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On Differentiability of Solutions with respect to Parameters in State-Dependent Delay Equations

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Abstract

In this paper we study differentiability of solutions with respect to parameters in state-dependent delay equations. In particular, we give sufficient conditions for differentiability of solutions in the $W^{1,p}$ norm $(1 \le p < \infty)$. In establishing our main results we make use of a version of the Uniform Contraction Principle for quasi-Banach spaces.

1 Introduction

In this paper we study differentiability of solutions of the state-dependent delay system

$$\dot{x}(t) = f\Big(t, x(t), x(t - \tau(t, x_t, \sigma)), \theta\Big), \qquad t \in [0, T], \tag{1.1}$$

with initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0] \tag{1.2}$$

with respect to (wrt) parameters of the equation. Here $\theta \in \Theta$ and $\sigma \in \Sigma$ represent parameters in the equation (f) and in the delay function, τ, where Θ and Σ are normed linear spaces with norms $|\cdot|_\Theta$ and $|\cdot|_\Sigma$, respectively. In this paper we restrict our attention to differentiability of solutions wrt the parameters φ, θ and σ. The notation x_t denotes the solution segment function, i.e., $x_t : [-r, 0] \to \mathbb{R}^n$, $x_t(s) \equiv x(t + s)$. (See Section 4 below for the detailed assumptions on the initial value problem (IVP) (1.1)-(1.2).)

Differentiability results wrt parameters, beside the obvious theoretical importance, have a natural application in the problem of identification of unknown parameters of the equation (such as the initial function, some coefficients in the equation, or for a constant delay equation, the delay itself). In this direction it is important to know if the solution is differentiable wrt the parameters in some sense, since many identification methods require the use of optimization techniques, in which the knowledge of the derivative of the solution wrt the parameter is essential.

Clearly, to be able to prove differentiability of the solution, we need to have some kind of smoothness of the delay term, $x(t-\tau(t,x_t,\sigma))$, of the equation wrt x_t and σ. More precisely, we need to discuss the differentiability of the function $\Lambda(t,\psi,\sigma)\equiv\psi(-\tau(t,\psi,\sigma))$ wrt ψ and σ, where ψ represents a function $[-r,0]\to\mathbb{R}^n$. The main question here is the selection of the space (i.e., the norm) for ψ (i.e., the state-space of solutions) in which $\Lambda(t,\psi,\sigma)$ is differentiable wrt ψ. Since $\Lambda(t,\psi,\sigma)=\lambda(t,\psi,\sigma,\psi)$, where $\lambda(t,\psi,\sigma,\xi)\equiv\xi(-\tau(t,\psi,\sigma))$, we need to assume differentiability of $\lambda(t,\psi,\sigma,\xi)$ wrt ψ, σ and ξ in some sense. The latter is relatively easy, since $\lambda(t,\psi,\sigma,\xi)$ is linear in ξ, therefore it is differentiable wrt ξ (in any norm) with derivative $\frac{\partial\lambda}{\partial\xi}(t,\psi,\sigma,\xi)h=\lambda(t,\psi,\sigma,h)$. It is easy to see that in order to have continuous differentiability of λ wrt ξ, we need to consider, e.g., the space $W^{1,\infty}$ (see Section 2 for definition), since the inequality

$$|\lambda(t,\psi,\sigma,h)-\lambda(t,\bar\psi,\bar\sigma,h)|\le L_2|h|_{W^{1,\infty}}\Big(|\psi-\bar\psi|_C+|\sigma-\bar\sigma|_\Sigma\Big),$$

(provided by Lemma 4.1 below), guarantees the continuous differentiability of $\lambda(t,\psi,\sigma,\xi)$ wrt ξ for $\xi\in W^{1,\infty}$. This suggests the use of $W^{1,\infty}$ for the state-space of solutions. It looks as a natural choice, since the solutions of IVP (1.1)-(1.2) are $W^{1,\infty}$ functions (see, e.g., [5] or [7]). The difficulty with $W^{1,\infty}$ is that for $\psi,\xi\in W^{1,\infty}$, the function $\lambda(t,\psi,\sigma,\xi)$ is a composition of ξ and ψ, and therefore we need to guarantee differentiability, or preferably, continuous differentiability of the composition of $W^{1,\infty}$ functions, which is, in general, impossible. But in the case when the two functions are C^1 functions, differentiability follows immediately from the Chain Rule, assuming that $\tau(t,\psi,\sigma)$ is continuously differentiable wrt ψ and σ. We refer to [5], where, under restrictive conditions, differentiability of solutions wrt parameters was obtained in the $W^{1,\infty}$ norm.

Since in $W^{1,\infty}$ the assumption for differentiability is too strong, we will explore different spaces for the more general case, i.e., when the solution, (and the initial function) is a $W^{1,\infty}$ function only.

Hale and Ladeira [4] investigated differentiability of solutions of the constant delay equation

$$\dot x(t)=f(x(t),x(t-\tau))$$

wrt the delay, τ. They have shown, using an extension of the Uniform Contraction Principle to quasi-Banach spaces (see Theorem 3.1 below, and see Section 3 below for the definition of quasi-Banach spaces), that the map

$$[0,r]\to W^{1,1}([-r,\alpha];\mathbb{R}^n),\quad \tau\mapsto x(\cdot;\tau)$$

is differentiable. This result suggests that $W^{1,p}$ (more precisely, the set $W^{1,\infty}$ equipped with the norm $|\cdot|_{W^{1,p}}$) could possibly be used as the state-space for solutions. It might be a reasonable choice, since (see e.g. [5]), the map $(\varphi,\theta,\sigma)\mapsto x(\cdot;\varphi,\theta,\sigma)_t$ is Lipschitz-continuous in both the $|\cdot|_{W^{1,\infty}}$ and $|\cdot|_{W^{1,p}}$ norms, but the map $t\mapsto x(\cdot;\varphi,\theta,\sigma)_t$ is continuous only in the $|\cdot|_{W^{1,p}}$ norm, not in the $|\cdot|_{W^{1,\infty}}$ norm. This indicates that the set $W^{1,\infty}$ equipped with the $|\cdot|_{W^{1,p}}$ norm (which is not a Banach-space, it is only a quasi-Banach space) could be considered as a "natural" state-space for state-dependent delay equations. The method used in [4] is the following: transform the IVP into an equivalent integral equation, introduce the new variable $y(t)=x(t)-\tilde\varphi(t)$, and then reformulate the problem as to find the fixed point of an operator, and obtain differentiability of the fixed point wrt parameters. We will follow the same procedure. The transformed integral equation in our case will be (4.1), and the operator $S(y,\varphi,\theta,\sigma)$ will be defined by (4.3). If we use the $|\cdot|_{W^{1,p}}$ norm for y, then we need continuous differentiability of $S(y,\varphi,\theta,\sigma)$ wrt y, φ, θ and σ in the $W^{1,p}$ norm. It turns out that instead of the pointwise differentiability of $\Lambda(t,\psi,\sigma)$ wrt ψ and σ it is enough to have the differentiability of the composite function $t\mapsto\Lambda(t,x_t,\sigma)$ wrt x and σ in "an L^p-type of norm", where $x\in W^{1,\infty}([-r,\alpha];\mathbb{R}^n)$.

Brokate and Colonius [1] studied linearization of the equation

$$\dot x(t)=f\Big(t,x(t-\tau(t,x(t)))\Big),\qquad t\in[0,\alpha].$$

In particular, they investigated differentiability of the composition operator

$$A\ :\ \Big(\bar X\subset W^{1,\infty}_\alpha\Big)\to L^p([0,\alpha];\mathbb{R}^n),\quad A(x)(t)\equiv x(t-\tau(t,x(t))),$$

where $W^{1,\infty}_\alpha \equiv W^{1,\infty}([-r,\alpha];\mathbb{R}^n)$. It was assumed that $\tau(t,v)$ is twice continuously differentiable satisfying $-r \le t-\tau(t,v) \le \alpha$ for all $t\in[0,\alpha]$ and $v\in\mathbb{R}^n$, and

$$\bar X \equiv \Big\{ x\in W^{1,\infty}_\alpha \;:\; \text{there exists } \varepsilon>0 \text{ s.t. } \frac{d}{dt}\Big(t-\tau(t,x(t))\Big)\ge\varepsilon \;\text{ a.e. } t\in[0,\alpha]\Big\}.$$

It was shown in [1], that under these assumptions, A is continuously (Frechét-)differentiable on its domain with derivative

$$(A'(x)h)(t) = h(t-\tau(t,x(t))) + \dot x(t-\tau(t,x(t)))\frac{\partial\tau}{\partial x}(t,x(t))h(t). \tag{1.3}$$

The key assumption of obtaining the results in [1], and which was suggested in [7] as well, is the choice of the domain, $\bar X$.

To obtain continuous differentiability of the operator $S(y,\varphi,\theta,\sigma)$ in $W^{1,p}$ we need continuous differentiability of the composition map $(x,\sigma)\mapsto\Lambda(\cdot,x_\cdot,\sigma)$ wrt x and σ, but using the $|\cdot|_{W^{1,p}_\alpha}$ norm on the space of x. It turns out that the right choice for our purposes is "in between the $|\cdot|_{W^{1,\infty}_\alpha}$ norm and the $|\cdot|_{W^{1,p}_\alpha}$ norm". We will introduce a "product norm" in Section 3. Let $x\in W^{1,\infty}_\alpha$ (since all solutions are $W^{1,\infty}_\alpha$ functions, this should be the space of the solutions), and decompose x as $x=y+\tilde\varphi$, (where $\varphi(t)=x(t)$ for $t\in[-r,0]$, and $\tilde\varphi$ is the extension of φ to $[-r,\alpha]$ by $\tilde\varphi(t)=\varphi(0)$), and define the norm of x by

$$|x|_{\mathbb{X}^p_\alpha} \equiv \left(\int_0^\alpha |\dot y(u)|^p\,du\right)^{1/p} + |\varphi|_{W^{1,\infty}},$$

and consider the normed linear space $\mathbb{X}^p_\alpha \equiv (W^{1,\infty}_\alpha, |\cdot|_{\mathbb{X}^p_\alpha})$. The norm $|\cdot|_{\mathbb{X}^p_\alpha}$ is weaker than the $|\cdot|_{W^{1,\infty}_\alpha}$ norm, but stronger than the $|\cdot|_{W^{1,p}_\alpha}$ norm (see Lemma 3.8 below). But it is still "strong enough" that the methods of [1], with minor modifications, provide differentiability of the composition map

$$B_\Lambda \;:\; \Big(A_1\times A_2\subset\mathbb{X}^p_\alpha\times\Sigma\Big)\to L^p([0,\alpha];\mathbb{R}^n), \quad B_\Lambda(x,\sigma)(t)\equiv\Lambda(t,x_t,\sigma).$$

(See Section 5 below.) On the other hand, $|\cdot|_{\mathbb{X}^p_\alpha}$ is "weak enough" that using the differentiability of the operator B_Λ above, we can obtain obtain differentiability of the operator $S(y,\varphi,\theta,\sigma) \;:\; (B_1\times B_2\times B_3\times B_4\subset\mathbb{X}^p_\alpha\times W^{1,\infty}\times\Theta\times\Sigma)\to\mathbb{X}^p_\alpha$ wrt y, φ, θ and σ (see Lemma 6.1 below), and be able to use a variation of the Uniform Contraction Principle (see Theorem 3.5 below) to get differentiability of the fixed point (the solution of the IVP) wrt the parameters φ, θ and σ in the $|\cdot|_{\mathbb{X}^p_\alpha}$ norm (see Theorem 6.2 below). Since this product norm is stronger than the $|\cdot|_{W^{1,p}_\alpha}$ norm, the result implies the differentiability of solutions in the latter norm as well (see Corollary 6.3 below).

We close this section by noting that differentiability of solutions of delay equations of the form

$$\dot x(t) = f(t,x_t)$$

wrt parameters has been studied, e.g., in [3], where it was shown differentiability of solutions wrt initial function and f, using C as the state-space of the solution, and the Uniform Contraction Principle. Differentiability of solutions of state-dependent delay equations wrt parameters (to the best knowledge of the authors) has not been studied in the literature yet.

2 Notations, preliminaries

Throughout this paper a norm on $\mathbb{R}^n$ and the corresponding matrix norm on $\mathbb{R}^{n\times n}$ are denoted by $|\cdot|$ and $\|\cdot\|$, respectively. (The constant n is fixed throughout this paper.)

The notation $f \;:\; \Big(A\subset X\Big)\to Y$ will be used to denote that the function maps the subset A of the normed linear space X to Y. This notation emphasizes that the topology on A is defined by the norm of X.

We denote the open ball around a point x_0 with radius R in a normed linear space $(X, |\cdot|_X)$ by $\mathcal{G}_X(x_0;\, R)$, i.e., $\mathcal{G}_X(x_0;\, R) \equiv \{x \in X \,:\, |x - x_0|_X < R\}$, and the corresponding closed ball by $\overline{\mathcal{G}}_X(x_0;\, R)$. If the ball is centered at the origin, we use simply $\mathcal{G}_X(R)$ and $\overline{\mathcal{G}}_X(R)$, respectively.

$W^{1,p}([a,b];\, \mathbb{R}^n)$, $(1 \le p \le \infty)$ denote spaces of absolutely continuous functions $\psi \,:\, [a,b] \to \mathbb{R}^n$ of finite norm

$$|\psi|_{W^{1,p}([a,b];\,\mathbb{R}^n)} \equiv \left(\int_a^b |\psi(s)|^p + |\dot\psi(s)|^p \, ds \right)^{1/p}, \qquad 1 \le p < \infty,$$

and

$$|\psi|_{W^{1,\infty}([a,b];\,\mathbb{R}^n)} \equiv \max\left\{ \sup_{a \le s \le b} |\psi(s)|,\, \operatorname*{ess\,sup}_{a \le s \le b} |\dot\psi(s)| \right\}, \qquad p = \infty,$$

respectively.

The constant $r > 0$ is fixed throughout this paper. We will mainly work with functions defined on $[-r, 0]$ or $[-r, \alpha]$. To keep the notation simple, the function spaces $C([-r,0];\, \mathbb{R}^n)$, $L^p([-r,0];\, \mathbb{R}^n)$, $W^{1,p}([-r,0];\, \mathbb{R}^n)$ and the corresponding norms will be denoted by C, L^p, $W^{1,p}$ and $|\cdot|_C$, $|\cdot|_{L^p}$ and $|\cdot|_{W^{1,p}}$, respectively. Similarly, the spaces $C([-r,\alpha];\, \mathbb{R}^n)$, $L^p([-r,\alpha];\, \mathbb{R}^n)$, $W^{1,p}([-r,\alpha];\, \mathbb{R}^n)$ and the corresponding norms will be denoted by C_α, L^p_α, $W^{1,p}_\alpha$ and $|\cdot|_{C_\alpha}$, $|\cdot|_{L^p_\alpha}$ and $|\cdot|_{W^{1,p}_\alpha}$, respectively. We will use $L^p_{0,\alpha}$ and $|\cdot|_{L^p_{0,\alpha}}$ to denote the space $L^p([0,\alpha];\, \mathbb{R}^n)$ and the norm on it.

Finally, we recall a result for later reference concerning differentiability of functions. Note that in this paper all the derivatives we use are Frechét-derivatives.

Lemma 2.1 (see, e.g., [8]) *Suppose that X and Y are normed linear spaces, and U is an open subset of X, and $F \,:\, U \to Y$ is differentiable. Let $x, y \in U$ and $y + \nu(x-y) \in U$ for $\nu \in [0,1]$. Then*

$$|F(y) - F(x) - F'(x)(y-x)|_Y \le |x-y|_X \sup_{0<\nu<1} \|F'(y + \nu(x-y)) - F'(x)\|_{\mathcal{L}(X,Y)}.$$

3 The Uniform Contraction Principle in quasi-Banach spaces

Let Y be a linear space, and let $|\cdot|$ and $\|\cdot\|$ denote norms defined on Y. We say that $(Y, |\cdot|)$ is a quasi-Banach space with respect to the norm $\|\cdot\|$, if for all $R > 0$, $\left(\overline{\mathcal{G}}_{(Y,\|\cdot\|)}(R), |\cdot|\right)$ is a complete metric space, i.e., all the closed balls of Y at the origin corresponding to the $\|\cdot\|$ norm are complete sets in the $|\cdot|$ norm. We consider Y with the topology defined by the norm $|\cdot|$, i.e., by open, closed sets in Y we mean open, closed sets of Y in the norm $|\cdot|$. Introduce $\tilde{\mathcal{L}}(Y)$, the quasi-Banach space of linear operators $S \,:\, Y \to Y$ which are bounded in both $|\cdot|$ and $\|\cdot\|$ norms. (See [4].)

The following generalization of the Uniform Contraction Principle holds for quasi-Banach spaces:

Theorem 3.1 (see [4]) *Let Z be a normed space, and assume that $(Y, |\cdot|)$ is a quasi-Banach space with respect to the norm $\|\cdot\|$. Let $U \subset Y$ be open, and $V \subset Z$ be open, and assume that $S \,:\, \overline{U} \times V \to \overline{U}$ satisfies*

(i) S is a uniform $|\cdot|$ and $\|\cdot\|$ contraction, i.e., there exists $0 \le c < 1$ such that

$$|S(y,z) - S(\bar y, z)| \le c|y - \bar y|, \qquad \text{for } y, \bar y \in \overline{U},\ z \in V,$$

and

$$\|S(y,z) - S(\bar y, z)\| \le c\|y - \bar y\|, \qquad \text{for } y, \bar y \in \overline{U},\ z \in V.$$

(ii) For each $\rho > 0$ there exists $R > 0$ such that

$$S\Big((\overline{\mathcal{G}}_{(Y,\|\cdot\|)}(R) \cap \overline{U}) \times (\mathcal{G}_Z(\rho) \cap V)\Big) \subset (\overline{\mathcal{G}}_{(Y,\|\cdot\|)}(R) \cap \overline{U}).$$

(iii) $S \in C^k(\overline{U} \times V; Y)$ for some $k \geq 1$.

Then for each $z \in V$, there exists a unique fixed point $g(z)$ of $S(\cdot, z)$ in $\overline{U}$, and the map g is in $C^k(V; Y)$.

The following notion of the (Frechét-)derivative wrt to a set in a linear space which is equipped with two norms (e.g., a quasi-Banach space) will be crucial for our future purposes. Let X_1 be a linear space, and assume that $|\cdot|_{X_1}$ and $\|\cdot\|$ are two norms on X_1, and let X_2 be a normed linear space. We consider the normed linear space X_1 as the space $(X_1, |\cdot|_{X_1})$, i.e., with the topology generated by the $|\cdot|_{X_1}$ norm, and denote the normed linear space of bounded linear operators from X_1 to X_2 with the norm $\|A\|_{\mathcal{L}(X_1,X_2)} \equiv \sup\{|Ax|_{X_2} : |x|_{X_1} \leq 1\}$ by $\mathcal{L}(X_1, X_2)$. We define differentiability of a map over a set which is not open in the $|\cdot|_{X_1}$ norm, but open in the $\|\cdot\|$ norm.

Definition 3.2 *Let U be an $\|\cdot\|$-open subset of X_1, and $F : \Big(U \subset X_1\Big) \to X_2$. We say that F is differentiable with respect to the set U, if for every $x \in U$ there exists $A \in \mathcal{L}(X_1, X_2)$, such that*

$$\lim_{\substack{|h|\to 0,\\ x+h\in U}} \frac{|F(x+h) - F(x) - Ah|_{X_2}}{|h|_{X_1}} = 0. \tag{3.1}$$

The map A is uniquely determined, called the derivative of F at x, and denoted by $F'(x)$. If, moreover, the map $F' : \Big(U \subset X_1\Big) \to \mathcal{L}(X_1, X_2)$ is continuous, then we say that F is continuously differentiable wrt U.

In (3.1) the limit is computed for h such that $x + h \in U$, or equivalently, for h such that $h \in U - x \equiv \{u - x : u \in U\}$. The uniqueness of A in (3.1) follows from the assumption that U is $\|\cdot\|$-open, and therefore there exists $\beta > 0$ such that $h \in U - x$ for $\|h\| < \beta$. Let $x \in U$ be fixed, and suppose there exist $A, \tilde{A} \in \mathcal{L}(X_1, X_2)$ both satisfying (3.1). It is easy to see that (3.1) yields that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|(A - \tilde{A})h|_{X_2} \leq \varepsilon |h|_{X_1}, \qquad \text{for } |h|_{X_1} < \delta, \quad h \in U - x. \tag{3.2}$$

Let $h^* \in X_1$ be such that $|h^*|_{X_1} < \delta$ and $h^* \notin U - x$. Then there exists $\nu \in (0, 1)$ such that $\|\nu h^*\| < \beta$, therefore $\nu h^* \in U - x$. Hence (3.2) yields $|(A - \tilde{A})\nu h|_{X_2} \leq \varepsilon |\nu h|_{X_1}$, and therefore $|(A - \tilde{A})h|_{X_2} \leq \varepsilon |h|_{X_1}$ for all $|h|_{X_1} < \delta$. Since ε was arbitrary, we get $A = \tilde{A}$.

Let X_1 be a linear space equipped with two norms, $|\cdot|_{X_1}$ and $\|\cdot\|$, as before, and let X_2 and X_3 be normed linear spaces.

Definition 3.3 *Let U be an $\|\cdot\|$-open subset of X_1, and V be an open subset of X_2, $F : \Big(U \times V \subset X_1 \times X_2\Big) \to X_3$. We say that $F(u, v)$ is continuously differentiable wrt to u and wrt the set U, if for every $v \in V$ the function $F(\cdot, v) : \Big(U \subset X_1\Big) \to X_3$ is differentiable wrt the set U (in the sense of Definition 3.2), and the derivative, $\frac{\partial F}{\partial u} : \Big(U \times V \subset X_1 \times X_2\Big) \to \mathcal{L}(X_1, X_3)$, is continuous.*

We will use the following result in the sequel.

