$

where we take the minimum over the N-dimensional linear spline functions with equidistant mesh points in $[-0.5, 0]$. One can obtain a priori estimates for $\operatorname{ess\,sup}|\dot{\varphi}(t)|$ and then for $|\varphi|_C$ from (4.13), and use them as a constraint in this minimization, but here, for simplicity, we solved the unconstrained minimization problem.

Table 3 and 4 show the value of the cost function and the maximal error of the approximate solution for several h and N. Figure 2 shows the approximate solution for $N = 3$, 5 and 7. We used the initial guess $\tau^N(t) = 0.5$.

Trying to identify φ on an interval larger than $[-0.5, 0]$ introduces unnecessary parameters to the equation. On the other hand, our experience with our minimization routine is that the solution remains equal to the initial guess between those mesh points which do not belong to the "true" initial interval. That is, if $-r^N$ denotes the first mesh point, where the approximate solution is not equal to the initial guess, then $-r^N \to -r$, i.e., one can recover the beginning of the "true" initial interval. See [14] for a more detailed example.

Example 5.3 Consider again the IVP of Example 5.1:

$$\begin{aligned}
\frac{d}{dt}\Big(x(t) + (0.5t^2 - t - 0.5)x(t - \tau(t, x(t)))\Big) & \\
= \;& 0.0003tx(t) - 0.0255x(t - |x(t)|) + (0.5088t - 1.4895)x(t) + 2.99t, \quad t \geq 0, \\
x(t) \;=\;& t^2, \qquad t \in [-r, 0], \\
\tau(t, x) \;=\;& \min\Big\{a + bt + c|x|,\ 50\Big\},
\end{aligned}$$

where a, b and c are parameters, satisfying $\min\{a, b, c\} \geq 0.001$. The solution of the IVP is $x(t) = t^2$ corresponding to parameter values $a = 0.5$, $b = 0.5$ and $c = 0.01$. In this example we identify these parameters. We use the measurements of Example 5.2, and the initial parameter values $a = 0.25$, $b = 0.25$ and $c = 0.25$. The numerical results are presented in Table 5.

The main problem in identifying $\tau(t, x)$ is that there is no hope to recover τ as a function of x using only one set of measurements, since the approximation uses values of τ along the

Table 3: $J^{N,h}(\bar{\varphi}^N)$

h	N=3	N=5	N=7	N=9	N=11	N=13
0.10000	6.8331e-03	1.1835e-03	7.3757e-04	7.7210e-04	7.1118e-04	4.3269e-01
0.01000	4.6050e-04	3.6782e-05	1.1412e-05	7.2736e-06	6.0434e-06	5.3722e-06
0.00100	4.5456e-04	2.7043e-05	5.7129e-06	1.5963e-06	9.4420e-07	4.0088e-07
0.00010	4.5985e-04	2.7181e-05	5.7585e-06	1.6632e-06	9.8119e-07	4.6465e-07

Table 4: Maximal error, i.e., $\max_{i=0,1,\ldots,N-1} |\bar{\varphi}^N(\xi_i) - \varphi(\xi_i)|, \quad \xi_i = -0.5i/(N-1)$

h	N=3	N=5	N=7	N=9	N=11	N=13
0.10000	2.2714e-01	2.6463e-01	2.6520e-01	2.6636e-01	2.6273e-01	2.3274e+01
0.01000	2.9694e-02	2.8321e-02	2.7231e-02	2.6925e-02	2.6533e-02	2.6347e-02
0.00100	1.0787e-02	4.5946e-03	3.3874e-03	2.7903e-03	2.6643e-03	2.5311e-03
0.00010	1.0274e-02	2.5566e-03	1.2419e-03	6.2784e-04	3.6297e-04	3.4763e-04

solution only. Therefore one has to assume a certain form of the dependence of τ on t and x, and identify unknown parameters of the formula (like we did in this example), or assuming, e.g., that $\tau(t,x) = \tau_1(t) + \tau_2(x)$, where τ_2 is known, identify only the time dependent part, τ_1.