Lemma 3.4 *Let X_1 be a normed linear space with norm $|\cdot|_{X_1}$, and let $\|\cdot\|$ be an other norm defined on X_1. Let X_2 and X_3 be normed linear spaces. Let U be an $\|\cdot\|$-open subset of X_1, and V be an open subset of X_2. Let $F : \Big(U \times V \subset X_1 \times X_2\Big) \to X_3$, be continuously differentiable wrt u and wrt the set U, and continuously differentiable wrt v on its domain. Let $(\bar{u}, \bar{v}) \in U \times V$ be fixed. Then the function*

$$\omega(\bar{u}, \bar{v}; u, v) \equiv F(u,v) - F(\bar{u}, \bar{v}) - \frac{\partial F}{\partial u}(\bar{u}, \bar{v})(u - \bar{u}) - \frac{\partial F}{\partial v}(\bar{u}, \bar{v})(v - \bar{v})$$

satisfies

$$\frac{|\omega(\bar{u}, \bar{v}; u, v)|_{X_3}}{|u - \bar{u}|_{X_1} + |v - \bar{v}|_{X_2}} \to 0, \qquad \text{as } |u - \bar{u}|_{X_1} \to 0,\ u \in U,\ \text{and } |v - \bar{v}|_{X_2} \to 0.$$

Proof The definition of ω and elementary manipulations give

$$\begin{aligned} |\omega(\bar{u}, \bar{v}; u, v)|_{X_3} \quad \leq \quad & \left| F(u,v) - F(u,\bar{v}) - \frac{\partial F}{\partial v}(u,\bar{v})(v - \bar{v}) \right|_{X_3} + \left| \left(\frac{\partial F}{\partial v}(u,\bar{v}) - \frac{\partial F}{\partial v}(\bar{u},\bar{v}) \right) (v - \bar{v}) \right|_{X_3} \\ & + \left| F(u,\bar{v}) - F(\bar{u},\bar{v}) - \frac{\partial F}{\partial u}(\bar{u},\bar{v})(u - \bar{u}) \right|_{X_3} \end{aligned}$$

Applying Lemma 2.1 to the function $F(u,\cdot) : \mathcal{G}_{X_2}(\bar{v};\, \delta) \to X_3$ (for some $\delta > 0$), we get

$$\begin{aligned} |\omega(\bar{u}, \bar{v}; u, v)|_{X_3} \quad \leq \quad & |v - \bar{v}|_{X_2} \sup_{0<\nu<1} \left\| \frac{\partial F}{\partial v}(u, \bar{v} + \nu(v - \bar{v})) - \frac{\partial F}{\partial v}(u,\bar{v}) \right\|_{\mathcal{L}(X_2,X_3)} \\ & + \left\| \frac{\partial F}{\partial v}(u,\bar{v}) - \frac{\partial F}{\partial v}(\bar{u},\bar{v}) \right\|_{\mathcal{L}(X_2,X_3)} |v - \bar{v}|_{X_2} \\ & + \left| F(u,\bar{v}) - F(\bar{u},\bar{v}) - \frac{\partial F}{\partial u}(\bar{u},\bar{v})(u - \bar{u}) \right|_{X_3}, \end{aligned}$$

which, using the continuity of $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ on $U \times V$, proves the lemma. □

Since Theorem 3.1 is not applicable to the class of equations considered here, we state the following result (a weaker version of Theorem 3.1), and introduce some new spaces essential for our future purposes.

Theorem 3.5 *Let Z be a normed space, and $(Y, |\cdot|)$ be a quasi-Banach space wrt the norm $\|\cdot\|$. Let U be an $\|\cdot\|$-open subset of Y, W be a $(|\cdot|$-$)$closed subset of U, and V be an open subset of Z, and assume that $S : U \times V \to Y$ satisfies the following conditions:*

(i) $S(W \times V) \subset W$,

(ii) *S is a uniform $|\cdot|$ and $\|\cdot\|$ contraction on $W \times V$, i.e., there exists $0 \leq c < 1$ such that*

$$|S(y,z) - S(\bar{y},z)| \leq c|y - \bar{y}|, \qquad \text{for } y, \bar{y} \in W,\ z \in V,$$

and

$$\|S(y,z) - S(\bar{y},z)\| \leq c\|y - \bar{y}\|, \qquad \text{for } y, \bar{y} \in W,\ z \in V.$$

(iii) *For each $\rho > 0$ there exists $R > 0$ such that*

$$S\Big((\overline{\mathcal{G}}_{(Y,\|\cdot\|)}(R) \cap W) \times (\mathcal{G}_Z(\rho) \cap V) \Big) \subset (\overline{\mathcal{G}}_{(Y,\|\cdot\|)}(R) \cap W).$$

(iv) *For all $y \in W$ the function $S(y,\cdot) : \Big(V \subset Z\Big) \to Y$ is continuous.*

Then for each $z \in V$, there exists a unique fixed point $g(z)$ of $S(\cdot, z)$ in W, which depends continuously on z. Moreover, if in addition

(v) S is continuously differentiable wrt y and z on $U \times V$ in the following sense:

(a) for each $z \in V$, the function $S(\cdot, z) : \Big(U \subset (Y, |\cdot|)\Big) \to Y$ is differentiable wrt U in the sense of Definition 3.2,

(b) for each $y \in U$, the function $S(y, \cdot) : \Big(V \subset Z\Big) \to Y$ is differentiable, and

(c) the partial derivatives $\frac{\partial S}{\partial y} : \Big(U \times V \subset (Y, |\cdot|) \times Z\Big) \to \mathcal{L}(Y, Y)$ and $\frac{\partial S}{\partial z} : \Big(U \times V \subset (Y, |\cdot|) \times Z\Big) \to \mathcal{L}(Z, Y)$ are continuous functions,

then the map $g : \Big(V \subset Z\Big) \to Y$ is continuously differentiable.

Proof The proof is essentially the same as that of Theorem 3.1 (see [4]), and therefore only the main steps are presented here, and we point out the difference in the respective arguments due to the fact that here differentiability is required in a weaker sense.

For a fixed $z \in V$, assumption (iii) implies that there exists an $R > 0$ such that

$$S(\cdot, z) : (\overline{\mathcal{G}}_{(Y, \|\cdot\|)}(R) \cap W) \to (\overline{\mathcal{G}}_{(Y, \|\cdot\|)}(R) \cap W),$$

and since $\overline{\mathcal{G}}_{(Y, \|\cdot\|)}(R)$ is a complete subset of Y, the existence of a unique fixed point of $S(\cdot, z)$, $g(z)$, follows from (ii). A standard argument (using (ii) and (iv)) shows that $g(\cdot) : V \to Y$ is continuous.

Assumption (ii) yields that $\left\|\frac{\partial S}{\partial y}(y, z)\right\|_{\mathcal{L}(Y, Y)} \leq c$ and $\left\|\frac{\partial S}{\partial y}(y, z)\right\|_{\mathcal{L}\left((Y, \|\cdot\|), (Y, \|\cdot\|)\right)} \leq c$ for all $(y, z) \in W \times V$, and therefore (by using a series of Lemmas in [4]), $\left(I - \frac{\partial S}{\partial y}(y, z)\right)^{-1} \in \tilde{\mathcal{L}}(Y)$ exists and is continuous in (y, z). Define

$$M(z) \equiv \left(I - \frac{\partial S}{\partial y}(g(z), z)\right)^{-1} \frac{\partial S}{\partial z}(g(z), z).$$

We will show that $g'(z) = M(z)$. Let $\gamma = \gamma(h) \equiv g(z + h) - g(z)$. Then it is easy to see that

$$\gamma = \frac{\partial S}{\partial y}(g(z), z)\gamma + \frac{\partial S}{\partial z}(g(z), z)h + \Delta,$$

where

$$\Delta \equiv S(g(z) + \gamma, z + h) - S(g(z), z) - \frac{\partial S}{\partial y}(g(z), z)\gamma - \frac{\partial S}{\partial z}(g(z), z)h.$$

Since $g(z) \in W$, $g(z) + \gamma = g(z + h) \in W$, and $W \subset U$, Lemma 3.4 implies that $|\Delta| \leq \varepsilon(|\gamma| + |h|_Z)$, for some $\varepsilon > 0$ and for sufficiently small γ and h. The remaining part of the proof is identical to that of Theorem 3.1. In particular, it is possible to obtain an estimate of the form

$$|g(z + h) - g(z) - M(z)h| < \frac{\varepsilon(1 + k)}{1 - c}|h|_Z,$$

which proves the statement. The details are omitted. $\square$

Let $\alpha > 0$. We define the space

$$\mathbb{Y}^p_\alpha \equiv \Big\{y \in W^{1,\infty}_\alpha \ : \ y(t) = 0 \text{ on } [-r, 0]\Big\},$$

with corresponding norms

$$|y|_{\mathbb{Y}^p_\alpha} \equiv \left(\int_0^\alpha |\dot y(s)|^p\,ds\right)^{1/p}, \qquad \text{for } 1 \le p < \infty,$$

and

$$|y|_{\mathbb{Y}^\infty_\alpha} \equiv \operatorname*{ess\,sup}_{s\in[0,\alpha]} |\dot y(s)|, \qquad \text{for } p = \infty,$$

respectively. Note, that $\mathbb{Y}^p_\alpha$ is the same set for all p, but it is equipped with different norms. Clearly, $\mathbb{Y}^p_\alpha$ is a normed linear space, and $\mathbb{Y}^\infty_\alpha$ is a Banach-space.

The following lemma lists some basic properties of these norms.

Lemma 3.6 *Let $y \in \mathbb{Y}^p_\alpha$, $1 \le p \le \infty$, and q be the conjugate to p, i.e., $1/p + 1/q = 1$. Then the following estimates hold:*

(i) $|y(t)| \le \alpha^{1/q}|y|_{\mathbb{Y}^p_\alpha}$, *for* $t \in [-r, \alpha]$, $1 \le p < \infty$,

(ii) $|y(t)| \le \alpha|y|_{\mathbb{Y}^\infty_\alpha}$, *for* $t \in [-r, \alpha]$,

(iii) $|y_t|_C \le \alpha^{1/q}|y|_{\mathbb{Y}^p_\alpha}$, *for* $t \in [0, \alpha]$, $1 \le p < \infty$,

(iv) $|y_t|_C \le \alpha|y|_{\mathbb{Y}^\infty_\alpha}$, *for* $t \in [0, \alpha]$,

(v) $|y|_{\mathbb{Y}^p_\alpha} \le \alpha^{1/p}|y|_{\mathbb{Y}^\infty_\alpha}$, *for* $1 \le p < \infty$,

(vi) $|y|_{\mathbb{Y}^p_\alpha} \le |y|_{W^{1,p}_\alpha} \le (\alpha^p + 1)^{1/p}|y|_{\mathbb{Y}^p_\alpha}$, *i.e.,* $|\cdot|_{\mathbb{Y}^p_\alpha}$ *is equivalent to the norm* $|\cdot|_{W^{1,p}_\alpha}$ *on* $\mathbb{Y}^p_\alpha$, *for* $1 \le p < \infty$,

(vii) $|y|_{\mathbb{Y}^\infty_\alpha} \le |y|_{W^{1,\infty}_\alpha} \le \max\{\alpha, 1\}|y|_{\mathbb{Y}^\infty_\alpha}$, *i.e.,* $|\cdot|_{\mathbb{Y}^\infty_\alpha}$ *is equivalent to the norm* $|\cdot|_{W^{1,\infty}_\alpha}$ *on* $\mathbb{Y}^\infty_\alpha$,

(viii) $|y|_{L^p_\alpha} \le \alpha|y|_{\mathbb{Y}^p_\alpha}$, *for* $1 \le p < \infty$.

For $1 \le p < \infty$, $\mathbb{Y}^p_\alpha$ is not a Banach-space, but, as the next lemma shows, it is a quasi-Banach space wrt the $|\cdot|_{\mathbb{Y}^\infty_\alpha}$ norm. We comment that Hale and Ladeira [4] applied the extension of the Uniform Contraction Theorem (Theorem 3.1) in this space (with $p = 1$) to show differentiability of solutions wrt the delay in constant delay equations. This space was also used in [6] to establish continuous dependence of solutions on parameters in a class of neutral differential equations.

Lemma 3.7 *Let $\bar y \in W^{1,\infty}_\alpha$, $\delta > 0$, $1 \le p < \infty$. Then the set $\overline{\mathcal{G}}_{\mathbb{Y}^\infty_\alpha}(\bar y; \delta)$ is a closed and complete subset of $\mathbb{Y}^p_\alpha$.*

Proof Let $y^k \in \overline{\mathcal{G}}_{\mathbb{Y}^\infty_\alpha}(\bar y; \delta)$ be a Cauchy-sequence in the $|\cdot|_{\mathbb{Y}^p_\alpha}$ norm. By Lemma 3.6 (vi) the $|\cdot|_{\mathbb{Y}^p_\alpha}$ and $|\cdot|_{W^{1,p}_\alpha}$ norms are equivalent, therefore $\{y^k\}$ is a Cauchy-sequence in $W^{1,p}_\alpha$ as well. Since $W^{1,p}_\alpha$ is a Banach-space, there exists a function $y \in W^{1,p}_\alpha$ such that $|y^k - y|_{W^{1,p}_\alpha} \to 0$ as $k \to \infty$, and therefore $|y^k - y|_{\mathbb{Y}^p_\alpha} \to 0$ as $k \to \infty$. Lemma 3.6 (i) yields that $|y^k(t) - y^l(t)| \le \alpha^{1/q}|y^k - y^l|_{\mathbb{Y}^p_\alpha} \to 0$, as $k, l \to \infty$, so $\{y^k(t)\}$ is a Cauchy-sequence in $\mathbb{R}^n$ for all $t \in [0, \alpha]$, and hence $\{y^k(t)\}$ is pointwise convergent to $y(t)$.

Suppose that $y \notin \overline{\mathcal{G}}_{\mathbb{Y}^\infty_\alpha}(\bar y; \delta)$, i.e., $\operatorname*{ess\,sup}_{0\le u\le\alpha}|\dot y(u) - \dot{\bar y}(u)| > \delta + \varepsilon$ for some $\varepsilon > 0$. Then the set $A \equiv \{u \;:\; |\dot y(u) - \dot{\bar y}(u)| > \delta + \varepsilon\}$ has positive measure. Since $\operatorname*{ess\,sup}_{0\le u\le\alpha}|\dot{y^k}(u) - \dot{\bar y}(u)| \le \delta$ for all $k \in \mathbb{N}$, and hence $\operatorname{meas}(\{u \;:\; |\dot{y^k}(u) - \dot{\bar y}(u)| > \delta\}) = 0$, we have that the set

$$B \equiv [0, \alpha] \setminus \bigcup_{k=1}^\infty \Big\{u \;:\; |\dot{y^k}(u) - \dot{\bar y}(u)| > \delta\Big\} = \Big\{u \;:\; |\dot{y^k}(u) - \dot{\bar y}(u)| \le \delta,\ k \in \mathbb{N}\Big\}$$

has measure α. Then elementary estimates imply for all k that

$$|y-y^k|_{\mathbb{Y}^p_\alpha} \geq \left(\int\limits_{A\cap B} \Big(|\dot{y}(u)-\dot{\bar{y}}(u)| - |\dot{\bar{y}}(u)-\dot{y}^k(u)|\Big)^p \, du \right)^{1/p} \geq \varepsilon\Big(\mathrm{meas}(A\cap B)\Big)^{1/p} > 0,$$

which is a contradiction. Therefore $y \in \overline{\mathcal{G}}_{\mathbb{Y}^\infty_\alpha}(\bar{y};\,\delta)$, i.e., $\overline{\mathcal{G}}_{\mathbb{Y}^\infty_\alpha}(\bar{y};\,\delta)$ is complete, and hence also closed in $\mathbb{Y}^p_\alpha$. □

Next we introduce a new norm on $W^{1,\infty}_\alpha$. We define the projection operators

$$\mathrm{Pr}_\varphi \;:\; W^{1,\infty}_\alpha \to W^{1,\infty}, \qquad (\mathrm{Pr}_\varphi\, x)(s) \equiv x(s), \quad s\in[-r,0], \tag{3.3}$$

and

$$\mathrm{Pr}_y \;:\; W^{1,\infty}_\alpha \to \mathbb{Y}^p_\alpha, \qquad (\mathrm{Pr}_y\, x)(u) \equiv \begin{cases} 0, & -r\leq u\leq 0, \\ x(u)-x(0), & 0\leq u\leq\alpha. \end{cases} \tag{3.4}$$

Conversely, if $\varphi \in W^{1,\infty}$ and $y\in\mathbb{Y}^p_\alpha$, then the function $x = y+\tilde{\varphi}$ is in $W^{1,\infty}_\alpha$, where $\tilde{\varphi}$ denotes the extension of φ to $[-r,\alpha]$ defined by

$$\tilde{\varphi}(t) \equiv \begin{cases} \varphi(t), & t\in[-r,0] \\ \varphi(0), & t\in[0,\alpha]. \end{cases} \tag{3.5}$$

We define a "product norm" on the set $W^{1,\infty}_\alpha$ for $1\leq p<\infty$ by

$$|x|_{\mathbb{X}^p_\alpha} \equiv |\mathrm{Pr}_y\, x|_{\mathbb{Y}^p_\alpha} + |\mathrm{Pr}_\varphi\, x|_{W^{1,\infty}}, \tag{3.6}$$

and denote the corresponding normed linear space by $\mathbb{X}^p_\alpha \equiv \Big(W^{1,\infty}_\alpha,\, |\cdot|_{\mathbb{X}^p_\alpha}\Big)$.

Part (i) and (ii) of the following lemma shows that this "product" norm is stronger than the $|\cdot|_{W^{1,p}_\alpha}$ norm, and weaker than the $|\cdot|_{W^{1,\infty}_\alpha}$ norm on $W^{1,\infty}_\alpha$. Estimate (iii) will be used later.

Lemma 3.8 *Let $1\leq p<\infty$. There exist positive constants c_1, c_2 and c_3 such that for all $x\in W^{1,\infty}_\alpha$*

(i) $|x|_{W^{1,p}_\alpha} \leq c_1 |x|_{\mathbb{X}^p_\alpha}$,

(ii) $|x|_{\mathbb{X}^p_\alpha} \leq c_2 |x|_{W^{1,\infty}_\alpha}$,

(iii) $|x|_{C_\alpha} \leq c_3 |x|_{\mathbb{X}^p_\alpha}$.

Proof Let $y = \mathrm{Pr}_y\, x$ and $\varphi = \mathrm{Pr}_\varphi\, x$, i.e., $x = y+\tilde{\varphi}$ be the direct sum decomposition of x. Using the inequality $(a+b)^p \leq 2^{p-1}(a^p+b^p)$ and Lemma 3.6 (i) we get

$$\begin{aligned} |x|^p_{W^{1,p}_\alpha} &= \int_{-r}^0 |\varphi(s)|^p + |\dot{\varphi}(s)|^p\, ds + \int_0^\alpha |y(u)+\varphi(0)|^p + |\dot{y}(u)|^p\, du \\ &\leq 2r|\varphi|^p_{W^{1,\infty}} + 2^{p-1}\int_0^\alpha |y(u)|^p\, du + \alpha 2^{p-1}|\varphi(0)|^p + \int_0^\alpha |\dot{y}(u)|^p\, du \\ &\leq (2^{p-1}\alpha + 2r)|\varphi|^p_{W^{1,\infty}} + 2^{p-1}\alpha^{p/q+1}|y|^p_{\mathbb{Y}^p_\alpha} + |y|^p_{\mathbb{Y}^p_\alpha} \\ &\leq (2^{p-1}\alpha + 2r + 2^{p-1}\alpha^p + 1)|x|^p_{\mathbb{X}^p_\alpha}, \end{aligned}$$

which proves the first statement of the lemma with $c_1 = (2^{p-1}\alpha + 2r + 2^{p-1}\alpha^p + 1)^{1/p}$.

To show the second inequality, consider the elementary estimates

$$|x|_{\mathbb{X}^p_\alpha} = \left(\int_0^\alpha |\dot{y}(u)|^p\, du\right)^{1/p} + |\varphi|_{W^{1,\infty}} \leq \alpha^{1/p}|\dot{y}|_{L^\infty_\alpha} + |\varphi|_{W^{1,\infty}} \leq (\alpha^{1/p}+1)|x|_{W^{1,\infty}_\alpha},$$

therefore $c_2 = (\alpha^{1/p}+1)$ in (ii).

Consider (iii). Then by Lemma 3.6 (i) we get $|x|_{C_\alpha} \leq |y|_{C_\alpha} + |\tilde{\varphi}|_{C_\alpha} \leq \alpha^{1/q}|y|_{\mathbb{Y}^p_\alpha} + |\varphi|_{W^{1,\infty}} \leq \max\{\alpha^{1/q},1\}|x|_{\mathbb{X}^p_\alpha}$, therefore (iii) is satisfied with $c_3 = \max\{\alpha^{1/q},1\}$. This completes the proof of the lemma. □

4 A Class of State-Dependent Delay Equations

In this section we consider a set of technical conditions, guaranteeing well-posedness and differentiability of solutions wrt parameters, for the state-dependent delay differential equation (1.1) with initial condition (1.2). In particular, we make the following assumptions:

Let $\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^n$, $\Omega_3 \subset \Theta$, $\Omega_4 \subset C$, and $\Omega_5 \subset \Sigma$ be open subsets of the respective spaces. $T > 0$ is finite or $T = \infty$, in which case $[0, T]$ denotes the interval $[0, \infty)$.

(A1) (i) $f : [0, T] \times \Omega_1 \times \Omega_2 \times \Omega_3 \to \mathbb{R}^n$ is continuous,

(ii) $f(t, v, w, \theta)$ is locally Lipschitz-continuous in v, w and θ in the following sense: for every $\alpha > 0$, $M_1 \subset \Omega_1$, $M_2 \subset \Omega_2$, $M_3 \subset \Omega_3$, where M_1 and M_2 are compact subsets of $\mathbb{R}^n$ and M_3 is a closed, bounded subset of Θ, there exists a constant $L_1 = L_1(\alpha, M_1, M_2, M_3)$ such that

$$|f(t, v, w, \theta) - f(t, \bar{v}, \bar{w}, \bar{\theta})| \le L_1\Big(|v - \bar{v}| + |w - \bar{w}| + |\theta - \bar{\theta}|_\Theta\Big),$$

for $t \in [0, \alpha]$, $v, \bar{v} \in M_1$, $w, \bar{w} \in M_2$, and $\theta, \bar{\theta} \in M_3$,

(iii) $f(t, v, w, \theta) : \Big([0, T] \times \Omega_1 \times \Omega_2 \times \Omega_3 \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \Theta\Big) \to \mathbb{R}^n$ is continuously differentiable wrt v, w and θ,

(A2) (i) $\tau : [0, T] \times \Omega_4 \times \Omega_5 \to [0, \infty)$ is continuous, and

$$t - \tau(t, \psi, \sigma) \ge -r, \qquad \text{for } t \in [0, T],\ \psi \in \Omega_4,\ \text{and } \sigma \in \Omega_5,$$

(ii) $\tau(t, \psi, \sigma)$ is locally Lipschitz-continuous in ψ and σ in the following sense: for every $\alpha > 0$, $M_4 \subset \Omega_4$ and $M_5 \subset \Omega_5$, where M_4 is a compact subset of C, and M_5 is a closed, bounded subset of Σ, there exists a constant $L_2 = L_2(\alpha, M_4, M_5)$ such that

$$|\tau(t, \psi, \sigma) - \tau(t, \bar{\psi}, \bar{\sigma})| \le L_2\Big(|\psi - \bar{\psi}|_C + |\sigma - \bar{\sigma}|_\Sigma\Big)$$

for $t \in [0, \alpha]$, $\psi, \bar{\psi} \in M_4$, and $\sigma, \bar{\sigma} \in M_5$,

(iii) $\tau(t, \psi, \sigma) : \Big([0, T] \times \Omega_4 \times \Omega_5 \subset [0, \alpha] \times C \times \Sigma\Big) \to \mathbb{R}$ is continuously differentiable wrt t, ψ and σ,

(iv) $\frac{\partial \tau}{\partial t}(t, \psi, \sigma)$, $\frac{\partial \tau}{\partial \psi}(t, \psi, \sigma)$ and $\frac{\partial \tau}{\partial \sigma}(t, \psi, \sigma)$ are locally Lipschitz-continuous in ψ and σ, i.e., for every $\alpha > 0$, $M_4 \subset \Omega_4$ and $M_5 \subset \Omega_5$, where M_4 is a compact subset of C, and M_5 is a closed, bounded subset of Σ, there exists $L_3 = L_3(\alpha, M_4, M_5)$ such that

$$\begin{aligned} \left|\frac{\partial \tau}{\partial t}(t, \psi, \sigma) - \frac{\partial \tau}{\partial t}(t, \bar{\psi}, \bar{\sigma})\right| &\le L_3\Big(|\psi - \bar{\psi}|_C + |\sigma - \bar{\sigma}|_\Sigma\Big), \\ \left\|\frac{\partial \tau}{\partial \psi}(t, \psi, \sigma) - \frac{\partial \tau}{\partial \psi}(t, \bar{\psi}, \bar{\sigma})\right\|_{\mathcal{L}(C, \mathbb{R})} &\le L_3\Big(|\psi - \bar{\psi}|_C + |\sigma - \bar{\sigma}|_\Sigma\Big), \end{aligned}$$

and

$$\left\|\frac{\partial \tau}{\partial \sigma}(t, \psi, \sigma) - \frac{\partial \tau}{\partial \sigma}(t, \bar{\psi}, \bar{\sigma})\right\|_{\mathcal{L}(\Sigma, \mathbb{R})} \le L_3\Big(|\psi - \bar{\psi}|_C + |\sigma - \bar{\sigma}|_\Sigma\Big)$$

hold for all $t \in [0, \alpha]$, $\psi, \bar{\psi} \in M_4$, and $\sigma, \bar{\sigma} \in M_5$,

(A3) $\varphi \in W^{1,\infty}$.

Assumptions (A1) (iii), (A2) (ii) and (iv) are equivalent to the usual local Lipschitz-continuity properties if Θ and Σ are finite dimensional spaces. On the other hand, they can also be satisfied in special cases when Θ and Σ are infinite dimensional. For example, let $\Theta = C([0, T]; \mathbb{R}^k)$, $\Omega_3 = \Theta$, and $f(t,v,w,\theta) = g(t,v,w,\theta(t))$, where $g : [0,T] \times \Omega_1 \times \Omega_2 \times \mathbb{R}^k \to \mathbb{R}^n$. Then if g is continuous, and continuously differentiable wrt its last three arguments, then (A1) is satisfied. Similarly, let, e.g., $\Sigma = C([0,T];\mathbb{R}^k)$, $\Omega_5 = \Sigma$, and $\tau(t,\psi,\sigma) = \bar{\tau}(t,\psi(0),\sigma(t))$. Then if $\bar{\tau} : [0,T] \times \mathbb{R}^n \times \mathbb{R}^k \to [0,r]$ is twice continuously differentiable wrt its arguments, then (A2) is satisfied.

For future notational convenience we introduce the functions

$$\Lambda(t,\psi,\sigma) \equiv \psi(-\tau(t,\psi,\sigma)) \quad \text{and} \quad \lambda(t,\psi,\sigma,\xi) \equiv \xi(-\tau(t,\psi,\sigma))$$

for $t \in [0,T]$, $\psi, \xi \in C$ and $\sigma \in \Sigma$. With this notation we can rewrite (1.1) shortly as

$$\dot{x}(t) = f(t, x(t), \Lambda(t, x_t, \sigma), \theta).$$

The definitions of λ and Λ, assumption (A2) (ii), and the Mean Value Theorem imply immediately the following inequalities, which we will need later.

Lemma 4.1 *Assume (A2) (ii), and let $0 < \alpha \le T$, $M_4 \subset \Omega_4$ be a compact subset of C, and $M_5 \subset \Omega_5$ be a closed, bounded subset of Σ. Let $L_2 = L_2(\alpha, M_4, M_5)$ be the corresponding constant from (A2) (ii). Then the inequalities*

$$|\lambda(t,\psi,\sigma,\xi) - \lambda(t,\bar{\psi},\bar{\sigma},\xi)| \le L_2 |\dot{\xi}|_{L^\infty} \Big(|\psi - \bar{\psi}|_C + |\sigma - \bar{\sigma}|_\Sigma \Big),$$

and

$$|\Lambda(t,\psi,\sigma) - \Lambda(t,\bar{\psi},\bar{\sigma})| \le |\psi - \bar{\psi}|_C + L_2 |\dot{\psi}|_{L^\infty} \Big(|\psi - \bar{\psi}|_C + |\sigma - \bar{\sigma}|_\Sigma \Big)$$

hold for $t \in [0,\alpha]$, $\psi \in W^{1,\infty}$, $\psi, \bar{\psi} \in M_4$, and $\sigma, \bar{\sigma} \in M_5$.

Assumptions (A1)–(A3) yield that for any $x \in C([-r,T];\mathbb{R}^n)$ the map $t \mapsto \Lambda(t, x_t, \sigma)$ is continuous, and hence so is the map $t \to f\Big(t, x(t), \Lambda(t, x_t, \sigma), \theta\Big)$. Therefore, using the new variable $y(t) \equiv x(t) - \tilde{\varphi}(t)$, IVP (1.1)-(1.2) can be transformed to an equivalent integral equation

$$y(t) = \begin{cases} 0, & t \in [-r,0] \\ \displaystyle\int_0^t f\Big(u,\, y(u) + \tilde{\varphi}(u),\, \Lambda(u, y_u + \tilde{\varphi}_u, \sigma), \theta\Big)\, du, & t \in [0,T]. \end{cases} \tag{4.1}$$

In this section we study well-posedness of (4.1) corresponding to parameters φ, θ and σ satisfying the domain conditions

$$\varphi(0) \in \Omega_1, \quad \varphi(-\tau(0,\varphi,\sigma)) \in \Omega_2, \quad \theta \in \Omega_3, \quad \varphi \in \Omega_4, \quad \text{and } \sigma \in \Omega_5. \tag{4.2}$$

We assume for the rest of this paper that $\varphi^* \in W^{1,\infty}$, $\theta^* \in \Theta$ and $\sigma^* \in \Sigma$ are fixed parameter values satisfying (4.2).

Our goal is to define an operator S by

$$S(y,\varphi,\theta,\sigma)(t) = \begin{cases} 0, & t \in [-r,0] \\ \displaystyle\int_0^t f\Big(u,\, y(u) + \tilde{\varphi}(u),\, \Lambda(u, y_u + \tilde{\varphi}_u, \sigma), \theta\Big)\, du, & t \in [0,\alpha], \end{cases} \tag{4.3}$$

and, using Theorem 3.5, obtain existence of a unique fixed point of S, i.e., of a unique solution of (4.1). The next lemma gives the precise definition of the domain, where S can be defined, and where the conditions of Theorem 3.5 are satisfied.

Lemma 4.2 *Assume (A1) (i),(ii), (A2) (i),(ii) and (A3). Let $1 \le p < \infty$ and assume that φ^*, θ^* and σ^* satisfy (4.2). Then there exist positive constants δ_1, δ_2, δ_3, α, and sets M_1, M_2, M_3, M_4, M_5, $\mathcal{U}$ and $\mathcal{W}$, such that*

(1) $M_1 \subset \Omega_1$, $M_2 \subset \Omega_2$ *are compact subsets of* $\mathbb{R}^n$, $M_3 \subset \Omega_3$ *is a closed, bounded subset of* Θ, $M_4 \subset \Omega_4$ *is a compact subset of* C, *and* $M_5 \subset \Omega_5$ *is a closed, bounded subset of* Σ,

(2) $\mathcal{U}$ *is an open subset of* $\mathbb{Y}^\infty_\alpha$, $\mathcal{W}$ *is a closed subset of* $\mathbb{Y}^p_\alpha$, *and* $\mathcal{W} \subset \mathcal{U}$,

(3) for $u \in [0,\alpha]$, $y \in \mathcal{U}$, $\varphi \in \mathcal{G}_{W^{1,\infty}}(\varphi^*; \delta_1)$, $\theta \in \mathcal{G}_\Theta(\theta^*; \delta_2)$, *and* $\sigma \in \mathcal{G}_\Sigma(\sigma^*; \delta_3)$

$$y(u) + \tilde\varphi(u) \in M_1, \quad \Lambda(u, y_u + \tilde\varphi_u, \sigma) \in M_2, \quad \theta \in M_3, \quad y_u + \tilde\varphi_u \in M_4, \quad \text{and } \sigma \in M_5 \tag{4.4}$$

hold, and

(4) the operator

$$S \;:\; \Big(\mathcal{U} \times \mathcal{G}_{W^{1,\infty}}(\varphi^*; \delta_1) \times \mathcal{G}_\Theta(\theta^*; \delta_2) \times \mathcal{G}_\Sigma(\sigma^*; \delta_3) \subset \mathbb{Y}^p_\alpha \times W^{1,\infty} \times \Theta \times \Sigma\Big) \to \mathbb{Y}^p_\alpha, \tag{4.5}$$

defined by (4.3) satisfies

(i) $S\Big(\mathcal{W} \times \mathcal{G}_{W^{1,\infty}}(\varphi^*; \delta_1) \times \mathcal{G}_\Theta(\theta^*; \delta_2) \times \mathcal{G}_\Sigma(\sigma^*; \delta_3)\Big) \subset \mathcal{W}$,

(ii) S *is a uniform contraction on* $\mathcal{W}$ *both in* $|\cdot|_{\mathbb{Y}^\infty_\alpha}$ *and* $|\cdot|_{\mathbb{Y}^p_\alpha}$ *norms, i.e., there exists* $0 \le c < 1$ *such that for all* $y, \bar y \in \mathcal{W}$, $\varphi \in \mathcal{G}_{W^{1,\infty}}(\varphi^*; \delta_1)$, $\theta \in \mathcal{G}_\Theta(\theta^*; \delta_2)$, $\sigma \in \mathcal{G}_\Sigma(\sigma^*; \delta_3)$

$$|S(y,\varphi,\theta,\sigma) - S(\bar y,\varphi,\theta,\sigma)|_{\mathbb{Y}^\infty_\alpha} \le c|y - \bar y|_{\mathbb{Y}^\infty_\alpha},$$

and

$$|S(y,\varphi,\theta,\sigma) - S(\bar y,\varphi,\theta,\sigma)|_{\mathbb{Y}^p_\alpha} \le c|y - \bar y|_{\mathbb{Y}^p_\alpha},$$

(iii) for all $y \in \mathcal{W}$ *the function* $S(y,\cdot,\cdot,\cdot) \;:\; \mathcal{G}_{W^{1,\infty}}(\varphi^*; \delta_1) \times \mathcal{G}_\Theta(\theta^*; \delta_2) \times \mathcal{G}_\Sigma(\sigma^*; \delta_3) \to \mathbb{Y}^p_\alpha$ *is continuous.*

Proof Since φ^*, θ^* and σ^* satisfy (4.2), and Ω_i $(i = 1,\dots,5)$ are open sets, there exist positive constants R_i $(i = 1,\dots,5)$ such that $M_1 \equiv \overline{\mathcal{G}}_{\mathbb{R}^n}(\varphi^*(0); R_1) \subset \Omega_1$, $M_2 \equiv \overline{\mathcal{G}}_{\mathbb{R}^n}(\varphi^*(-\tau(0,\varphi^*,\sigma^*)); R_1) \subset \Omega_2$, $M_3 \equiv \overline{\mathcal{G}}_\Theta(\theta^*; R_3) \subset \Omega_3$, $M_4^* \equiv \overline{\mathcal{G}}_C(\varphi^*; R_4) \subset \Omega_4$, and $M_5 \equiv \overline{\mathcal{G}}_\Sigma(\sigma^*; R_5) \subset \Omega_5$.

Let $0 < \bar T \le T$ be a fixed finite number, and $L_1 = L_1(\bar T, M_1, M_2, M_3)$ be the constants from (A1) (ii). Assumptions (A1) (i) and (ii), the compactness of M_1 and M_2, and the boundedness of M_3 yield

$$\begin{aligned} &\sup\{|f(u,v,w,\theta)| \;:\; u \in [0,\bar T],\, v \in M_1,\ w \in M_2,\ \theta \in M_3\} \\ &\qquad \le \ \sup\{|f(u,v,w,\theta^*)| \;:\; u \in [0,\bar T],\, v \in M_1,\ w \in M_2\} + L_1 \sup\{|\theta - \theta^*|_\Theta \;:\; \theta \in M_3\} \\ &\qquad < \ \infty, \end{aligned}$$

therefore the constant $\bar\beta \equiv \sup\{|f(u,v,w,\theta)| \;:\; u \in [0,\bar T],\, v \in M_1,\ w \in M_2,\ \theta \in M_3\}$ is finite. Let

$$\beta > \bar\beta, \quad \bar\alpha \equiv \min\left\{\bar T,\, \frac{R_1}{2\beta},\, \frac{R_4}{3\beta},\, \frac{R_4}{3(|\varphi^*|_{W^{1,\infty}} + 1)}\right\}, \quad \text{and} \quad \bar\delta_1 \equiv \min\left\{\frac{R_1}{2},\, \frac{R_4}{3}\right\}.$$

Let $u \in [0,\bar\alpha]$, $y \in \mathcal{G}_{\mathbb{Y}^\infty_{\bar\alpha}}(\beta)$ and $\varphi \in \mathcal{G}_{W^{1,\infty}}\big(\varphi^*; \bar\delta_1\big)$. Then Lemma 3.6 (ii) yields

$$|y(u) + \tilde\varphi(u) - \varphi^*(0)| \le |y(u)| + |\varphi(0) - \varphi^*(0)| \le \bar\alpha|y|_{\mathbb{Y}^\infty_{\bar\alpha}} + |\varphi - \varphi|_{W^{1,\infty}} \le \bar\alpha\beta + \bar\delta_1 \le R_1,$$

i.e., $y(u) + \tilde\varphi(u) \in M_1$. Similarly, using Lemma 3.6 (ii) and the Mean Value Theorem we get

$$\begin{aligned} |y_u + \tilde\varphi_u - \varphi^*|_C \ &\le\ |y_u|_C + |\tilde\varphi_u - \tilde{\varphi^*}_u|_C + |\tilde{\varphi^*}_u - \varphi^*|_C \\ &\le\ \bar\alpha|y|_{\mathbb{Y}^\infty_{\bar\alpha}} + |\varphi - \varphi^*|_C + u|\varphi^*|_{W^{1,\infty}} \\ &\le\ \bar\alpha\beta + \bar\delta_1 + \bar\alpha|\varphi^*|_{W^{1,\infty}} \\ &\le\ R_4, \end{aligned} \tag{4.6}$$

i.e., $y_u+\tilde\varphi_u\in M_4^*\subset\Omega_4$. Define $M_4\equiv\{y_u+\tilde\varphi_u\,:\,u\in[0,\bar\alpha],\ y\in\mathcal{G}_{\mathbb{Y}^\infty_{\bar\alpha}}(\beta),\ \varphi\in\mathcal{G}_{W^{1,\infty}}\big(\varphi^*;\,\bar\delta_1\big)\}$. Then $M_4\subset M_4^*\subset\Omega_4$, and Arsela-Ascoli's lemma implies that M_4 is a compact subset of C. Let $L_2=L_2(\bar\alpha,M_4,M_5)$ be the constant from (A2) (ii), and define

$$\delta_1\equiv\min\left\{\bar\delta_1,\ \frac{R_2}{5L_2(|\varphi^*|_{W^{1,\infty}}+1)}\right\},\quad \delta_2\equiv R_3,\quad\text{and}\quad \delta_3\equiv\min\left\{\frac{R_2}{5L_2(|\varphi^*|_{W^{1,\infty}}+1)},\ R_5\right\}.$$

Select α such that

$$\alpha\le\min\left\{\bar\alpha,\ \frac{R_2}{5\beta L_2(|\varphi^*|_{W^{1,\infty}}+1)},\ \frac{R_2}{5L_2(|\varphi^*|^2_{W^{1,\infty}}+1)}\right\},\quad \alpha L_1(2+L_2(\bar\beta+|\varphi^*|_{W^{1,\infty}}+\delta_1))<1,$$

and

$$|\varphi^*|_{W^{1,\infty}}|\tau(u,\varphi^*,\sigma^*)-\tau(0,\varphi^*,\sigma^*)|\le R_2/5\qquad\text{for }u\in[0,\alpha].$$

Define the sets $\mathcal{U}\equiv\mathcal{G}_{\mathbb{Y}^\infty_\alpha}(\beta)$ and $\mathcal{W}\equiv\overline{\mathcal{G}}_{\mathbb{Y}^\infty_\alpha}(\bar\beta)$. Then $\mathcal{W}\subset\mathcal{U}$, $\mathcal{U}$ is an open subset of $\mathbb{Y}^\infty_\alpha$, and it follows from Lemma 3.7 that $\mathcal{W}$ is a closed subset of $\mathbb{Y}^p_\alpha$, so part (2) of the lemma holds.

Let $u\in[0,\alpha]$, $y\in\mathcal{U}$, $\varphi\in\mathcal{G}_{W^{1,\infty}}(\varphi^*;\,\delta_1)$, $\sigma\in\mathcal{G}_\Sigma(\sigma^*;\,\delta_3)$. Then Lemma 4.1, $\Lambda(0,\varphi^*,\sigma^*)=\varphi^*(-\tau(0,\varphi^*,\sigma^*))$, $y_u+\tilde\varphi_u\in M_4$, and an estimate similar to (4.6) yield

$$\begin{aligned}
&|\Lambda(u,y_u+\tilde\varphi_u,\sigma)-\Lambda(0,\varphi^*,\sigma^*)|\\
&\quad\le\ |\Lambda(u,y_u+\tilde\varphi_u,\sigma)-\Lambda(u,\varphi^*,\sigma^*)|+|\Lambda(u,\varphi^*,\sigma^*)-\Lambda(0,\varphi^*,\sigma^*)|\\
&\quad\le\ L_2|\varphi^*|_{W^{1,\infty}}(|y_u+\varphi_u-\varphi^*|_C+|\sigma-\sigma^*|_\Sigma)+|\varphi^*(-\tau(u,\varphi^*,\sigma^*))-\varphi^*(-\tau(0,\varphi^*,\sigma^*))|\\
&\quad\le\ L_2|\varphi^*|_{W^{1,\infty}}(|y_u+\varphi_u-\varphi^*|_C+|\sigma-\sigma^*|_\Sigma)+|\varphi^*|_{W^{1,\infty}}|\tau(u,\varphi^*,\sigma^*)-\tau(0,\varphi^*,\sigma^*)|\\
&\quad\le\ L_2|\varphi^*|_{W^{1,\infty}}(\alpha\beta+\delta_1+\alpha|\varphi^*|_{W^{1,\infty}}+\delta_3)+|\varphi^*|_{W^{1,\infty}}|\tau(u,\varphi^*,\sigma^*)-\tau(0,\varphi^*,\sigma^*)|\\
&\quad\le\ R_2,
\end{aligned}$$

i.e., $\Lambda(u,y_u+\tilde\varphi_u,\sigma)\in M_2$. Clearly, $\theta\in M_3$ for $\theta\in\mathcal{G}_\Theta(\theta^*;\,\delta_2)$, and $\sigma\in M_5$ for $\sigma\in\mathcal{G}_\Sigma(\sigma^*;\,\delta_3)$, therefore part (1) and (3) of the lemma is proved.

(1)–(3) imply that the operator S defined by (4.5) and (4.3) is well-defined on its domain. Let $y\in\mathcal{W}$, $\varphi\in\mathcal{G}_{W^{1,\infty}}(\varphi^*;\,\delta_1)$, $\theta\in\mathcal{G}_\Theta(\theta^*;\,\delta_2)$, and $\sigma\in\mathcal{G}_\Sigma(\sigma^*;\,\delta_3)$. Then the definition of $\bar\beta$ yields that $|S(y,\varphi,\theta,\sigma)|_{\mathbb{Y}^\infty_\alpha}\le\bar\beta$, i.e. $S(y,\varphi,\theta,\sigma)\in\mathcal{W}$, which shows (4) (i). Lemma 4.1 and Lemma 3.6 (ii) and (iv) yield

$$\begin{aligned}
&|S(y,\varphi,\theta,\sigma)-S(\bar y,\varphi,\theta,\sigma)|_{\mathbb{Y}^\infty_\alpha}\\
&\quad=\ \operatorname*{ess\,sup}_{0\le u\le\alpha}\Big|f(u,y(u)+\tilde\varphi(u),\Lambda(u,y_u+\tilde\varphi_u,\sigma),\theta)-f(u,\bar y(u)+\tilde\varphi(u),\Lambda(u,\bar y_u+\tilde\varphi_u,\sigma),\theta)\Big|\\
&\quad\le\ L_1\operatorname*{ess\,sup}_{0\le u\le\alpha}\Big(|y(u)-\bar y(u)|+|\Lambda(u,y_u+\tilde\varphi_u,\sigma)-\Lambda(u,\bar y_u+\tilde\varphi_u,\sigma)|\Big)\\
&\quad\le\ L_1\operatorname*{ess\,sup}_{0\le u\le\alpha}\Big(|y(u)-\bar y(u)|+|y_u-\bar y_u|_C+L_2|\dot y_u+\dot{\tilde\varphi}_u|_{L^\infty}|y_u-\bar y_u|_C\Big)\\
&\quad\le\ L_1\alpha(2+L_2(|y|_{\mathbb{Y}^\infty_\alpha}+|\varphi|_{W^{1,\infty}}))|y-\bar y|_{\mathbb{Y}^\infty_\alpha}\\
&\quad\le\ L_1\alpha(2+L_2(\bar\beta+|\varphi^*|_{W^{1,\infty}}+\delta_1))|y-\bar y|_{\mathbb{Y}^\infty_\alpha}.
\end{aligned}$$

Similarly, using Lemma 3.6 (i) and (iii), we have that

$$\begin{aligned}
&|S(y,\varphi,\theta,\sigma)-S(\bar y,\varphi,\theta,\sigma)|^p_{\mathbb{Y}^p_\alpha}\\
&\quad=\ \int_0^\alpha\Big|f(u,y(u)+\tilde\varphi(u),\Lambda(u,y_u+\tilde\varphi_u,\sigma),\theta)-f(u,\bar y(u)+\tilde\varphi(u),\Lambda(u,\bar y_u+\tilde\varphi_u,\sigma),\theta)\Big|^p\,du\\
&\quad\le\ L_1^p\int_0^\alpha\Big(|y(u)-\bar y(u)|+|y_u-\bar y_u|_C+L_2|\dot y_u+\dot{\tilde\varphi}_u|_{L^\infty}|y_u-\bar y_u|_C\Big)^p\,du\\
&\quad\le\ L_1^p\alpha^p(2+L_2(\bar\beta+|\varphi^*|_{W^{1,\infty}}+\delta_1))^p|y-\bar y|^p_{\mathbb{Y}^p_\alpha}.
\end{aligned}$$

Therefore (4) (ii) is satisfied with $c = \alpha(L_1(2 + L_2(\bar{\beta} + |\varphi^*|_{W^{1,\infty}} + \delta_1)) < 1$.

Statement (4) (iii) follows easily from the continuity of f and Λ, and the Lebesgue's Dominated Convergence Theorem. $\square$

Lemma 4.2 and Theorem 3.5 yield the well-posedness of IVP (1.1)-(1.2). For comparison, we refer to [2] as a standard reference for well-posedness of differential equations with state-dependent delays.

Theorem 4.3 *Assume (A1) (i),(ii), (A2) (i),(ii) and (A3). Let $1 \le p < \infty$, and assume that φ^*, θ^*, and σ^* satisfy (4.2). Then there exist $\alpha > 0$ and a neighborhood of the parameters, where IVP (1.1)-(1.2) has a unique solution, $x(\varphi, \theta, \sigma)(\cdot)$, on $[0, \alpha]$, which depends continuously on the parameters φ, θ and σ in the $|\cdot|_{\mathbb{Y}^p_\alpha}$ norm, or equivalently, in the $|\cdot|_{W^{1,p}_\alpha}$ norm.*

We comment that, under our assumptions, its is easy to show that the solution, $x(\varphi, \theta, \sigma)$, depends continuously on φ, θ and σ in the $W^{1,\infty}_\alpha$ norm, in fact, the map $(\varphi, \theta, \sigma) \mapsto x(\varphi, \theta, \sigma)$ is locally Lipschitz-continuous as a map $W^{1,\infty} \times \Theta \times \Sigma \to W^{1,\infty}_\alpha$. (See, e.g., [5].)

5 Differentiability of the composition operator

Clearly, in order to apply Lemma 4.2 and Theorem 3.5 to obtain differentiability of solutions wrt parameters, i.e., to obtain differentiability of the operator $S(y, \varphi, \theta, \sigma)$ wrt its arguments, it is necessary to have some kind of continuous differentiability of $\Lambda(t, \psi, \sigma)$ wrt ψ and σ. It turns out that we need differentiability of the following composition operator. Fix $1 \le p < \infty$, $0 < \alpha \le T$ finite, δ_4, $\delta_5 > 0$, and let $x^* \in W^{1,\infty}_\alpha$ and $\sigma^* \in \Omega_5$ such that $x_t \in \Omega_4$ for $t \in [0, \alpha]$. We define the composition operator B_Λ corresponding to the delayed term Λ by

$$B_\Lambda : \Big(\mathcal{G}_{W^{1,\infty}_\alpha}(x^*; \delta_4) \times \mathcal{G}_\Sigma(\sigma^*; \delta_5) \subset \mathbb{X}^p_\alpha \times \Sigma\Big) \to L^p_{0,\alpha}, \quad B_\Lambda(x, \sigma)(t) \equiv \Lambda(t, x_t, \sigma), \quad t \in [0, \alpha]. \tag{5.1}$$

Similarly, we define the composition map B_λ corresponding to λ:

$$B_\lambda : \Big(\mathcal{G}_{W^{1,\infty}_\alpha}(x^*; \delta_4) \times \mathcal{G}_\Sigma(\sigma^*; \delta_5) \times W^{1,\infty}_\alpha \subset \mathbb{X}^p_\alpha \times \Sigma \times \mathbb{X}^p_\alpha\Big) \to L^p_{0,\alpha},$$
$$B_\lambda(x, \sigma, z)(t) \equiv \lambda(t, x_t, \sigma, z_t), \qquad t \in [0, \alpha]. \tag{5.2}$$

Our goal in this section is to give conditions guaranteeing that

(P) $x^* \in W^{1,\infty}_\alpha$, $\sigma^* \in \Omega_5$, $\delta_4 > 0$ and $\delta_5 > 0$ are such that the composition operator B_Λ is continuously differentiable wrt x wrt the set $\mathcal{G}_{W^{1,\infty}_\alpha}(x^*; \delta_4)$ (in the sense of Definition 3.3), and wrt σ on $\mathcal{G}_{W^{1,\infty}_\alpha}(x^*; \delta_4) \times \mathcal{G}_\Sigma(\sigma^*; \delta_5)$.

Assuming that $B_\lambda(x, \sigma, z)$ has continuous partial derivatives wrt x and wrt the set $\mathcal{G}_{W^{1,\infty}_\alpha}(x^*; \delta_4)$, and wrt σ and z, relation $B_\Lambda(x, \sigma) = B_\lambda(x, \sigma, x)$ yields that

$$\frac{\partial B_\Lambda}{\partial x}(x, \sigma) = \frac{\partial B_\lambda}{\partial x}(x, \sigma, x) + \frac{\partial B_\lambda}{\partial z}(x, \sigma, x), \tag{5.3}$$

and

$$\frac{\partial B_\Lambda}{\partial \sigma}(x, \sigma) = \frac{\partial B_\lambda}{\partial \sigma}(x, \sigma, x). \tag{5.4}$$

Therefore, to obtain (P), it is enough to show that $B_\lambda(x, \sigma, z)$ has continuous partial derivatives wrt x and wrt the set $\mathcal{G}_{W^{1,\infty}_\alpha}(x^*; \delta_4)$, and wrt σ and z on $\mathcal{G}_{W^{1,\infty}_\alpha}(x^*; \delta_4) \times \mathcal{G}_\Sigma(\sigma^*; \delta_5) \times W^{1,\infty}_\alpha$ for some $x^* \in W^{1,\infty}_\alpha$, $\sigma^* \in \Omega_5$, $\delta_4 > 0$ and $\delta_5 > 0$.

Since $B_\lambda(x, \sigma, z)(t) = z(t - \tau(t, x_t, \sigma))$, first we study the smoothness of the map $t \mapsto \tau(t, x_t, \sigma)$.

Lemma 5.1 *Assume (A2) (i)–(iii), and let $x \in W^{1,\infty}_\alpha$ and $\sigma \in \Omega_5$ be such that $x_t \in \Omega_4$ for $t \in [0,\alpha]$. Then the function $t \mapsto \tau(t, x_t, \sigma)$ is Lipschitz-continuous, and therefore a.e. differentiable on $[0,\alpha]$.*

Proof Let $M_4 \equiv \{x_t \, : \, t \in [0,\alpha]\}$, and $M_5 \equiv \{\sigma\}$. Then M_4 is a compact subset of C, and $M_4 \subset \Omega_4$. Let $L_2 = L_2(\alpha, M_4, M_5)$ be the constant from (A2) (ii), and $t, \bar t \in [0,\alpha]$. Since the set M_4 is compact, the inequalities

$$\begin{aligned}|\tau(t,x_t,\sigma) - \tau(\bar t, x_{\bar t}, \sigma)| &\leq |\tau(t,x_t,\sigma) - \tau(\bar t, x_t,\sigma)| + |\tau(\bar t, x_t, \sigma) - \tau(\bar t, x_{\bar t}, \sigma)| \\ &\leq \left(\sup\left\{\left|\frac{\partial\tau}{\partial t}(u,\psi,\sigma)\right| \, : \, u \in [0,\alpha], \; \psi \in M_4\right\} + L_2 |x|_{W^{1,\infty}_\alpha}\right)|t - \bar t|,\end{aligned}$$

prove the lemma. □

For $\varepsilon > 0$, $\sigma \in \Omega_5$ and $0 < \alpha \leq T$ we define the set

$$\begin{aligned}X(\varepsilon,\sigma,\alpha) \;\equiv\; \Big\{x \in W^{1,\infty}_\alpha \, : \, x_t \in \Omega_4 \text{ for } t \in [0,\alpha], \text{ the map } t \mapsto t - \tau(t,x_t,\sigma) \text{ is differentiable} \\ \text{for a.e. } t \in [0,\alpha], \text{ and } \frac{d}{dt}\Big(t - \tau(t,x_t,\sigma)\Big) \geq \varepsilon \text{ for a.e. } t \in [0,\alpha]\Big\}.\end{aligned} \tag{5.5}$$

Lemma 5.2 *Assume (A2) (i)–(iv), and let $x^* \in X(\varepsilon^*, \sigma^*, \alpha)$ for some $\varepsilon^* > 0$ and $\sigma^* \in \Omega_5$. Then there exist positive constants δ_4, δ_5 and ε such that $\mathcal{G}_{W^{1,\infty}_\alpha}(x^*; \delta_4) \subset X(\varepsilon,\sigma,\alpha)$ for all $\sigma \in \mathcal{G}_\Sigma(\sigma^*; \delta_5)$, and $\overline{\mathcal{G}}_\Sigma(\sigma^*; \delta_5) \subset \Omega_5$.*

Proof It is enough to show that there exist $\delta_4 > 0$ and $\delta_5 > 0$ such that

$$\{x_t \, : \, t \in [0,\alpha], \; x \in \mathcal{G}_{W^{1,\infty}_\alpha}(x^*; \delta_4)\} \subset \Omega_4, \qquad \overline{\mathcal{G}}_\Sigma(\sigma^*; \delta_5) \subset \Omega_5, \tag{5.6}$$

and there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$\begin{aligned}\frac{\tau(t+h, x_{t+h}, \sigma) - \tau(t, x_t, \sigma)}{h} \leq 1 - \varepsilon, \quad \text{for} \quad 0 < |h| \leq \delta, \quad x \in \mathcal{G}_{W^{1,\infty}_\alpha}(x^*; \delta_4), \\ \sigma \in \mathcal{G}_\Sigma(\sigma^*; \delta_5), \quad \text{and} \quad \text{a.e. } t \in [0,\alpha].\end{aligned} \tag{5.7}$$

In fact, if (5.6) holds, then the map $t \mapsto \tau(t,x_t,\sigma)$ is defined for $t \in [0,\alpha]$, hence Lemma 5.1 yields that it is a.e. differentiable, and therefore, by (5.7), $\frac{d}{dt}\tau(t,x_t,\sigma) \leq 1 - \varepsilon$ for a.e. $t \in [0,\alpha]$, i.e., $x \in X(\varepsilon,\sigma,\alpha)$ for all $x \in \mathcal{G}_{W^{1,\infty}_\alpha}(x^*; \delta_4)$ and $\sigma \in \mathcal{G}_\Sigma(\sigma^*; \delta_5)$.

The set $M_4^* \equiv \{x_t^* \, : \, t \in [0,\alpha]\} \subset \Omega_4$ is a compact subset of C, Ω_4 is open in C, therefore there exists $\delta_4^* > 0$ such that $\mathcal{G}_{C_\alpha}(M_4^*; \delta_4^*) \subset \Omega_4$, and hence $\{x_t \, : \, t \in [0,\alpha], \; x \in \mathcal{G}_{W^{1,\infty}_\alpha}(x^*; \delta_4^*)\} \subset \Omega_4$. The existence of δ_5^* satisfying $\overline{\mathcal{G}}_\Sigma(\sigma^*; \delta_5^*) \subset \Omega_5$ is obvious since Ω_5 is open.

The continuity of $\frac{\partial\tau}{\partial t}$ and $\frac{\partial\tau}{\partial\psi}$ yield that the function $(t,\psi) \mapsto \tau(t,\psi,\sigma)$ is differentiable, i.e., the function

$$\omega(\bar t, \bar\psi, \sigma; t, \psi) \equiv \tau(t,\psi,\sigma) - \tau(\bar t, \bar\psi, \sigma) - \frac{\partial\tau}{\partial t}(\bar t, \bar\psi, \sigma)(t - \bar t) - \frac{\partial\tau}{\partial\psi}(\bar t, \bar\psi, \sigma)(\psi - \bar\psi)$$

satisfies

$$\frac{|\omega(\bar t, \bar\psi, \sigma; t, \psi)|}{|t - \bar t| + |\psi - \bar\psi|_C} \to 0, \qquad \text{as } t \to \bar t, \quad |\psi - \bar\psi|_C \to 0. \tag{5.8}$$

We have

$$\tau(t+h, x^*_{t+h}, \sigma^*) - \tau(t, x^*_t, \sigma^*) = \frac{\partial\tau}{\partial t}(t, x^*_t, \sigma^*)h + \frac{\partial\tau}{\partial\psi}(t, x^*_t, \sigma^*)(x^*_{t+h} - x^*_t) + \omega(t, x^*_t, \sigma^*; t+h, x^*_{t+h}). \tag{5.9}$$

Relations (5.8) and $|x^*_{t+h} - x^*_t|_C \to 0$ as $h \to 0$ imply that

$$
\begin{aligned}
\frac{\omega(t, x^*_t, \sigma^*; t+h, x^*_{t+h})}{h} &= \frac{\omega(t, x^*_t, \sigma^*; t+h, x^*_{t+h})}{|h| + |x^*_{t+h} - x^*_t|_C} \cdot \frac{|h| + |x^*_{t+h} - x^*_t|_C}{h} \\
&\leq \frac{\omega(t, x^*_t, \sigma^*; t+h, x^*_{t+h})}{|h| + |x^*_{t+h} - x^*_t|_C}(1 + |x^*|_{W^{1,\infty}_\alpha}) \qquad (5.10) \\
&\to 0, \qquad \text{as } h \to 0. \qquad (5.11)
\end{aligned}
$$

Since $x^* \in X(\varepsilon^*, \sigma^*, \alpha)$, i.e., $\frac{d}{dt}\tau(t, x^*_t, \sigma^*) \leq 1 - \varepsilon^*$ for a.e. $t \in [0, \alpha]$, it follows from (5.9) and (5.11) that there exist $\varepsilon^{**} > 0$ and $\delta^* > 0$ such that

$$
\frac{\partial \tau}{\partial t}(t, x^*_t, \sigma^*) + \frac{\partial \tau}{\partial \psi}(t, x^*_t, \sigma^*)\frac{x^*_{t+h} - x^*_t}{h} \leq 1 - \varepsilon^{**}, \qquad 0 < |h| < \delta^*, \quad \text{a.e. } t \in [0, \alpha]. \qquad (5.12)
$$

Consider

$$
\begin{aligned}
&\tau(t+h, x_{t+h}, \sigma) - \tau(t, x_t, \sigma) \\
&\quad = \frac{\partial \tau}{\partial t}(t, x_t, \sigma)h + \frac{\partial \tau}{\partial \psi}(t, x_t, \sigma)(x_{t+h} - x_t) + \omega(t, x_t, \sigma; t+h, x_{t+h}) \\
&\quad = \frac{\partial \tau}{\partial t}(t, x^*_t, \sigma^*)h + \frac{\partial \tau}{\partial \psi}(t, x^*_t, \sigma^*)(x^*_{t+h} - x^*_t) \\
&\qquad + \left(\frac{\partial \tau}{\partial t}(t, x_t, \sigma) - \frac{\partial \tau}{\partial t}(t, x^*_t, \sigma^*)\right)h + \left(\frac{\partial \tau}{\partial \psi}(t, x_t, \sigma) - \frac{\partial \tau}{\partial \psi}(t, x^*_t, \sigma^*)\right)(x_{t+h} - x_t) \\
&\qquad + \frac{\partial \tau}{\partial \psi}(t, x^*_t, \sigma^*)(x_{t+h} - x^*_{t+h} - (x_t - x^*_t)) + \omega(t, x_t, \sigma; t+h, x_{t+h}). \qquad (5.13)
\end{aligned}
$$

Let $M_4 \equiv \{x_t : t \in [0, \alpha],\ x \in \mathcal{G}_{W^{1,\infty}_\alpha}(x^*;\, \delta^*_4)\}$, and $M_5 \equiv \overline{\mathcal{G}}_\Sigma(\sigma^*;\, \delta^*_5)$. Then, by Arsela-Ascoli's lemma, the set M_4 is compact in C. Let $L_2 = L_2(\alpha, M_4, M_5)$ and $L_3 = L_3(\alpha, M_4, M_5)$ be the constants from (A2) (ii) and (iii), respectively. Let $x \in \mathcal{G}_{W^{1,\infty}_\alpha}(x^*;\, \delta^*_4)$, and $\sigma \in \overline{\mathcal{G}}_\Sigma(\sigma^*;\, \delta^*_5)$. Then assumption (A2) (ii) and (iii), the Mean Value Theorem, Lemma 3.6 (iv) and (vii), the compactness of M_4, (5.12) and (5.13) imply for $0 < |h| < \delta^*$:

$$
\begin{aligned}
&\frac{\tau(t+h, x_{t+h}, \sigma) - \tau(t, x_t, \sigma)}{h} \\
&\leq 1 - \varepsilon^{**} + \left|\frac{\partial \tau}{\partial t}(t, x_t, \sigma) - \frac{\partial \tau}{\partial t}(t, x^*_t, \sigma^*)\right| + \left\|\frac{\partial \tau}{\partial \psi}(t, x_t, \sigma) - \frac{\partial \tau}{\partial \psi}(t, x^*_t, \sigma^*)\right\|_{\mathcal{L}(C,\mathbb{R})} \frac{|x_{t+h} - x_t|_C}{|h|} \\
&\quad + \left\|\frac{\partial \tau}{\partial \psi}(t, x^*_t, \sigma^*)\right\|_{\mathcal{L}(C,\mathbb{R})} \frac{|x_{t+h} - x^*_{t+h} - (x_t - x^*_t)|_C}{|h|} + \frac{|\omega(t, x_t, \sigma; t+h, x_{t+h})|}{|h|} \\
&\leq 1 - \varepsilon^{**} + L_3\Big(|x_t - x^*_t|_C + |\sigma - \sigma^*|_\Sigma\Big) + L_3\Big(|x_t - x^*_t|_C + |\sigma - \sigma^*|_\Sigma\Big)|\dot{x}|_{L^\infty_\alpha} \\
&\quad + \left\|\frac{\partial \tau}{\partial \psi}(t, x^*_t, \sigma^*)\right\|_{\mathcal{L}(C,\mathbb{R})} |\dot{x} - \dot{x^*}|_{L^\infty_\alpha} + \frac{|\omega(t, x_t, \sigma; t+h, x_{t+h})|}{|h|} \\
&\leq 1 - \varepsilon^{**} + L_3\Big(\alpha|x - x^*|_{W^{1,\infty}_\alpha} + |\sigma - \sigma^*|_\Sigma\Big) + L_3\Big(\alpha|x - x^*|_{W^{1,\infty}_\alpha} + |\sigma - \sigma^*|_\Sigma\Big)(|x^*|_{W^{1,\infty}_\alpha} + \delta^*_4) \\
&\quad + \sup\left\{\left\|\frac{\partial \tau}{\partial \psi}(u, \psi, \sigma^*)\right\|_{\mathcal{L}(C,\mathbb{R})} : u \in [0, \alpha],\ \psi \in M_4\right\} |x - x^*|_{W^{1,\infty}_\alpha} + \frac{|\omega(t, x_t, \sigma; t+h, x_{t+h})|}{|h|}.
\end{aligned}
\qquad (5.14)
$$

Since, similarly to (5.11), $|\omega(t, x_t, \sigma; t+h, x_{t+h})|/|h| \to 0$ as $h \to 0$ for all $x \in \mathcal{G}_{W^{1,\infty}_\alpha}(x^*;\, \delta^*_4)$, and $\sigma \in \mathcal{G}_\Sigma(\sigma^*;\, \delta^*_5)$, (5.14) yields the existence of $\varepsilon > 0$, $\delta > 0$, $0 < \delta_4 \leq \delta^*$ and $0 < \delta_5 \leq \delta^*_5$ satisfying (5.7). This concludes the proof of the lemma. □

We recall the following result from [1].

Lemma 5.3 *Let $g \in L^p_\alpha$, $\varepsilon > 0$, and $u \in \mathcal{A} \equiv \{v \in W^{1,\infty}([0,\alpha]; [-r,\alpha]) : \dot v(s) \ge \varepsilon$ for a.e. $s \in [0,\alpha]\}$. Then*

$$\int_0^\alpha |g(u(s))|^p\, ds \le \frac{1}{\varepsilon}|g|^p_{L^p_\alpha}.$$

Moreover, if $u^k \in \mathcal{A}$ is such that $|u^k - u|_{C([0,\alpha],\mathbb{R})} \to 0$ as $k \to \infty$, then

$$\lim_{k\to\infty} \int_0^\alpha \Big| g(u^k(s)) - g(u(s)) \Big|^p\, ds = 0.$$

Note that the second part of the lemma was stated in [1] with the assumption that $u^k \to u$ in the $W^{1,\infty}$-norm, but in the proof it was used only that $u^k \to u$ in the C-norm.

Let $x^* \in W^{1,\infty}_\alpha$ be such that $x^* \in X(\varepsilon^*, \sigma^*, \alpha)$ for some $\varepsilon^* > 0$, $\sigma^* \in \Omega_5$ and $\alpha > 0$, and δ_4 and δ_5 be the constants corresponding to x^* and σ^* from Lemma 5.2. The next lemma shows that x^*, σ^*, δ_4 and δ_5 satisfy property (P).

Lemma 5.4 *Assume (A2), and let $x^* \in X(\varepsilon^*, \sigma^*, \alpha)$ for some $\varepsilon^* > 0$, $\sigma^* \in \Omega_5$ and $\alpha > 0$. Let δ_4 and δ_5 be the constants corresponding to x^* and σ^* from Lemma 5.2. Then the composition operator $B_\lambda(x,\sigma,z)$ defined by (5.2) has continuous partial derivatives wrt x and wrt the set $\mathcal{G}_{W^{1,\infty}_\alpha}(x^*; \delta_4)$, and wrt σ and z for $x \in \mathcal{G}_{W^{1,\infty}_\alpha}(x^*; \delta_4)$, $\sigma \in \mathcal{G}_\Sigma(\sigma^*; \delta_5)$ and $z \in \mathbb{X}^p_\alpha$. Moreover, $\frac{\partial B_\lambda}{\partial z}(x,\sigma,z)z = \mathcal{B}_z(x,\sigma,z)$, $\frac{\partial B_\lambda}{\partial x}(x,\sigma,z) = \mathcal{B}_x(x,\sigma,z)$ and $\frac{\partial B_\lambda}{\partial \sigma}(x,\sigma,z) = \mathcal{B}_\sigma(x,\sigma,z)$, where*

$$\mathcal{B}_z(x,\sigma,z)h \equiv B_\lambda(x,\sigma,h), \qquad h \in \mathbb{X}^p_\alpha, \tag{5.15}$$

$$(\mathcal{B}_x(x,\sigma,z)h)(t) \equiv -\dot z(t - \tau(t,x_t,\sigma))\frac{\partial \tau}{\partial \psi}(t,x_t,\sigma)h_t, \qquad h \in \mathbb{X}^p_\alpha, \quad a.e.\ t \in [0,\alpha], \tag{5.16}$$

and

$$(\mathcal{B}_\sigma(x,\sigma,z)h)(t) \equiv -\dot z(t - \tau(t,x_t,\sigma))\frac{\partial \tau}{\partial \sigma}(t,x_t,\sigma)h, \qquad h \in \Sigma, \quad a.e.\ t \in [0,\alpha]. \tag{5.17}$$

Proof We will use the notations $M_4 \equiv \{x_t : t \in [0,\alpha],\ x \in \mathcal{G}_{W^{1,\infty}_\alpha}(x^*; \delta_4)\}$ and $M_5 \equiv \overline{\mathcal{G}}_\Sigma(\sigma^*; \delta_5)$ throughout this proof. Arsela-Ascoli's lemma implies that M_4 is a compact subset of C, and Lemma 5.2 yields that $M_4 \subset \Omega_4$ and $M_5 \subset \Omega_5$. Let $L_2 = L_2(\alpha, M_4, M_5)$ and $L_3 = L_3(\alpha, M_4, M_5)$ be the constants from (A2) (ii) and (iii), respectively.

First we show that the linear operator $\mathcal{B}_z(x,\sigma,z) : \mathbb{X}^p_\alpha \to L^p_{0,\alpha}$ defined by (5.15) is bounded. Let $h \in \mathbb{X}^p_\alpha$, $x \in \mathcal{G}_{W^{1,\infty}_\alpha}(x^*; \delta_4)$, $\sigma \in \mathcal{G}_\Sigma(\sigma^*; \delta_5)$, and $z \in \mathbb{X}^p_\alpha$. Since, by Lemma 5.2, $x \in X(\varepsilon, \sigma, \alpha)$, Lemma 5.3 and Lemma 3.8 (i) imply

$$|\mathcal{B}_z(x,\sigma,z)h|_{L^p_{0,\alpha}} = \left(\int_0^\alpha |h(t - \tau(t,x_t,\sigma))|^p dt\right)^{1/p} \le \frac{1}{\varepsilon^{1/p}}|h|_{L^p_\alpha} \le \frac{c_1}{\varepsilon^{1/p}}|h|_{\mathbb{X}^p_\alpha},$$

which shows the boundedness of $\mathcal{B}_z(x,\sigma,z)$. Since the map $z \mapsto B_\lambda(x,\sigma,z)$ is linear, it is obvious that the bounded linear operator $\mathcal{B}_z(x,\sigma,z)$ defined by (5.15) is the partial derivative of $B_\lambda(x,\sigma,z)$ wrt z.

Next we show the continuity of $\frac{\partial B_\lambda}{\partial z}(x,\sigma,z)$ wrt x, σ and z. First we comment that $\frac{\partial B_\lambda}{\partial z}(x,\sigma,z)$ is independent of z. Let $x, \bar x \in \mathcal{G}_{W^{1,\infty}_\alpha}(x^*; \delta_4)$, $\sigma, \bar\sigma \in \mathcal{G}_\Sigma(\sigma^*; \delta_5)$, $z, \bar z \in \mathbb{X}^p_\alpha$, and $h \in \mathbb{X}^p_\alpha$. Since h is absolutely continuous, the definition of $\frac{\partial B_\lambda}{\partial z}$ yields

$$\begin{aligned}
\left|\frac{\partial B_\lambda}{\partial z}(x,\sigma,z)h - \frac{\partial B_\lambda}{\partial z}(\bar x,\bar\sigma,\bar z)h\right|^p_{L^p_{0,\alpha}} &= |B_\lambda(x,\sigma,h) - B_\lambda(\bar x,\bar\sigma,h)|^p_{L^p_{0,\alpha}} \\
&= \int_0^\alpha |h(t - \tau(t,x_t,\sigma)) - h(t - \tau(t,\bar x_t,\bar\sigma))|^p\, dt \\
&= \int_0^\alpha \left|\int_{t-\tau(t,\bar x_t,\bar\sigma)}^{t-\tau(t,x_t,\sigma)} \dot h(s)\, ds\right|^p dt.
\end{aligned}$$

Using the substitution $s(u) = t - \tau(t, \bar{x}_t, \bar{\sigma}) + u(\tau(t, \bar{x}_t, \bar{\sigma}) - \tau(t, x_t, \sigma))$, we get

$$\begin{aligned}
&\left|\frac{\partial B_\lambda}{\partial z}(x,\sigma,z)h - \frac{\partial B_\lambda}{\partial z}(\bar{x},\bar{\sigma},\bar{z})h\right|^p_{L^p_{0,\alpha}} \\
&\quad = \int_0^\alpha \left|\int_0^1 \dot{h}\Big(t - \tau(t,\bar{x}_t,\bar{\sigma}) + u(\tau(t,\bar{x}_t,\bar{\sigma}) - \tau(t,x_t,\sigma))\Big)\Big(\tau(t,\bar{x}_t,\bar{\sigma}) - \tau(t,x_t,\sigma)\Big)du\right|^p dt \\
&\quad \le \int_0^\alpha \Big|\tau(t,x_t,\sigma) - \tau(t,\bar{x}_t,\bar{\sigma})\Big|^p \left|\int_0^1 \Big|\dot{h}\Big(t - \tau(t,\bar{x}_t,\bar{\sigma}) + u(\tau(t,\bar{x}_t,\bar{\sigma}) - \tau(t,x_t,\sigma))\Big)\Big|\, du\right|^p dt.
\end{aligned}$$

Using that $x_t, \bar{x}_t \in M_4$ for $t \in [0,\alpha]$, $x, \bar{x} \in \mathcal{G}_{W^{1,\infty}_\alpha}(x^*; \delta_4)$, the fact that $\sigma, \bar{\sigma} \in M_5$, and the function $(u,t) \mapsto \dot{h}\Big(t - \tau(t,\bar{x}_t,\bar{\sigma}) + u(\tau(t,\bar{x}_t,\bar{\sigma}) - \tau(t,x_t,\sigma))\Big)$ is integrable on $[0,1]\times[0,\alpha]$, assumption (A2) (ii), Hölder's inequality and Fubini's theorem we obtain

$$\begin{aligned}
&\left|\frac{\partial B_\lambda}{\partial z}(x,\sigma,z)h - \frac{\partial B_\lambda}{\partial z}(\bar{x},\bar{\sigma},\bar{z})h\right|^p_{L^p_{0,\alpha}} \\
&\quad \le L_2^p\Big(|x-\bar{x}|_{C_\alpha} + |\sigma-\bar{\sigma}|_\Sigma\Big)^p \int_0^\alpha\int_0^1 \Big|\dot{h}\Big(t - \tau(t,\bar{x}_t,\bar{\sigma}) + u(\tau(t,\bar{x}_t,\bar{\sigma}) - \tau(t,x_t,\sigma))\Big)\Big|^p du\, dt \\
&\quad = L_2^p\Big(|x-\bar{x}|_{C_\alpha} + |\sigma-\bar{\sigma}|_\Sigma\Big)^p \int_0^1\int_0^\alpha \Big|\dot{h}\Big(t - \tau(t,\bar{x}_t,\bar{\sigma}) + u(\tau(t,\bar{x}_t,\bar{\sigma}) - \tau(t,x_t,\sigma))\Big)\Big|^p dt\, du.
\end{aligned} \tag{5.18}$$

Since $x \in X(\varepsilon,\sigma,\alpha)$ and $\bar{x} \in X(\varepsilon,\bar{\sigma},\alpha)$, it follows for $u \in [0,1]$ and a.e. $t \in [0,\alpha]$ that

$$\frac{d}{dt}\Big(t-\tau(t,\bar{x}_t,\bar{\sigma})+u(\tau(t,\bar{x}_t,\bar{\sigma})-\tau(t,x_t,\sigma))\Big) = u\frac{d}{dt}\Big(t-\tau(t,x_t,\sigma)\Big)+(1-u)\frac{d}{dt}\Big(t-\tau(t,\bar{x}_t,\bar{\sigma})\Big) > \varepsilon, \tag{5.19}$$

therefore (5.18), Lemma 3.8 (i), (iii) and Lemma 5.3 imply that

$$\begin{aligned}
\left|\frac{\partial B_\lambda}{\partial z}(x,\sigma,z)h - \frac{\partial B_\lambda}{\partial z}(\bar{x},\bar{\sigma},\bar{z})h\right|_{L^p_{0,\alpha}} &\le \frac{L_2}{\varepsilon^{1/p}}\Big(|x-\bar{x}|_{C_\alpha} + |\sigma-\bar{\sigma}|_\Sigma\Big)|\dot{h}|_{L^p_\alpha} \\
&\le \frac{L_2 c_1}{\varepsilon^{1/p}}\Big(c_3|x-\bar{x}|_{\mathbb{X}^p_\alpha} + |\sigma-\bar{\sigma}|_\Sigma\Big)|h|_{\mathbb{X}^p_\alpha},
\end{aligned}$$

i.e.,

$$\left\|\frac{\partial B_\lambda}{\partial z}(x,\sigma,z) - \frac{\partial B_\lambda}{\partial z}(\bar{x},\bar{\sigma},\bar{z})\right\|_{\mathcal{L}(\mathbb{X}^p_\alpha, L^p_{0,\alpha})} \le \frac{L_2 c_1}{\varepsilon^{1/p}}\Big(c_3|x-\bar{x}|_{\mathbb{X}^p_\alpha} + |\sigma-\bar{\sigma}|_\Sigma\Big). \tag{5.20}$$

Hence $\frac{\partial B_\lambda}{\partial z}$ is continuous (in fact it is Lipschitz-continuous) on its domain.

Now we show that the linear operator $\mathcal{B}_x(x,\sigma,z) : \mathbb{X}^p_\alpha \to L^p_{0,\alpha}$ defined by (5.16) is the partial derivative of B_λ wrt x. The boundedness of $\mathcal{B}_x(x,\sigma,z)$ follows from Lemma 3.8 (iii) and from the estimates

$$\begin{aligned}
|\mathcal{B}_x(x,\sigma,z)h|_{L^p_{0,\alpha}} &= \left(\int_0^\alpha \left|\dot{z}(t-\tau(t,x_t,\sigma))\frac{\partial \tau}{\partial \psi}(t,x_t,\sigma)h_t\right|^p dt\right)^{1/p} \\
&\le |z|_{W^{1,\infty}_\alpha} \sup_{0\le t\le\alpha}\left\|\frac{\partial \tau}{\partial \psi}(t,x_t,\sigma)\right\|_{\mathcal{L}(C,\mathbb{R})} \alpha^{1/p}|h|_{C_\alpha} \\
&\le |z|_{W^{1,\infty}_\alpha} \sup_{0\le t\le\alpha}\left\|\frac{\partial \tau}{\partial \psi}(t,x_t,\sigma)\right\|_{\mathcal{L}(C,\mathbb{R})} \alpha^{1/p}c_3|h|_{\mathbb{X}^p_\alpha}.
\end{aligned}$$

Let $x \in \mathcal{G}_{W^{1,\infty}_\alpha}(x^*;\, \delta_4)$ and $h \in \mathbb{X}^p_\alpha$ such that $x + h \in \mathcal{G}_{W^{1,\infty}_\alpha}(x^*;\, \delta_4)$. Elementary manipulations yield that

$$
\begin{aligned}
&|B_\lambda(x+h,\sigma,z) - B_\lambda(x,\sigma,z) - \mathcal{B}_x(x,\sigma,z)h|^p_{L^p_{0,\alpha}} \\
&\quad = \int_0^\alpha \Big| z(t-\tau(t,x_t+h_t,\sigma)) - z(t-\tau(t,x_t,\sigma)) + \dot z(t-\tau(t,x_t,\sigma))\frac{\partial \tau}{\partial \psi}(t,x_t,\sigma)h_t\Big|^p\, dt \\
&\quad = \int_0^\alpha \Big| \int_{t-\tau(t,x_t,\sigma)}^{t-\tau(t,x_t+h_t,\sigma)} \Big(\dot z(s) - \dot z(t-\tau(t,x_t,\sigma))\Big)\, ds \\
&\qquad\quad + \dot z(t-\tau(t,x_t,\sigma))\Big(\tau(t,x_t,\sigma) - \tau(t,x_t+h_t,\sigma) + \frac{\partial\tau}{\partial\psi}(t,x_t,\sigma)h_t\Big)\Big|^p\, dt \\
&\quad = \int_0^\alpha \Big| \int_0^1 \Big(\dot z\Big(t-\tau(t,x_t,\sigma) + u(\tau(t,x_t,\sigma) - \tau(t,x_t+h_t,\sigma))\Big) - \dot z(t-\tau(t,x_t,\sigma))\Big) \\
&\qquad\quad \cdot (\tau(t,x_t,\sigma) - \tau(t,x_t+h_t,\sigma))\, du \\
&\qquad\quad + \dot z(t-\tau(t,x_t,\sigma))\Big(\tau(t,x_t,\sigma) - \tau(t,x_t+h_t,\sigma) + \frac{\partial\tau}{\partial\psi}(t,x_t,\sigma)h_t\Big)\Big|^p\, dt.
\end{aligned}
$$

Then by the triangle and Hölder's inequalities it follows that

$$
\begin{aligned}
&|B_\lambda(x+h,\sigma,z) - B_\lambda(x,\sigma,z) - \mathcal{B}_x(x,\sigma,z)h|_{L^p_{0,\alpha}} \\
&\quad \le \Big(\int_0^\alpha \Big|\tau(t,x_t,\sigma) - \tau(t,x_t+h_t,\sigma)\Big|^p \int_0^1 \Big|\dot z\Big(t-\tau(t,x_t,\sigma) + u(\tau(t,x_t,\sigma) - \tau(t,x_t+h_t,\sigma))\Big) \\
&\qquad\quad - \dot z(t-\tau(t,x_t,\sigma))\Big|^p\, du\, dt\Big)^{1/p} \\
&\qquad + \Big(\int_0^\alpha \Big|\dot z(t-\tau(t,x_t,\sigma))\Big|^p \Big|\tau(t,x_t,\sigma) - \tau(t,x_t+h_t,\sigma) + \frac{\partial\tau}{\partial\psi}(t,x_t,\sigma)h_t\Big|^p\, dt\Big)^{1/p}. \qquad (5.21)
\end{aligned}
$$

Consider the first term of the right hand side of (5.21). Since $x + h \in \mathcal{G}_{W^{1,\infty}_\alpha}(x^*;\, \delta_4)$, we have that $x_t, x_t + h_t \in M_4$ for $t \in [0, \alpha]$. Then (A2) (ii), Fubini's theorem, and Lemma 3.8 (iii) imply that

$$
\begin{aligned}
&\Big(\int_0^\alpha \Big|\tau(t,x_t,\sigma) - \tau(t,x_t+h_t,\sigma)\Big|^p \int_0^1 \Big|\dot z\Big(t-\tau(t,x_t,\sigma) + u(\tau(t,x_t,\sigma) - \tau(t,x_t+h_t,\sigma))\Big) \\
&\qquad - \dot z(t-\tau(t,x_t,\sigma))\Big|^p\, du\, dt\Big)^{1/p} \\
&\le L_2|h|_{C_\alpha}\Big(\int_0^\alpha\int_0^1 \Big|\dot z\Big(t-\tau(t,x_t,\sigma) + u(\tau(t,x_t,\sigma) - \tau(t,x_t+h_t,\sigma))\Big) - \dot z(t-\tau(t,x_t,\sigma))\Big|^p du\, dt\Big)^{1/p} \\
&\le L_2 c_3|h|_{\mathbb{X}^p_\alpha}\Big(\int_0^1\int_0^\alpha \Big|\dot z\Big(t-\tau(t,x_t,\sigma) + u(\tau(t,x_t,\sigma) - \tau(t,x_t+h_t,\sigma))\Big) - \dot z(t-\tau(t,x_t,\sigma))\Big|^p dt\, du\Big)^{1/p}.
\end{aligned}
\qquad (5.22)
$$

Lemma 5.3 yields that

$$
\int_0^\alpha \Big|\dot z\Big(t-\tau(t,x_t,\sigma) + u(\tau(t,x_t,\sigma) - \tau(t,x_t+h_t,\sigma))\Big) - \dot z(t-\tau(t,x_t,\sigma))\Big|^p\, dt \to 0,
$$

as $|h|_{\mathbb{X}^p_\alpha} \to 0$, since, using Lemma 3.8 (iii),

$$
\begin{aligned}
\Big|t-\tau(t,x_t,\sigma) + u(\tau(t,x_t,\sigma) - \tau(t,x_t+h_t,\sigma)) - (t-\tau(t,x_t,\sigma))\Big| &= u\Big|\tau(t,x_t,\sigma) - \tau(t,x_t+h_t,\sigma)\Big| \\
&\le uL_2|h_t|_C \\
&\le uL_2c_3|h|_{\mathbb{X}^p_\alpha} \\
&\to 0, \qquad \text{as } |h|_{\mathbb{X}^p_\alpha} \to 0,
\end{aligned}
$$

and because, similarly to (5.19), we can show that

$$\frac{d}{dt}\Big(t-\tau(t,x_t,\sigma)+u(\tau(t,x_t,\sigma)-\tau(t,x_t+h_t,\sigma))\Big)\geq\varepsilon\qquad\text{for a.e. } t\in[0,\alpha].$$

Since $z\in W^{1,\infty}_\alpha$, we get that the function

$$u\mapsto\int_0^\alpha\Big|\dot z\Big(t-\tau(t,x_t,\sigma)+u(\tau(t,x_t,\sigma)-\tau(t,x_t+h_t,\sigma))\Big)-\dot z(t-\tau(t,x_t,\sigma))\Big|^p\,dt$$

is bounded on $[0,1]$, therefore the Lebesgue's Dominated Convergence Theorem yields that

$$\int_0^1\int_0^\alpha\Big|\dot z\Big(t-\tau(t,x_t,\sigma)+u(\tau(t,x_t,\sigma)-\tau(t,x_t+h_t,\sigma))\Big)-\dot z(t-\tau(t,x_t,\sigma))\Big|^p\,dt\,du\to 0,\ \text{ as } |h|_{\mathbb{X}^p_\alpha}\to 0. \tag{5.23}$$

Consider the second term of the right hand side of (5.21). Applying Lemma 2.1, (A2) (iv), Lemma 3.8 (i) and (iii), Lemma 5.3, and that $x\in X(\varepsilon,\sigma,\alpha)$, we get

$$\begin{aligned}
&\left(\int_0^\alpha\Big|\dot z(t-\tau(t,x_t,\sigma))\Big|^p\Big|\tau(t,x_t,\sigma)-\tau(t,x_t+h_t,\sigma)+\frac{\partial\tau}{\partial\psi}(t,x_t,\sigma)h_t\Big|^p\,dt\right)^{1/p}\\
&\quad\leq\ \left(\int_0^\alpha\Big|\dot z(t-\tau(t,x_t,\sigma))\Big|^p\sup_{0\leq\nu\leq1}\Big\|\frac{\partial\tau}{\partial\psi}(t,x_t+\nu h_t,\sigma)-\frac{\partial\tau}{\partial\psi}(t,x_t,\sigma)\Big\|^p_{\mathcal{L}(C,\mathbb{R})}|h_t|^p_C\,dt\right)^{1/p}\\
&\quad\leq\ L_3\left(\int_0^\alpha\Big|\dot z(t-\tau(t,x_t,\sigma))\Big|^p|h_t|^{2p}_C\,dt\right)^{1/p}\\
&\quad\leq\ L_3|h|^2_{C_\alpha}\left(\int_0^\alpha\Big|\dot z(t-\tau(t,x_t,\sigma))\Big|^p\,dt\right)^{1/p}\\
&\quad\leq\ \frac{L_3}{\varepsilon^{1/p}}|h|^2_{C_\alpha}|\dot z|_{L^p_\alpha}\\
&\quad\leq\ \frac{L_3c_1c_3^2}{\varepsilon^{1/p}}|h|^2_{\mathbb{X}^p_\alpha}|z|_{\mathbb{X}^p_\alpha}.
\end{aligned}\tag{5.24}$$

Combining (5.21), (5.22), (5.23) and (5.24), we get that

$$\begin{aligned}
&\frac{1}{|h|_{\mathbb{X}^p_\alpha}}\left|B_\lambda(x+h,\sigma,z)-B_\lambda(x,\sigma,z)-\mathcal{B}_x(x,\sigma,z)h\right|_{L^p_{0,\alpha}}\\
&\leq\ L_2c_3\left(\int_0^1\int_0^\alpha\Big|\dot z\Big(t-\tau(t,x_t,\sigma)+u(\tau(t,x_t,\sigma)-\tau(t,x_t+h_t,\sigma))\Big)-\dot z(t-\tau(t,x_t,\sigma))\Big|^p\,dt\,du\right)^{1/p}\\
&\quad+\ \frac{L_3c_1c_3^2}{\varepsilon^{1/p}}|h|_{\mathbb{X}^p_\alpha}|z|_{\mathbb{X}^p_\alpha}\\
&\to\ 0,\qquad\text{as } |h|_{\mathbb{X}^p_\alpha}\to 0,
\end{aligned}$$

which proves that $\frac{\partial B_\lambda}{\partial x}(x,\sigma,z)=\mathcal{B}_x(x,\sigma,z)$.

Next we show that $\frac{\partial B_\lambda}{\partial x}$ is continuous on $\mathcal{G}_{W^{1,\infty}_\alpha}(x^*;\,\delta_4)\times\mathcal{G}_\Sigma(\sigma^*;\,\delta_5)\times\mathbb{X}^p_\alpha$. Consider

$$\begin{aligned}
&\left|\frac{\partial B_\lambda}{\partial x}(x,\sigma,z)h-\frac{\partial B_\lambda}{\partial x}(\bar x,\bar\sigma,\bar z)h\right|_{L^p_{0,\alpha}}\\
&\quad=\ \left(\int_0^\alpha\Big|\dot z(t-\tau(t,x_t,\sigma))\frac{\partial\tau}{\partial\psi}(t,x_t,\sigma)h_t-\dot{\bar z}(t-\tau(t,\bar x_t,\bar\sigma))\frac{\partial\tau}{\partial\psi}(t,\bar x_t,\bar\sigma)h_t\Big|^p\,dt\right)^{1/p}\\
&\quad\leq\ \left(\int_0^\alpha\Big|\dot z(t-\tau(t,x_t,\sigma))-\dot{\bar z}(t-\tau(t,\bar x_t,\bar\sigma))\Big|^p\Big|\frac{\partial\tau}{\partial\psi}(t,x_t,\sigma)h_t\Big|^p\,dt\right)^{1/p}\\
&\qquad+\ \left(\int_0^\alpha\Big|\dot{\bar z}(t-\tau(t,\bar x_t,\bar\sigma))\Big|^p\Big|\frac{\partial\tau}{\partial\psi}(t,x_t,\sigma)h_t-\frac{\partial\tau}{\partial\psi}(t,\bar x_t,\bar\sigma)h_t\Big|^p\,dt\right)^{1/p}.
\end{aligned}\tag{5.25}$$

Assumption (A2) (iv), Lemma 5.3, and Lemma 3.8 (i) and (iii) yield

$$
\begin{aligned}
&\left|\frac{\partial B_\lambda}{\partial x}(x,\sigma,z)h - \frac{\partial B_\lambda}{\partial x}(\bar{x},\bar{\sigma},\bar{z})h\right|_{L^p_{0,\alpha}} \\
&\quad\le \left(\int_0^\alpha \left|\dot{z}(t-\tau(t,x_t,\sigma)) - \dot{\bar{z}}(t-\tau(t,\bar{x}_t,\bar{\sigma}))\right|^p \left\|\frac{\partial \tau}{\partial \psi}(t,x_t,\sigma)\right\|^p_{\mathcal{L}(C,\mathbb{R})} |h_t|^p_C\,dt\right)^{1/p} \\
&\qquad + L_3\left(\int_0^\alpha \left|\dot{\bar{z}}(t-\tau(t,\bar{x}_t,\bar{\sigma}))\right|^p \Big(|x_t-\bar{x}_t|_C + |\sigma-\bar{\sigma}|_\Sigma\Big)^p |h_t|^p_C\,dt\right)^{1/p} \\
&\quad\le \left(\max_{0\le t\le\alpha}\left\|\frac{\partial \tau}{\partial \psi}(t,\bar{x}_t,\bar{\sigma})\right\|_{\mathcal{L}(C,\mathbb{R})} + L_3 2(\delta_4+\delta_5)\right)|h|_{C_\alpha} \\
&\qquad\cdot\Bigg(\left(\int_0^\alpha \left|\dot{z}(t-\tau(t,x_t,\sigma)) - \dot{\bar{z}}(t-\tau(t,x_t,\sigma))\right|^p dt\right)^{1/p} \\
&\qquad\qquad + \left(\int_0^\alpha \left|\dot{\bar{z}}(t-\tau(t,x_t,\sigma)) - \dot{\bar{z}}(t-\tau(t,\bar{x}_t,\bar{\sigma}))\right|^p dt\right)^{1/p}\Bigg) \\
&\qquad + L_3\Big(|x-\bar{x}|_{C_\alpha} + |\sigma-\bar{\sigma}|_\Sigma\Big)\alpha^{1/p}|h|_{C_\alpha}|\bar{z}|_{W^{1,\infty}_\alpha} \\
&\quad\le \left(\max_{0\le t\le\alpha}\left\|\frac{\partial \tau}{\partial \psi}(t,\bar{x}_t,\bar{\sigma})\right\|_{\mathcal{L}(C,\mathbb{R})} + L_3 2(\delta_4+\delta_5)\right)|h|_{C_\alpha} \\
&\qquad\cdot\left(\frac{1}{\varepsilon^{1/p}}|\dot{z}-\dot{\bar{z}}|_{L^p_\alpha} + \left(\int_0^\alpha \left|\dot{\bar{z}}(t-\tau(t,x_t,\sigma)) - \dot{\bar{z}}(t-\tau(t,\bar{x}_t,\bar{\sigma}))\right|^p dt\right)^{1/p}\right) \\
&\qquad + L_3\Big(|x-\bar{x}|_{C_\alpha} + |\sigma-\bar{\sigma}|_\Sigma\Big)\alpha^{1/p}|h|_{C_\alpha}|\bar{z}|_{W^{1,\infty}_\alpha} \\
&\quad\le \left(\max_{0\le t\le\alpha}\left\|\frac{\partial \tau}{\partial \psi}(t,\bar{x}_t,\bar{\sigma})\right\|_{\mathcal{L}(C,\mathbb{R})} + L_3 2(\delta_4+\delta_5)\right)c_3|h|_{\mathbb{X}^p_\alpha} \\
&\qquad\cdot\left(\frac{c_1}{\varepsilon^{1/p}}|z-\bar{z}|_{\mathbb{X}^p_\alpha} + \left(\int_0^\alpha \left|\dot{\bar{z}}(t-\tau(t,x_t,\sigma)) - \dot{\bar{z}}(t-\tau(t,\bar{x}_t,\bar{\sigma}))\right|^p dt\right)^{1/p}\right) \\
&\qquad + L_3\Big(c_3|x-\bar{x}|_{\mathbb{X}^p_\alpha} + |\sigma-\bar{\sigma}|_\Sigma\Big)\alpha^{1/p}c_3|h|_{\mathbb{X}^p_\alpha}|\bar{z}|_{W^{1,\infty}_\alpha},
\end{aligned}
$$

which, together with the continuity of $\frac{\partial\tau}{\partial\psi}$, the relation

$$
\begin{aligned}
|\tau(t,x_t,\sigma)-\tau(t,\bar{x}_t,\bar{\sigma})| &\le L_2(c_3|x-\bar{x}|_{\mathbb{X}^p_\alpha} + |\sigma-\bar{\sigma}|_\Sigma) \\
&\to 0, \qquad \text{as } |x-\bar{x}|_{\mathbb{X}^p_\alpha}\to 0, \text{ and } \sigma\to\bar{\sigma},
\end{aligned}
$$

and Lemma 5.3, implies the continuity of $\frac{\partial B_\lambda}{\partial x}$.

The proof of $\frac{\partial B_\lambda}{\partial \sigma}(x,\sigma,z) = \mathcal{B}_\sigma(x,\sigma,z)$ is analogous to that of $\frac{\partial B_\lambda}{\partial x}(x,\sigma,z) = \mathcal{B}_x(x,\sigma,z)$, and therefore it is omitted here. □

6 Differentiability of solutions wrt parameters

The following lemma shows that property (P) of the previous section yields the existence of continuous partial derivatives of $S(y,\varphi,\theta,\sigma)$ wrt y, φ, θ and σ if we restrict y to a certain subset of its domain, and the derivative wrt y is taken in the sense of Definition 3.3.

Lemma 6.1 *Assume (A1)–(A3), and let $1\le p<\infty$. Assume that φ^*, θ^* and σ^* satisfy (4.2). Let α, δ_1, δ_2, δ_3 be the constants, and let M_1, M_2, M_3, M_4, M_5, $\mathcal{U}$ and $\mathcal{W}$ be the sets from Lemma 4.2. Let $x^*\in W^{1,\infty}_\alpha$ be such that $Pr_\varphi\, x^* = \varphi^*$, $x^*\in X(\varepsilon,\sigma^*,\alpha)$ for some $\varepsilon>0$, and $y^*\in\mathcal{U}$, where $y^*\equiv Pr_y\, x^*$.*

Then there exist constants $0<\bar\delta_1\le\delta_1$, $0<\bar\delta_2\le\delta_2$, and $0<\bar\delta_3\le\delta_3$, and an open subset, $\mathcal{U}^*$, of $\mathbb{Y}^\infty_\alpha$, such that $\mathcal{U}^*\subset\mathcal{U}$, and the operator

$$S(y,\varphi,\theta,\sigma)\,:\,\Big(\mathcal{U}^*\times\mathcal{G}_{W^{1,\infty}}\big(\varphi^*;\,\bar\delta_1\big)\times\mathcal{G}_\Theta\big(\theta^*;\,\bar\delta_2\big)\times\mathcal{G}_\Sigma\big(\sigma^*;\,\bar\delta_3\big)\subset\mathbb{Y}^p_\alpha\times W^{1,\infty}\times\Theta\times\Sigma\Big)\to\mathbb{Y}^p_\alpha$$

defined by (4.3) has continuous partial derivatives wrt y and wrt the set $\mathcal{U}^*$, and wrt φ, θ and σ on its domain. Moreover, let $y\in\mathcal{U}^*$, $\varphi\in\mathcal{G}_{W^{1,\infty}}\big(\varphi^*;\,\bar\delta_1\big)$, $\theta\in\mathcal{G}_\Theta\big(\theta^*;\,\bar\delta_2\big)$, and $\sigma\in\mathcal{G}_\Sigma\big(\sigma^*;\,\bar\delta_3\big)$. Then $\frac{\partial S}{\partial y}(y,\varphi,\theta,\sigma)=\mathcal{S}_y(y,\varphi,\theta,\sigma)$, $\frac{\partial S}{\partial\varphi}(y,\varphi,\theta,\sigma)=\mathcal{S}_\varphi(y,\varphi,\theta,\sigma)$, $\frac{\partial S}{\partial\theta}(y,\varphi,\theta,\sigma)=\mathcal{S}_\theta(y,\varphi,\theta,\sigma)$ and $\frac{\partial S}{\partial\sigma}(y,\varphi,\theta,\sigma)=\mathcal{S}_\sigma(y,\varphi,\theta,\sigma)$, where

$$\begin{aligned}&(\mathcal{S}_y(y,\varphi,\theta,\sigma)h)\,(t)\\&\quad\equiv\begin{cases}0, & t\in[-r,0],\\ \displaystyle\int_0^t\frac{\partial f}{\partial v}\Big(u,y(u)+\tilde\varphi(u),\Lambda(u,y_u+\tilde\varphi_u,\sigma),\theta\Big)h(u)\\ \quad+\displaystyle\frac{\partial f}{\partial w}\Big(u,y(u)+\tilde\varphi(u),\Lambda(u,y_u+\tilde\varphi_u,\sigma),\theta\Big)\left(\frac{\partial B_\Lambda}{\partial x}(y+\tilde\varphi,\sigma)h\right)(u)\,du, & t\in[0,\alpha],\end{cases}\end{aligned}\tag{6.1}$$

$h\in\mathbb{Y}^p_\alpha$;

$$\begin{aligned}&(\mathcal{S}_\varphi(y,\varphi,\theta,\sigma)h)\,(t)\\&\quad\equiv\begin{cases}0, & t\in[-r,0],\\ \displaystyle\int_0^t\frac{\partial f}{\partial v}\Big(u,y(u)+\tilde\varphi(u),\Lambda(u,y_u+\tilde\varphi_u,\sigma),\theta\Big)h(0)\\ \quad+\displaystyle\frac{\partial f}{\partial w}\Big(u,y(u)+\tilde\varphi(u),\Lambda(u,y_u+\tilde\varphi_u,\sigma),\theta\Big)\left(\frac{\partial B_\Lambda}{\partial x}(y+\tilde\varphi,\sigma)\tilde h\right)(u)\,du, & t\in[0,\alpha],\end{cases}\end{aligned}\tag{6.2}$$

$h\in W^{1,\infty}$;

$$(\mathcal{S}_\theta(y,\varphi,\theta,\sigma)h)\,(t)\quad\equiv\quad\begin{cases}0, & t\in[-r,0],\\ \displaystyle\int_0^t\frac{\partial f}{\partial\theta}\Big(u,y(u)+\tilde\varphi(u),\Lambda(u,y_u+\tilde\varphi_u,\sigma),\theta\Big)h\,du, & t\in[0,\alpha],\end{cases}\tag{6.3}$$

$h\in\Theta$;

$$\begin{aligned}&(\mathcal{S}_\sigma(y,\varphi,\theta,\sigma)h)\,(t)\\&\quad\equiv\begin{cases}0, & t\in[-r,0],\\ \displaystyle\int_0^t\frac{\partial f}{\partial w}\Big(u,y(u)+\tilde\varphi(u),\Lambda(u,y_u+\tilde\varphi_u,\sigma),\theta\Big)\left(\frac{\partial B_\Lambda}{\partial\sigma}(y+\tilde\varphi,\sigma)h\right)(u)\,du, & t\in[0,\alpha],\end{cases}\end{aligned}\tag{6.4}$$

$h\in\Sigma$, where $\frac{\partial f}{\partial v}$, $\frac{\partial f}{\partial w}$ and $\frac{\partial f}{\partial\theta}$ denote the partial derivatives of $f(t,v,w,\theta)$ wrt v, w and θ, respectively.

Proof Let δ_4 and δ_5 be the constants corresponding to x^* and σ^* from Lemma 5.4. Define $\bar\delta_1\equiv\min\{\delta_1,\delta_4/2\}$, $\bar\delta_2\equiv\delta_2$ and $\bar\delta_3\equiv\min\{\delta_3,\delta_5\}$, and let $\bar\delta_6>0$ be such that $\bar\delta_6\le\delta_4/(2\max\{\alpha,1\})$ and $\mathcal{G}_{\mathbb{Y}^\infty_\alpha}\big(y^*;\,\bar\delta_6\big)\subset\mathcal{U}$. Let $\mathcal{U}^*\equiv\mathcal{G}_{\mathbb{Y}^\infty_\alpha}\big(y^*;\,\bar\delta_6\big)$. Then, clearly, $\mathcal{U}^*\subset\mathcal{U}$, and $\mathcal{U}^*$ is an open subset of $\mathbb{Y}^\infty_\alpha$.

First note that the definitions of $\mathcal{U}^*$ and $\bar\delta_1$, and Lemma 3.6 (vii) yield for $y\in\mathcal{U}^*$ and $\varphi\in\mathcal{G}_{W^{1,\infty}}\big(\varphi^*;\,\bar\delta_1\big)$ that $|y+\tilde\varphi-x^*|_{W^{1,\infty}_\alpha}\le|y-y^*|_{W^{1,\infty}_\alpha}+|\varphi-\varphi^*|_{W^{1,\infty}}\le\max\{\alpha,1\}|y-y^*|_{\mathbb{Y}^\infty_\alpha}+|\varphi-\varphi^*|_{W^{1,\infty}}<\delta_4$, i.e.,

$$y+\tilde\varphi\in\mathcal{G}_{W^{1,\infty}_\alpha}(x^*;\,\delta_4)\qquad\text{for } y\in\mathcal{U}^*\quad\text{and}\quad\varphi\in\mathcal{G}_{W^{1,\infty}}\big(\varphi^*;\,\bar\delta_1\big).\tag{6.5}$$

Therefore $\frac{\partial B_\Lambda}{\partial x}(y+\tilde\varphi,\sigma)$ and $\frac{\partial B_\Lambda}{\partial\sigma}(y+\tilde\varphi,\sigma)$ are well-defined for all $y\in\mathcal{U}^*$, $\varphi\in\mathcal{G}_{W^{1,\infty}}\big(\varphi^*;\,\bar\delta_1\big)$, and $\sigma\in\mathcal{G}_\Sigma\big(\sigma^*;\,\bar\delta_3\big)$. Also comment that the selections of $\bar\delta_1$, $\bar\delta_2$, $\bar\delta_3$ and $\mathcal{U}^*$ implies that (4.4) holds for all $u\in[0,\alpha]$, $y\in\mathcal{U}^*$, $\varphi\in\mathcal{G}_{W^{1,\infty}}\big(\varphi^*;\,\bar\delta_1\big)$, $\theta\in\mathcal{G}_\Theta\big(\theta^*;\,\bar\delta_2\big)$, and $\sigma\in\mathcal{G}_\Sigma\big(\sigma^*;\,\bar\delta_3\big)$. Let $L_1=L_1(\alpha,M_1,M_2,M_3)$,

$L_2 = L_2(\alpha, M_4, M_5)$ and $L_3 = L_3(\alpha, M_4, M_5)$ be the constants from (A1) (ii), (A2) (ii) and (iii), respectively. Assumption (A1) (ii) and (iii) imply that

$$\left\|\frac{\partial f}{\partial v}(t,v,w,\theta)\right\| \leq L_1, \quad \left\|\frac{\partial f}{\partial w}(t,v,w,\theta)\right\| \leq L_1, \quad \text{and} \quad \left\|\frac{\partial f}{\partial \theta}(t,v,w,\theta)\right\|_{\mathcal{L}(\Theta,\mathbb{R}^n)} \leq L_1 \tag{6.6}$$

for $t \in [0,\alpha]$, $v \in M_1$, $w \in M_2$, and $\theta \in M_3$.

Let $y \in \mathcal{U}^*$, $\varphi \in \mathcal{G}_{W^{1,\infty}}\big(\varphi^*;\, \bar\delta_1\big)$, $\theta \in \mathcal{G}_\Theta\big(\theta^*;\, \bar\delta_2\big)$, $\sigma \in \mathcal{G}_\Sigma\big(\sigma^*;\, \bar\delta_3\big)$. We show that the linear operator $\mathcal{S}_y(y,\varphi,\theta,\sigma) : \mathbb{Y}^p_\alpha \to \mathbb{Y}^p_\alpha$ defined by (6.1) is the partial derivative of $S(y,\varphi,\theta,\sigma)$ wrt y. Let $h \in \mathbb{Y}^p_\alpha$. The definition of $\mathcal{S}_y(y,\varphi,\theta,\sigma)$, (4.4) and (6.6), Lemma 3.6 (viii), and the relation $|h|_{\mathbb{X}^p_\alpha} = |h|_{\mathbb{Y}^p_\alpha}$ yield

$$\begin{aligned}
&|\mathcal{S}_y(y,\varphi,\theta,\sigma)h|_{\mathbb{Y}^p_\alpha} \\
&\quad\leq \Big(\int_0^\alpha \Big|\frac{\partial f}{\partial v}\Big(u, y(u)+\tilde\varphi(u), \Lambda(u, y_u+\tilde\varphi_u,\sigma),\theta\Big)h(u)\Big|^p\,du\Big)^{1/p} \\
&\qquad+ \Big(\int_0^\alpha \Big|\frac{\partial f}{\partial w}\Big(u, y(u)+\tilde\varphi(u), \Lambda(u, y_u+\tilde\varphi_u,\sigma),\theta\Big)\Big(\frac{\partial B_\Lambda}{\partial x}(y+\tilde\varphi,\sigma)h\Big)(u)\Big|^p\,du\Big)^{1/p} \\
&\quad\leq L_1|h|_{L^p_\alpha} + L_1\Big|\frac{\partial B_\Lambda}{\partial x}(y+\tilde\varphi,\sigma)h\Big|_{L^p_{0,\alpha}} \\
&\quad\leq L_1\alpha|h|_{\mathbb{Y}^p_\alpha} + L_1\Big\|\frac{\partial B_\Lambda}{\partial x}(y+\tilde\varphi,\sigma)\Big\|_{\mathcal{L}(\mathbb{X}^p_\alpha, L^p_{0,\alpha})}|h|_{\mathbb{Y}^p_\alpha},
\end{aligned} \tag{6.7}$$

which shows the boundedness of $\mathcal{S}_y(y,\varphi,\theta,\sigma)$.

Next we show that $\mathcal{S}_y(y,\varphi,\theta,\sigma)$ is the derivative of $S(y,\varphi,\theta,\sigma)$ wrt y and wrt the set $\mathcal{U}^*$ in the sense of Definition 3.3. Let $h \in \mathbb{Y}^p_\alpha$ be such that $y+h \in \mathcal{U}^*$, and consider

$$\begin{aligned}
&|S(y+h,\varphi,\theta,\sigma) - S(y,\varphi,\theta,\sigma) - \mathcal{S}_y(y,\varphi,\theta,\sigma)h|_{\mathbb{Y}^p_\alpha} \\
&\quad= \Big(\int_0^\alpha \Big| f\Big(u, y(u)+\tilde\varphi(u)+h(u), \Lambda(u, y_u+h_u+\tilde\varphi_u,\sigma),\theta\Big) \\
&\qquad\quad - f\Big(u, y(u)+\tilde\varphi(u), \Lambda(u, y_u+\tilde\varphi_u,\sigma),\theta\Big) - \frac{\partial f}{\partial v}\Big(u, y(u)+\tilde\varphi(u), \Lambda(u, y_u+\tilde\varphi_u,\sigma),\theta\Big)h(u) \\
&\qquad\quad - \frac{\partial f}{\partial w}\Big(u, y(u)+\tilde\varphi(u), \Lambda(u, y_u+\tilde\varphi_u,\sigma),\theta\Big)\Big(\frac{\partial B_\Lambda}{\partial x}(y+\tilde\varphi,\sigma)h\Big)(u)\Big|^p\,du\Big)^{1/p}.
\end{aligned} \tag{6.8}$$

Introduce the function

$$\begin{aligned}
\omega^1(u,\bar v,\bar w,\bar\theta; v,w,\theta) \;\equiv\;& f(u,v,w,\theta) - f(u,\bar v,\bar w,\bar\theta) - \frac{\partial f}{\partial v}(u,\bar v,\bar w,\bar\theta)(v-\bar v) \\
&- \frac{\partial f}{\partial w}(u,\bar v,\bar w,\bar\theta)(w-\bar w) - \frac{\partial f}{\partial \theta}(u,\bar v,\bar w,\bar\theta)(\theta-\bar\theta),
\end{aligned} \tag{6.9}$$

for $u \in [0,T]$, $v,\bar v \in \Omega_1$, $w,\bar w \in \Omega_2$, and $\theta,\bar\theta \in \Omega_3$. The continuity of $\frac{\partial f}{\partial v}$ $\frac{\partial f}{\partial w}$ and $\frac{\partial f}{\partial w}$ yield that

$$\frac{|\omega^1(u,\bar v,\bar w,\bar\theta; v,w,\theta)|}{|v-\bar v| + |w-\bar w| + |\theta-\bar\theta|_\Theta} \to 0, \qquad \text{as } v \to \bar v, \quad w \to \bar w \quad \text{and} \quad \theta \to \bar\theta. \tag{6.10}$$

Assumption (A1) (ii) and (6.6) imply

$$|\omega^1(u,\bar v,\bar w,\bar\theta; v,w,\theta)| \leq 2L_1\Big(|v-\bar v| + |w-\bar w| + |\theta-\bar\theta|_\Theta\Big), \tag{6.11}$$

for $u \in [0,\alpha]$, $v,\bar v \in M_1$, $w,\bar w \in M_2$, $\theta,\bar\theta \in M_3$.

Similarly, by property (P) (guaranteed by Lemma 5.4) and Lemma 3.4, the function

$$\omega^2(u,\bar{x},\bar{\sigma};x,\sigma) \equiv \Lambda(u,x_u,\sigma) - \Lambda(u,\bar{x}_u,\bar{\sigma}) - \left(\frac{\partial B_\Lambda}{\partial x}(\bar{x},\bar{\sigma})(x-\bar{x})\right)(u) - \left(\frac{\partial B_\Lambda}{\partial \sigma}(\bar{x},\bar{\sigma})(\sigma-\bar{\sigma})\right)(u), \tag{6.12}$$

which is defined for $u \in [0,\alpha]$, $x,\bar{x} \in \mathcal{G}_{W^{1,\infty}_\alpha}(x^*;\,\delta_4)$, $\sigma,\bar{\sigma} \in \mathcal{G}_\Sigma(\sigma^*;\,\bar{\delta}_3)$, satisfies

$$\frac{\left(\int_0^\alpha |\omega^2(u,\bar{x},\bar{\sigma};x,\sigma)|^p du\right)^{1/p}}{|x-\bar{x}|_{\mathbb{X}^p_\alpha} + |\sigma-\bar{\sigma}|_\Sigma} \to 0, \qquad \text{as } |x-\bar{x}|_{\mathbb{X}^p_\alpha} \to 0, \quad x \in \mathcal{G}_{W^{1,\infty}_\alpha}(x^*;\,\delta_4), \quad \text{and} \quad \sigma \to \bar{\sigma}. \tag{6.13}$$

The definitions of ω^1 and ω^2, and the relations (6.5) and (6.8) yield that

$$\begin{aligned}
&|S(y+h,\varphi,\theta,\sigma) - S(y,\varphi,\theta,\sigma) - \mathcal{S}_y(y,\varphi,\theta,\sigma)h|_{\mathbb{Y}^p_\alpha} \\
&\leq \left(\int_0^\alpha \left|\omega^1(u,y(u)+\tilde{\varphi}(u),\Lambda(u,y_u+\tilde{\varphi}_u,\sigma),\theta;y(u)+h(u)+\tilde{\varphi}(u),\Lambda(u,y_u+h_u+\tilde{\varphi}_u,\sigma),\theta)\right|^p du\right)^{1/p} \\
&\quad + \left(\int_0^\alpha \left|\frac{\partial f}{\partial w}\Big(u,y(u)+\tilde{\varphi}(u),\Lambda(u,y_u+\tilde{\varphi}_u,\sigma),\theta\Big)\omega^2(u,y+\tilde{\varphi},\sigma;y+h+\tilde{\varphi},\sigma)\right|^p du\right)^{1/p}.
\end{aligned} \tag{6.14}$$

Using (4.4) and (6.6), estimate (6.14) implies that

$$\begin{aligned}
&\frac{1}{|h|_{\mathbb{Y}^p_\alpha}}|S(y+h,\varphi,\theta,\sigma) - S(y,\varphi,\theta,\sigma) - \mathcal{S}_y(y,\varphi,\theta,\sigma)h|_{\mathbb{Y}^p_\alpha} \\
&\leq \left(\int_0^\alpha \left|\frac{\omega^1(u,y(u)+\tilde{\varphi}(u),\Lambda(u,y_u+\tilde{\varphi}_u,\sigma),\theta;y(u)+h(u)+\tilde{\varphi}(u),\Lambda(u,y_u+h_u+\tilde{\varphi}_u,\sigma),\theta)}{|h|_{\mathbb{Y}^p_\alpha}}\right|^p du\right)^{1/p} \\
&\quad + L_1\left(\int_0^\alpha \left|\frac{\omega^2(u,y+\tilde{\varphi},\sigma;y+h+\tilde{\varphi},\sigma)}{|h|_{\mathbb{Y}^p_\alpha}}\right|^p du\right)^{1/p}.
\end{aligned} \tag{6.15}$$

We show first that $|\omega^1(\cdot)|/|h|_{\mathbb{Y}^p_\alpha}$ in (6.15) converges to zero pointwise as $|h|_{\mathbb{Y}^p_\alpha} \to 0$. It follows from the inequality $|h(u)| \leq \alpha^{1/q}|h|_{\mathbb{Y}^p_\alpha}$ (guaranteed by Lemma 3.6 (i)) that $y(u)+h(u)+\tilde{\varphi}(u) \to y(u)+\tilde{\varphi}(u)$ as $|h|_{\mathbb{Y}^p_\alpha} \to 0$. Lemma 4.1 with $L_2 = L_2(\alpha,M_4,M_5)$, Lemma 3.6 (i), and (6.5) imply for $y, y+h \in \mathcal{U}^*$ that

$$\begin{aligned}
|\Lambda(u,y_u+h_u+\tilde{\varphi}_u,\sigma) - \Lambda(u,y_u+\tilde{\varphi}_u,\sigma)| &\leq |h_u|_C + L_2|\dot{y}_u + \dot{\tilde{\varphi}}_u|_{L^\infty}|h_u|_C \\
&\leq (1+L_2(|x^*|_{W^{1,\infty}_\alpha}+\delta_4))\alpha^{1/q}|h|_{\mathbb{Y}^p_\alpha} \\
&\to 0, \qquad \text{as } |h|_{\mathbb{Y}^p_\alpha} \to 0.
\end{aligned}$$

Therefore, relation (6.10), with an argument similar to (5.10), gives that $|\omega^1(\cdot)|/|h|_{\mathbb{Y}^p_\alpha}$ in (6.15) converges to zero pointwise as $|h|_{\mathbb{Y}^p_\alpha} \to 0$. Next we show that $|\omega^1(\cdot)|/|h|_{\mathbb{Y}^p_\alpha}$ in (6.15) is bounded on $[0,\alpha]$. The previous estimate and (6.11) yield for $y, y+h \in \mathcal{U}^*$ that

$$\begin{aligned}
&|\omega^1(u,y(u)+\tilde{\varphi}(u),\Lambda(u,y_u+\tilde{\varphi}_u,\sigma),\theta;y(u)+h(u)+\tilde{\varphi}(u),\Lambda(u,y_u+h_u+\tilde{\varphi}_u,\sigma),\theta)| \\
&\quad \leq 2L_1(|h(u)| + |\Lambda(u,y_u+h_u+\tilde{\varphi}_u,\sigma) - \Lambda(u,y_u+\tilde{\varphi}_u,\sigma)|) \\
&\quad \leq 2L_1\alpha^{1/q}(2+L_2(|x^*|_{W^{1,\infty}_\alpha}+\delta_4))|h|_{\mathbb{Y}^p_\alpha}.
\end{aligned}$$

Therefore the Lebesgue's Dominated Theorem yields that the first term in (6.15) goes to zero as $|h|_{\mathbb{Y}^p_\alpha} \to 0$. So does the second term by (6.13), therefore we have proved that $\frac{\partial S}{\partial y}(y,\varphi,\theta,\sigma) = \mathcal{S}_y(y,\varphi,\theta,\sigma)$.

Next we show that $\frac{\partial S}{\partial y}(y,\varphi,\theta,\sigma)$ is continuous on its domain. Select sequences $y^k \in \mathcal{U}^*$, $\varphi^k \in \mathcal{G}_{W^{1,\infty}}(\varphi^*;\,\bar{\delta}_1)$, $\theta^k \in \mathcal{G}_\Theta(\theta^*;\,\bar{\delta}_2)$, and $\sigma^k \in \mathcal{G}_\Sigma(\sigma^*;\,\bar{\delta}_3)$ such that $|y^k - y|_{\mathbb{Y}^p_\alpha} \to 0$, $|\varphi^k - \varphi|_{W^{1,\infty}} \to 0$,

$\theta^k \to \theta$ and $\sigma^k \to \sigma$ as $k \to \infty$. Let $h \in \mathbb{Y}^p_\alpha$. Elementary manipulations give

$$\begin{aligned}
&\left|\frac{\partial S}{\partial y}(y^k,\varphi^k,\theta^k,\sigma^k)h - \frac{\partial S}{\partial y}(y,\varphi,\theta,\sigma)h\right|_{\mathbb{Y}^p_\alpha} \\
&\quad\leq \quad \Big(\int_0^\alpha \Big\|\frac{\partial f}{\partial v}(u, y^k(u)+\tilde{\varphi^k}(u), \Lambda(u, y^k_u + (\tilde{\varphi^k})_u, \sigma^k), \theta^k\Big) \\
&\qquad\qquad - \frac{\partial f}{\partial v}(u, y(u)+\tilde{\varphi}(u), \Lambda(u, y_u+\tilde{\varphi}_u,\sigma),\theta\Big)\Big\|^p |h(u)|^p\, du\Big)^{1/p} \\
&\qquad\qquad + \Big(\int_0^\alpha \Big\|\frac{\partial f}{\partial w}\Big(u, y^k(u)+\tilde{\varphi^k}(u), \Lambda(u, y^k_u + (\tilde{\varphi^k})_u, \sigma^k), \theta^k\Big) \\
&\qquad\qquad - \frac{\partial f}{\partial w}(u, y(u)+\tilde{\varphi}(u), \Lambda(u, y_u+\tilde{\varphi}_u,\sigma),\theta\Big)\Big\|^p \Big|\Big(\frac{\partial B_\Lambda}{\partial x}(y^k+\tilde{\varphi^k},\sigma^k)h\Big)(u)\Big|^p\, du\Big)^{1/p} \\
&\qquad\qquad + \Big(\int_0^\alpha \Big\|\frac{\partial f}{\partial w}(u, y(u)+\tilde{\varphi}(u), \Lambda(u, y_u+\tilde{\varphi}_u,\sigma),\theta\Big)\Big\|^p \\
&\qquad\qquad \cdot \Big|\Big(\frac{\partial B_\Lambda}{\partial x}(y^k+\tilde{\varphi^k},\sigma^k)h - \frac{\partial B_\Lambda}{\partial x}(y+\tilde{\varphi},\sigma)h\Big)(u)\Big|^p\, du\Big)^{1/p}.
\end{aligned}$$

Therefore, using (4.4), (6.6) and Lemma 3.6 (viii), we get

$$\begin{aligned}
&\left\|\frac{\partial S}{\partial y}(y^k,\varphi^k,\theta^k,\sigma^k) - \frac{\partial S}{\partial y}(y,\varphi,\theta,\sigma)\right\|_{\mathcal{L}(\mathbb{Y}^p_\alpha,\mathbb{Y}^p_\alpha)} \\
&\quad\leq \alpha \sup_{0\leq u\leq\alpha}\Big\|\frac{\partial f}{\partial v}(u, y^k(u)+\tilde{\varphi^k}(u), \Lambda(u, y^k_u + (\tilde{\varphi^k})_u, \sigma^k), \theta^k\Big) - \frac{\partial f}{\partial v}(u, y(u)+\tilde{\varphi}(u), \Lambda(u, y_u+\tilde{\varphi}_u,\sigma),\theta\Big)\Big\| \\
&\qquad + \sup_{0\leq u\leq\alpha}\Big\|\frac{\partial f}{\partial w}\Big(u, y^k(u)+\tilde{\varphi^k}(u), \Lambda(u, y^k_u + (\tilde{\varphi^k})_u, \sigma^k), \theta^k\Big) \\
&\qquad\qquad - \frac{\partial f}{\partial w}(u, y(u)+\tilde{\varphi}(u), \Lambda(u, y_u+\tilde{\varphi}_u,\sigma),\theta\Big)\Big\|\Big\|\frac{\partial B_\Lambda}{\partial x}(y^k+\tilde{\varphi^k},\sigma^k)\Big\|_{\mathcal{L}(\mathbb{X}^p_\alpha, L^p_{0,\alpha})} \\
&\qquad + L_1\Big\|\frac{\partial B_\Lambda}{\partial x}(y^k+\tilde{\varphi^k},\sigma^k) - \frac{\partial B_\Lambda}{\partial x}(y+\tilde{\varphi},\sigma)\Big\|_{\mathcal{L}(\mathbb{X}^p_\alpha, L^p_{0,\alpha})}.
\end{aligned} \tag{6.16}$$

Lemma 3.6 (i) implies that

$$\begin{aligned}
|y^k(u)+\tilde{\varphi^k}(u) - y(u) - \tilde{\varphi}(u)| &\leq |y^k(u)-y(u)| + |\tilde{\varphi^k}(u) - \tilde{\varphi}(u)| \\
&\leq \alpha^{1/q}|y^k-y|_{\mathbb{Y}^p_\alpha} + |\varphi^k-\varphi|_{W^{1,\infty}} \\
&\to 0, \qquad \text{as } k\to\infty.
\end{aligned} \tag{6.17}$$

Since $y^k \in \mathcal{U}^*$, and $\varphi^k \in \mathcal{G}_{W^{1,\infty}}(\varphi^*;\bar{\delta}_1)$, relation (6.5), Lemma 3.6 (iii), and Lemma 4.1 with $L_2 = L_2(\alpha, M_4, M_5)$ yield that

$$\begin{aligned}
&|\Lambda(u, y^k_u + (\tilde{\varphi^k})_u, \sigma^k) - \Lambda(u, y_u+\tilde{\varphi}_u, \sigma)| \\
&\quad\leq |y^k_u - y_u|_C + |\varphi^k-\varphi|_C + L_2|\dot{y}_u + \dot{\tilde{\varphi}}_u|_{L^\infty}\Big(|y^k_u-y_u|_C + |\varphi^k-\varphi|_C + |\sigma^k-\sigma|_\Sigma\Big) \\
&\quad\leq (1+L_2(|x^*|_{W^{1,\infty}_\alpha}+\delta_4))\Big(\alpha^{1/q}|y^k-y|_{\mathbb{Y}^p_\alpha} + |\varphi^k-\varphi|_{W^{1,\infty}} + |\sigma^k-\sigma|_\Sigma\Big) \\
&\quad\to 0, \qquad \text{as } k\to\infty.
\end{aligned} \tag{6.18}$$

Since $\theta^k\to\theta$, the set $M_3^* \equiv \{\theta^k : k\in\mathbb{N}\}\cup\{\theta\}$ is a compact subset of Θ, and hence the functions $\frac{\partial f}{\partial v}$ and $\frac{\partial f}{\partial w}$ are uniformly continuous on the compact set $[0,\alpha]\times M_1\times M_2\times M_3^*$. Consequently, (6.17) and

(6.18) yield that the first and second terms in the right hand side of (6.16) go to zero as $k \to \infty$. So does the third term, since by (P), $\frac{\partial B_\Lambda}{\partial x}$ is continuous on $\mathcal{G}_{W^{1,\infty}_\alpha}(x^*;\,\delta_4) \times \mathcal{G}_\Sigma(\sigma^*;\,\bar\delta_3)$ (in the $\|\cdot\|_{\mathcal{L}(\mathbb{X}^p_\alpha, L^p_{0,\alpha})}$ norm). This completes the proof of the continuity of $\frac{\partial S}{\partial y}$.

The proof of $\frac{\partial S}{\partial \varphi}(y,\varphi,\theta,\sigma) = \mathcal{S}_\varphi(y,\varphi,\theta,\sigma)$ is similar. Clearly, the operator $\mathcal{S}_\varphi(y,\varphi,\theta,\sigma)$ defined by (6.2) is linear, and similarly to (6.7), we can get

$$|\mathcal{S}_\varphi(y,\varphi,\theta,\sigma)h|_{\mathbb{Y}^p_\alpha} \quad \le \quad L_1\alpha^{1/p}|h|_{W^{1,\infty}} + L_1\left\|\frac{\partial B_\lambda}{\partial x}(y+\tilde\varphi,\sigma)\right\|_{\mathcal{L}(\mathbb{X}^p_\alpha, L^p_{0,\alpha})}|h|_{W^{1,\infty}},$$

which implies the boundedness of $\mathcal{S}_\varphi(y,\varphi,\theta,\sigma)$.

Let $h \in W^{1,\infty}$, then using the definitions of ω^1 and ω^2, and the relations (4.4), (6.6) and (6.2), we get

$$\begin{aligned}
&\frac{1}{|h|_{W^{1,\infty}}}\,|S(y,\varphi+h,\theta,\sigma) - S(y,\varphi,\theta,\sigma) - \mathcal{S}_\varphi(y,\varphi,\theta,\sigma)h|_{\mathbb{Y}^p_\alpha} \\
&\le \left(\int_0^\alpha \left|\frac{\omega^1(u,y(u)+\tilde\varphi(u),\Lambda(u,y_u+\tilde\varphi_u,\sigma),\theta;y(u)+\tilde\varphi(u)+\tilde h(u),\Lambda(u,y_u+\tilde\varphi_u+\tilde h_u,\sigma),\theta)}{|h|_{W^{1,\infty}}}\right|^p du\right)^{1/p} \\
&\quad + L_1\left(\int_0^\alpha \left|\frac{\omega^2(u,y+\tilde\varphi,\sigma;y+\tilde\varphi+\tilde h,\sigma)}{|h|_{W^{1,\infty}}}\right|^p du\right)^{1/p}. \qquad (6.19)
\end{aligned}$$

Lemma 4.1 with $L_2 = L_2(\alpha, M_4, M_5)$ and (6.5) yield that for small h such that $\varphi + h \in \mathcal{G}_{W^{1,\infty}}(\varphi^*;\,\bar\delta_1)$

$$\begin{aligned}
|\tilde h(t)| + |\Lambda(t,y_t+\tilde\varphi_t+\tilde h_t,\sigma) - \Lambda(t,y_t+\tilde\varphi_t,\sigma)| \quad &\le \quad (2 + L_2(|x^*|_{W^{1,\infty}_\alpha} + \delta_4))|h|_C \\
&\to \quad 0, \qquad \text{as } |h|_{W^{1,\infty}} \to 0, \qquad (6.20)
\end{aligned}$$

therefore $|\omega^1(\cdot)|/|h|_{W^{1,\infty}}$ in (6.19) converges to zero pointwise as $|h|_{W^{1,\infty}} \to 0$, and since it is bounded by $2L_1(2 + L_2(|x^*|_{W^{1,\infty}_\alpha} + \delta_4))$, the Lebesgue's Dominated Convergence Theorem implies that the first term in the right hand side of (6.19) goes to zero as $|h|_{W^{1,\infty}} \to 0$. Since $|\tilde h|_{\mathbb{X}^p_\alpha} = |h|_{W^{1,\infty}}$, (6.13) yields that

$$\frac{\left(\int_0^\alpha |\omega^2(t,y+\tilde\varphi,\sigma;y+\tilde\varphi+\tilde h,\sigma)|^p dt\right)^{1/p}}{|h|_{W^{1,\infty}}} \to 0, \qquad \text{as } |h|_{W^{1,\infty}} \to 0,$$

therefore $\mathcal{S}_\varphi(y,\varphi,\theta,\sigma)$ defined by (6.2) is really the partial derivative of $S(y,\varphi,\theta,\sigma)$ wrt φ.

We show that $\frac{\partial S}{\partial \varphi}(y,\varphi,\theta,\psi)$ is continuous on its domain. Let $y^k \in \mathcal{U}^*$, $\varphi^k \in \mathcal{G}_{W^{1,\infty}}(\varphi^*;\,\bar\delta_1)$, $\theta^k \in \mathcal{G}_\Theta(\theta^*;\,\bar\delta_2)$, and $\sigma^k \in \mathcal{G}_\Sigma(\sigma^*;\,\bar\delta_3)$ be sequences such that $|y^k - y|_{\mathbb{Y}^p_\alpha} \to 0$, $|\varphi^k - \varphi|_{W^{1,\infty}} \to 0$, $\theta^k \to \theta$ and $\sigma^k \to \sigma$ as $k \to \infty$. Similarly to (6.16) we can show that

$$\begin{aligned}
&\left\|\frac{\partial S}{\partial \varphi}(y^k,\varphi^k,\theta^k,\sigma^k) - \frac{\partial S}{\partial \varphi}(y,\varphi,\theta,\sigma)\right\|_{\mathcal{L}(W^{1,\infty},\mathbb{Y}^p_\alpha)} \\
&\le \alpha^{1/p}\sup_{0\le u\le\alpha}\left\|\frac{\partial f}{\partial v}\Big(u,y^k(u)+\tilde{\varphi^k}(u),\Lambda(u,y^k_u+(\tilde{\varphi^k})_u,\sigma^k),\theta^k\Big)\right. \\
&\qquad\qquad \left. - \frac{\partial f}{\partial v}\Big(u,y(u)+\tilde\varphi(u),\Lambda(u,y_u+\tilde\varphi_u,\sigma),\theta\Big)\right\| \\
&\quad + \sup_{0\le u\le\alpha}\left\|\frac{\partial f}{\partial w}\Big(u,y^k(u)+\tilde{\varphi^k}(u),\Lambda(u,y^k_u+(\tilde{\varphi^k})_u,\sigma^k),\theta^k\Big)\right. \\
&\qquad\qquad \left. - \frac{\partial f}{\partial w}\Big(u,y(u)+\tilde\varphi(u),\Lambda(u,y_u+\tilde\varphi_u,\sigma),\theta\Big)\right\|\left\|\frac{\partial B_\Lambda}{\partial x}(y^k+\tilde{\varphi^k})\right\|_{\mathcal{L}(\mathbb{X}^p_\alpha, L^p_{0,\alpha})} \\
&\quad + L_1\left\|\frac{\partial B_\Lambda}{\partial x}(y^k+\tilde{\varphi^k},\sigma^k) - \frac{\partial B_\Lambda}{\partial x}(y+\tilde\varphi,\sigma)\right\|_{\mathcal{L}(\mathbb{X}^p_\alpha, L^p_{0,\alpha})},
\end{aligned}$$

which implies the continuity of $\frac{\partial S}{\partial \varphi}$, since it is essentially the same as (6.16).

Next we prove $\frac{\partial S}{\partial \theta}(y,\varphi,\theta,\sigma)=\mathcal{S}_\theta(y,\varphi,\theta,\sigma)$. The estimate $|\mathcal{S}_\theta(y,\varphi,\theta,\sigma)h|_{\mathbb{Y}^p_\alpha}\le L_1\alpha^{1/p}|h|_\Theta$ implies the boundedness of the operator $\mathcal{S}_\theta(y,\varphi,\theta,\sigma)\,:\,\Theta\to\mathbb{Y}^p_\alpha$, defined by (6.3). Let $h\in\Theta$. One can obtain

$$\begin{aligned}
&\frac{1}{|h|_\Theta}\,|S(y,\varphi,\theta+h,\sigma)-S(y,\varphi,\theta,\sigma)-\mathcal{S}_\theta(y,\varphi,\theta,\sigma)h|_{\mathbb{Y}^p_\alpha}\\
&\le \left(\int_0^\alpha\left|\frac{\omega^1(u,y(u)+\tilde\varphi(u),\Lambda(u,y_u+\tilde\varphi_u,\sigma),\theta;y(u)+\tilde\varphi(u),\Lambda(u,y_u+\tilde\varphi_u,\sigma),\theta+h)}{|h|_\Theta}\right|^p du\right)^{1/p}\\
&\to\ 0,\qquad \text{as } |h|_\Theta\to 0,
\end{aligned}$$

using Lebesgue's Dominated Convergence Theorem and (6.10). To prove continuity of $\frac{\partial S}{\partial \theta}$, consider

$$\begin{aligned}
&\left\|\frac{\partial S}{\partial\theta}(y^k,\varphi^k,\theta^k,\sigma^k)-\frac{\partial S}{\partial\theta}(y,\varphi,\theta,\sigma)\right\|_{\mathcal{L}(\Theta,\mathbb{Y}^p_\alpha)}\\
&\qquad\le\ \alpha^{1/p}\sup_{0\le u\le\alpha}\left\|\frac{\partial f}{\partial\theta}\Big(u,y^k(u)+\tilde{\varphi^k}(u),\Lambda(u,y^k_u+(\tilde{\varphi^k})_u,\sigma^k),\theta^k\Big)\right.\\
&\qquad\qquad\left.-\frac{\partial f}{\partial\theta}\Big(u,y(u)+\tilde\varphi(u),\Lambda(u,y_u+\tilde\varphi_u,\sigma),\theta\Big)\right\|_{\mathcal{L}(\Theta,\mathbb{R}^n)}\\
&\qquad\to\ 0,\qquad \text{as } k\to\infty,
\end{aligned}$$

using a uniform continuity argument, as before.

It can be proved similarly that $\frac{\partial S}{\partial\sigma}(y,\varphi,\theta,\sigma)=\mathcal{S}_\sigma(y,\varphi,\theta,\sigma)$, the details are omitted. □

Theorem 6.2 *Assume (A1)–(A3). Let $1\le p<\infty$, and assume that φ^*, θ^* and σ^* satisfy (4.2). Then there exist $\alpha>0$, δ_1^*, δ_2^*, $\delta_3^*>0$ such that IVP (1.1)-(1.2) has a unique solution, $x(\varphi,\theta,\sigma)(\cdot)$, on $[0,\alpha]$ corresponding to $\varphi\in\mathcal{G}_{W^{1,\infty}}(\varphi^*;\,\delta_1^*)$, $\theta\in\mathcal{G}_\Theta(\theta^*;\,\delta_2^*)$ and $\sigma\in\mathcal{G}_\Sigma(\sigma^*;\,\delta_3^*)$. Moreover, if $x^*\equiv x(\varphi^*,\theta^*,\sigma^*)\in X(\varepsilon,\sigma^*,\alpha)$ for some $\varepsilon>0$, then the function*

$$\Big(\mathcal{G}_{W^{1,\infty}}(\varphi^*;\,\delta_1^*)\times\mathcal{G}_\Theta(\theta^*;\,\delta_2^*)\times\mathcal{G}_\Sigma(\sigma^*;\,\delta_3^*)\subset W^{1,\infty}\times\Theta\times\Sigma\Big)\to\mathbb{X}^p_\alpha,\qquad(\varphi,\theta,\sigma)\mapsto x(\varphi,\theta,\sigma)$$

is continuously differentiable wrt φ, θ and σ on its domain.

Proof Let the constants δ_1, δ_2, δ_3, α, c, and the sets $\mathcal{U}$, $\mathcal{W}$, M_1, M_2, M_3, M_4 and M_5 be defined by Lemma 4.2. Let $L_1=L_1(\alpha,M_1,M_2,M_3)$ and $L_2=L_2(\alpha,M_4,M_5)$ be the constants from (A1) (ii) and (A2) (ii), respectively.

Theorem 4.3 implies that IVP (1.1)-(1.2) has a unique solution on $[0,\alpha]$ for $\varphi\in\mathcal{G}_{W^{1,\infty}}(\varphi^*;\,\delta_1)$, $\theta\in\mathcal{G}_\Theta(\theta^*;\,\delta_2)$ and $\sigma\in\mathcal{G}_\Sigma(\sigma^*;\,\delta_3)$. Assume that $x^*\equiv x(\varphi^*,\theta^*,\sigma^*)\in X(\varepsilon,\sigma^*,\alpha)$ for some $\varepsilon>0$. Let $y^*\equiv \mathrm{Pr}_y\,x^*$. Lemma 4.2 yields that y^* is the unique fixed point of the operator

$$S(\cdot,\varphi^*,\theta^*,\sigma^*)\,:\,\mathcal{W}\to\mathcal{W}$$

defined by (4.3). In particular, we get that $y^*\in\mathcal{W}\subset\mathcal{U}$. Let the constants $\bar\delta_1$, $\bar\delta_2$, $\bar\delta_3$ and the set $\mathcal{U}^*$ be defined by Lemma 6.1 corresponding to x^*. Recall that $\mathcal{U}^*$ was defined in the proof of Lemma 6.1 as $\mathcal{U}^*=\mathcal{G}_{\mathbb{Y}^\infty_\alpha}(y^*;\,\bar\delta_6)$ for some $\bar\delta_6>0$. Define $\mathcal{W}^*\equiv\mathcal{W}\cap\overline{\mathcal{G}}_{\mathbb{Y}^\infty_\alpha}(y^*;\,\delta_6^*)$ for some $0<\delta_6^*<\bar\delta_6$. Then $\mathcal{W}^*\subset\mathcal{W}$, $\mathcal{W}^*\subset\mathcal{U}^*$, and $\mathcal{W}^*$ is a closed subset of $\mathbb{Y}^p_\alpha$ by Lemma 3.7. Since $\mathcal{W}^*\subset\mathcal{W}$, Lemma 4.2 yields that $S(\cdot,\varphi,\theta,\sigma)$ restricted to $\mathcal{W}^*$ is a uniform contraction both in $|\cdot|_{\mathbb{Y}^\infty_\alpha}$ and $|\cdot|_{\mathbb{Y}^p_\alpha}$ norms, and the operator $S(y,\cdot,\cdot,\cdot)\ :\ \mathcal{G}_{W^{1,\infty}}\big(\varphi^*;\,\bar\delta_1\big)\times\mathcal{G}_\Theta\big(\theta^*;\,\bar\delta_2\big)\times\mathcal{G}_\Sigma\big(\sigma^*;\,\bar\delta_3\big)\to\mathbb{Y}^p_\alpha$ is continuous for all $y\in\mathcal{W}^*$. Define $\delta_1^*\equiv\min\Big\{\bar\delta_1,\delta_6(1-c)/(3L_1(2+L_2|x^*|_{W^{1,\infty}_\alpha}))\Big\}$, $\delta_2^*\equiv\min\Big\{\bar\delta_2,\delta_6(1-c)/(3L_1)\Big\}$ and $\delta_3^*\equiv\min\Big\{\bar\delta_3,\delta_6(1-c)/(3L_1L_2(|x^*|_{W^{1,\infty}_\alpha}+1))\Big\}$. Consider the operator S defined by (4.3) as

$$S(y,\varphi,\theta,\sigma)\,:\,\Big(\mathcal{U}^*\times\mathcal{G}_{W^{1,\infty}}(\varphi^*;\,\delta_1^*)\times\mathcal{G}_\Theta(\theta^*;\,\delta_2^*)\times\mathcal{G}_\Sigma(\sigma^*;\,\delta_3^*)\subset\mathbb{Y}^p_\alpha\times W^{1,\infty}\times\Theta\times\Sigma\Big)\to\mathbb{Y}^p_\alpha.$$

Then Lemma 6.1 yields that $S(y,\varphi,\theta,\sigma)$ is continuously differentiable wrt y and wrt the set $\mathcal{U}^*$, and wrt φ, θ and σ. Next we show that $S(\cdot,\varphi,\theta,\sigma)\,:\,\mathcal{W}^*\to\mathcal{W}^*$ for all φ, θ and σ of its domain. Let $y\in\mathcal{W}^*$, $\varphi\in\mathcal{G}_{W^{1,\infty}}(\varphi^*;\,\delta_1^*)$, $\theta\in\mathcal{G}_\Theta(\theta^*;\,\delta_2^*)$ and $\sigma\in\mathcal{G}_\Sigma(\sigma^*;\,\delta_3^*)$. Since $\mathcal{W}^*\subset\mathcal{W}$, it follows from Lemma 4.2 that $S(y,\varphi,\theta,\sigma)\in\mathcal{W}$, hence we have to show only that $S(y,\varphi,\theta,\sigma)\in\overline{\mathcal{G}}_{\mathbb{Y}^\infty_\alpha}(y^*;\,\delta_6^*)$. Using that $y^*=S(y^*,\varphi^*,\theta^*,\sigma^*)$, Lemma 4.2 (ii), assumptions (A1) (ii) and (A2) (ii), Lemma 4.1, and the definitions of δ_1^*, δ_2^* and δ_3^*, we get the estimates

$$\begin{aligned}
&|S(y,\varphi,\theta,\sigma)-y^*|_{\mathbb{Y}^\infty_\alpha}\\
&\quad= |S(y,\varphi,\theta,\sigma)-S(y^*,\varphi,\theta,\sigma)|_{\mathbb{Y}^\infty_\alpha}+|S(y^*,\varphi,\theta,\sigma)-S(y^*,\varphi^*,\theta^*,\sigma^*)|_{\mathbb{Y}^\infty_\alpha}\\
&\quad\le c|y-y^*|_{\mathbb{Y}^\infty_\alpha}\\
&\qquad+\operatorname*{ess\,sup}_{0\le u\le\alpha}\Big|f(u,y^*(u)+\tilde\varphi(u),\Lambda(u,y^*_u+\tilde\varphi_u,\sigma),\theta)-f(u,y^*(u)+\tilde{\varphi^*}(u),\Lambda(u,y^*_u+(\tilde{\varphi^*})_u,\sigma^*),\theta^*)\Big|\\
&\quad\le c|y-y^*|_{\mathbb{Y}^\infty_\alpha}+L_1\operatorname*{ess\,sup}_{0\le u\le\alpha}\Big(|\tilde\varphi(u)-\tilde{\varphi^*}(u)|+|\Lambda(u,y^*_u+\tilde\varphi_u,\sigma)-\Lambda(u,y^*_u+(\tilde{\varphi^*})_u,\sigma^*)|+|\theta-\theta^*|_\Theta\Big)\\
&\quad\le c|y-y^*|_{\mathbb{Y}^\infty_\alpha}+L_1\operatorname*{ess\,sup}_{0\le u\le\alpha}\Big(|\tilde\varphi(u)-\tilde{\varphi^*}(u)|+|\tilde\varphi_u-(\tilde{\varphi^*})_u|_C\\
&\qquad\quad+L_2|x^*|_{W^{1,\infty}_\alpha}(|\tilde\varphi_u-(\tilde{\varphi^*})_u|_C+|\sigma-\sigma^*|_\Sigma)+|\theta-\theta^*|_\Theta\Big)\\
&\quad\le c|y-y^*|_{\mathbb{Y}^\infty_\alpha}+L_1\Big(2|\varphi-\varphi^*|_{W^{1,\infty}}+L_2|x^*|_{W^{1,\infty}_\alpha}(|\varphi-\varphi^*|_{W^{1,\infty}}+|\sigma-\sigma^*|_\Sigma)+|\theta-\theta^*|_\Theta\Big)\\
&\quad< c\delta_6+L_1(2+L_2|x^*|_{W^{1,\infty}_\alpha})\delta_1^*+L_1\delta_2^*+L_1L_2|x^*|_{W^{1,\infty}_\alpha}\delta_3^*\\
&\quad\le \delta_6.
\end{aligned}$$

Therefore S satisfies the conditions of Theorem 3.5, and hence the unique fixed point, $y(\varphi,\theta,\sigma)$, of $S(\cdot,\varphi,\theta,\sigma)$ is continuously differentiable wrt $\varphi\in\mathcal{G}_{W^{1,\infty}_\alpha}(\sigma^*;\,\delta_1^*)$, $\theta\in\mathcal{G}_\Theta(\theta^*;\,\delta_2^*)$ and $\sigma\in\mathcal{G}_\Sigma(\sigma^*;\,\delta_3^*)$. The function $y(\varphi,\theta,\sigma)$ is the unique solution of (4.1), and therefore $x(\varphi,\theta,\sigma)\equiv y(\varphi,\theta,\sigma)+\tilde\varphi$ is the unique solution of IVP (1.1)-(1.2), and it has continuous partial derivatives

$$\frac{\partial x}{\partial\varphi}(\varphi,\theta,\sigma)h=\frac{\partial y}{\partial\varphi}(\varphi,\theta,\sigma)h+\tilde h,\qquad h\in W^{1,\infty},\tag{6.21}$$

and

$$\frac{\partial x}{\partial\theta}(\varphi,\theta,\sigma)=\frac{\partial y}{\partial\theta}(\varphi,\theta,\sigma),\quad\text{and}\quad\frac{\partial x}{\partial\sigma}(\varphi,\theta,\sigma)=\frac{\partial y}{\partial\sigma}(\varphi,\theta,\sigma).\tag{6.22}$$

To prove (6.21), it is enough to consider the obvious relation

$$\left|x(\varphi+h,\theta,\sigma)-x(\varphi,\theta,\sigma)-\frac{\partial x}{\partial\varphi}(\varphi,\theta,\sigma)h\right|_{\mathbb{X}^p_\alpha}=\left|y(\varphi+h,\theta,\sigma)-y(\varphi,\theta,\sigma)-\frac{\partial y}{\partial\varphi}(\varphi,\theta,\sigma)h\right|_{\mathbb{Y}^p_\alpha}.$$

□

Since by Lemma 3.8 (i) the $|\cdot|_{\mathbb{X}^p_\alpha}$ norm is stronger than the $|\cdot|_{W^{1,p}_\alpha}$ norm, the theorem has the following corollary.

Corollary 6.3 *Assume the conditions of Theorem 6.2. Then $x(\varphi,\theta,\sigma)$ is continuously differentiable wrt φ, θ and σ as a function*

$$\Big(\mathcal{G}_{W^{1,\infty}}(\varphi^*;\,\delta_1^*)\times\mathcal{G}_\Theta(\theta^*;\,\delta_2^*)\times\mathcal{G}_\Sigma(\sigma^*;\,\delta_3^*)\subset W^{1,\infty}\times\Theta\times\Sigma\Big)\to W^{1,p}_\alpha,\qquad(\varphi,\theta,\sigma)\mapsto x(\varphi,\theta,\sigma).$$

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