Table 5:

h	$\bar{a}$	error	$\bar{b}$	error	$\bar{c}$	error	$J^h(\bar{a},\bar{b},\bar{c})$
0.1000	0.001000	0.499000	0.323625	0.176375	0.240148	0.230148	2.5368e-04
0.0100	0.201232	0.298768	0.723703	0.223703	0.001000	0.009900	2.0015e-06
0.0010	0.521671	0.021671	0.481883	0.018117	0.072947	0.062947	3.3664e-11
0.0001	0.500604	0.000604	0.499510	0.000490	0.014029	0.004029	2.5390e-11

Example 5.4 We close this paper by an example where identifyability fails. Consider

$$\begin{aligned} \frac{d}{dt}\Big(x(t) + q(t)x(t-1)\Big) &= \cos t, \qquad t \in [0,2], \\ x(t) &= 1, \qquad t \in [-1,0] \end{aligned}$$

Clearly, $x(t) = 1$ is the solution of this IVP corresponding to $q(t) = \sin(t) + c$, where c is an arbitrary constant. Therefore, in this example, the inverse problem has no unique solution. We generated measurements $X_i = 1$ for $t_i = 0.05i$, $i = 0, 1, \ldots, 40$, and considered N-dimensional linear spline approximations of q on $[0,2]$. Figure 3 contains the numerical solution of the corresponding finite dimensional minimization problems for $N = 3$, 5, 7, 9, and 11 and $h = 0.001$ (the solid line is the function $\sin t$). In all runs we used $q^N = 0$ as the initial condition. Interestingly, the numerical results are approximately a shifted sin function, i.e., a possible "true parameter", where the magnitude of the shift depends on N, h and the initial guess.

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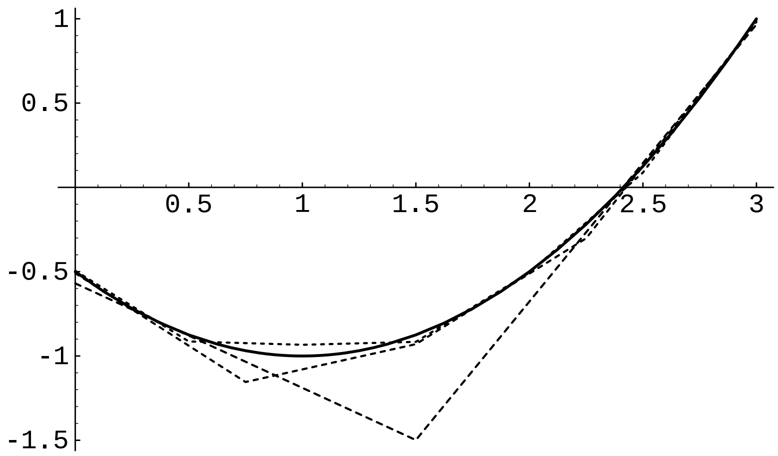

Figure 1:

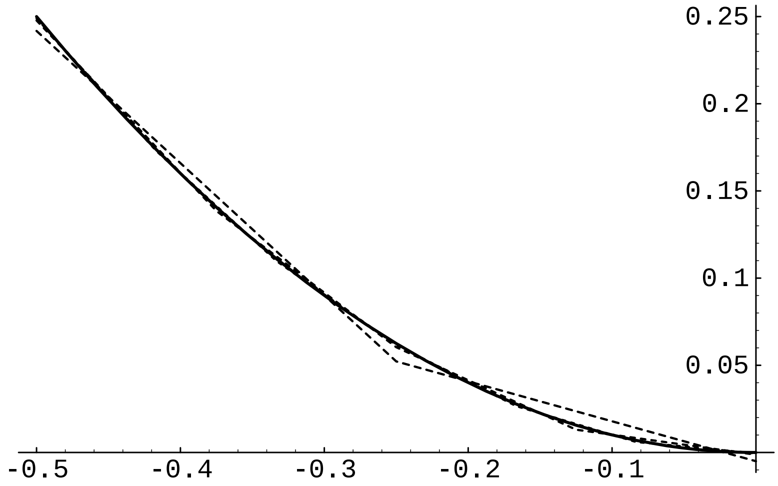

Figure 2:

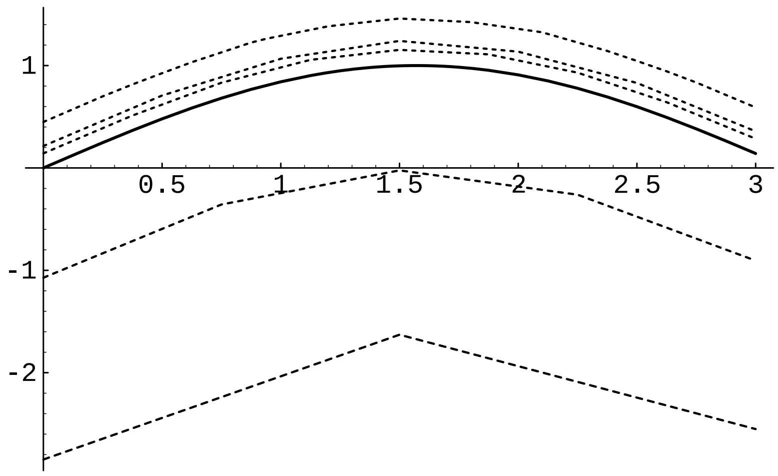

Figure 3: