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Control of Triangle Formations with a Mix of Angle and Distance Constraints

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Abstract— **A distributed control law for triangular formation control with a mixture of bearing and range measurements and relative pair-wise inter-agent angle constraints and a single range constraint is introduced. The control law is weak in the sense that two agents are free to choose their own heading within a relatively large range of values. Indeed, the agents can determine their heading independently at run-time given any criteria they desire as long as certain relaxed conditions are met. A convergence result is established that ensures the desired formation configuration is asymptotically stable. Illustrative examples are provided to demonstrate the claims.**

I. Introduction

The formation shape control problem involves a group of agents tasked with forming, and then maintaining, a prescribed geometric shape described in terms of relative geometrical constraints between some of the agents. There now exists a large body of literature on formation control and the references stated here represent only a subjective selection. The general formation problem remains of interest due to the various problem formulations, the distributed nature of the problem itself and the existence of undesired equilibria, see the discussions in [1], [2], [3], which prompts further investigation.

Much of the existing work considers range-only inter-agent control constraints and assumes that each agent measures the relative position (or state) of its neighbour agents; e.g. see [4], [5], [6], [7], [8], [9], [10], [1], [11], [2], [12]. Some authors have considered a related problem with range-only control constraints but relax the sensing requirement such that only range measurements are taken at the individual agents [13], [14]. There are some formation control problems in which the complete relative state (e.g. position and/or velocity) of certain neighbour agents are both sensed and controlled. The problems are variations of well-known consensus or flocking algorithms; see e.g. [15], [16].

More recent work assumes that agents only measure the inter-agent bearings and then seek to control certain angular constraints; hence the notion of bearing-only formation control [17], [18], [19], [20]. This work relates to a large literature on bearing-only localization [21], [22], [23], [24] and to the problem of vision-based distributed formation control, since video sensors act as bearing sensors; see [25], [26]. Each agent measures the bearing to the other two agents in a local coordinate system. The controlled constraint is the angle subtended at the agent by the other two agents. Each agent is given a desired value for this angle and is tasked with establishing and maintaining this constraint. Except for non-generic interior angles, the formation shape is completely controlled.

The main contribution of the present paper is to utilize a mixture of both range and bearing measurements along with a mixture of range and bearing constraints. Such a combination allows a formation in which agents have heterogeneous sensing capabilities. Moreover, the addition of bearings to a range-based scheme may be a way to deal with incorrect equilibrium formations that result from range-only constraints as discussed in [3], though this remains an open question. Conversely, the use of some range sensing removes the scaling ambiguity present in much of the bearing-only formation control literature [17], [18], [19], [20]. The combination of bearing and range (with some agents measuring only one or the other) is a natural extension to the formation control literature. Following [20], we use a *weak distributed control law*, in which the agents that control bearings are free to choose their own heading within a relatively large range of values. We prove local asymptotic stability of the desired formation shape and demonstrate our results via simulation.

In Section 2, the control problem is introduced. In Section 3, the distributed control law is proposed. The multi-agent system evolution is examined and local stability of the desired formation shape is proved. Illustrative examples are provided in Section 4. In Section 5 we provide a conclusion.

II. Triangular Formation Control

Consider a group of $n = 3$ agents in $\mathbb{R}^2$ which *interact* via an undirected topology $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with $\mathcal{V} = \{1, 2, 3\}$ and $\mathcal{E} = \mathcal{V} \times \mathcal{V}$. The position of each agent is $\mathbf{p}_i = [x_i \; y_i]^{\mathrm{T}} \in \mathbb{R}^2$. The neighbor set $\mathcal{N}_i \subset \mathcal{V}$ denotes the set of agents connected to agent i by a single (undirected) edge. In this case $\mathcal{N}_i = \{(i+1), (i-1)\}$ (taken modulo n).

The set of agent points $\mathbf{p}_i$, $\forall i \in \mathcal{V}$ and straight-line edges connecting points $\mathbf{p}_i$ and $\mathbf{p}_{i\pm1}$ define a triangle in the plane $\mathbb{R}^2$. This triangle is referred to as the *formation configuration* or, where no confusion is caused, simply as the *formation*.

Importantly, note that agents do *not* share a common coordinate frame. Each agent $i \in \{1, 2, 3\}$ measures the bearing $\phi_{ij} \in [-\pi, \pi)$, $\forall j \in \mathcal{N}_i$ positive (negative) counter-clockwise (clockwise) from its local x_i-direction to agent j.

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One of the agents, say agent 1 also measures the distance r_{1j} to agent j. We take this to be r_{12} for simplicity. Note, re-ordering the agents is possible. With this particular ordering, it will be the case that ϕ_{13} is never used and in general one bearing measurement is unnecessary to the development of the controller.

Let α_i denote the interior angle subtended at agent i by the two agents in $\mathcal{N}_i$. Then, the formation shape (not scale) is completely characterized by α_i, $\forall i \in \mathcal{V}$. In addition, given r_{12} then the formation scale is also defined.

Introduce the following angle

$$\vartheta_i = |\phi_{i(i+1)} - \phi_{i(i-1)}| \in [0, 2\pi) \tag{1}$$

which is the angle subtended at agent i by agents $i+1$ and $i-1$ (measured positive from the $\min(\phi_{i(i+1)}, \phi_{i(i-1)})$ to $\max(\phi_{i(i+1)}, \phi_{i(i-1)})$) in agent i's local coordinate frame. The interior α_i is given by

$$\alpha_i = \begin{cases} \vartheta_i & \text{if } \vartheta_i \leq \pi \\ 2\pi - \vartheta_i & \text{otherwise} \end{cases} \tag{2}$$

with $\alpha_i \in [0, \pi]$. Note the difference between $\alpha_i = 0$ and $\alpha_i = \pi$ implies agent i can ascertain whether or not it is in between agents $i+1$ and $i-1$ with all three collinear. Tacitly, it can be assumed that α_i is measured by agent i.

Define the desired steady-state angles $\alpha_i^* \in [0, \pi]$, $\forall i \in \mathcal{V}$. The α_i^* then completely characterize the shape (not scale) of the *desired* triangle formation. The following standing assumptions are adopted.

Assumption 1. *No two agents are initially collocated. The desired (i.e. control objective) interior angular separations α_i^*, obey $\alpha_1^* + \alpha_2^* + \alpha_3^* = \pi$. The case where $\alpha_i^* = 0$, $\alpha_j^* \neq 0$ and $\alpha_k^* = \pi - \alpha_j^*$ is excluded.*

Assumption 1 ensures the desired steady-state triangle is well-defined and the set of control objectives are simultaneously feasible. The case where $\alpha_i^* = 0$, $\alpha_j^* \neq 0$ and $\alpha_k^* = \pi - \alpha_j^*$ would place agent i infinitely far from the other two agents.

Assumption 2. *Suppose also that $0 < r_{12}^* < \infty$ is given as the desired steady-state range between agent 1 and agent 2.*

With $\alpha_i^* \in [0, \pi]$, $\forall i \in \mathcal{V}$ and r_{12}^* given the formation shape and scale is completely characterized. Note that given r_{12}^* we highlight that only agent 1 will be attempting to maintain this distance from agent 2 and not vice-versa. Agent 2 does not measure $r_{12} = r_{21}$. Thus r_{12}^* forms a directed constraint.

III. The Proposed Control Law

The motion of agent i is governed by

$$\dot{\mathbf{p}}_i = v_i \begin{bmatrix} \cos\beta_i \\ \sin\beta_i \end{bmatrix} \tag{3}$$

where $[v_i \ \beta_i]^\top$ are controls to be determined. The heading β_i is measured positive (negative) counter-clockwise (clockwise) from agent i's local x_i-direction.

In this paper we extend [18] and [17] by designing a distributed controller for each agent i that is relaxed in the sense that each agent has a relatively large degree of freedom in the design of its controller and the form of each controller can be different for each agent.

The speed control input of agent $i \in \{2, 3\}$ is defined as follows,

$$v_i = (\alpha_i^* - \alpha_i)k \tag{4}$$

where $k > 0$ is a constant. The heading β_i, $i \in \{2, 3\}$ is defined by

$$\beta_i = \begin{cases} \alpha_i\gamma_i + \min(\phi_{i(i+1)}, \phi_{i(i-1)}), & \text{if } \vartheta_i \leq \pi \\ \alpha_i\gamma_i + \max(\phi_{i(i+1)}, \phi_{i(i-1)}), & \text{if } \vartheta_i > \pi \end{cases} \tag{5}$$

where ϑ_i is given by (1) and $0 < \gamma_i < 1$. We allow γ_i to be a function of time and so long as it obeys $0 < \gamma_i < 1$ then it can be chosen at *run-time* by the agent itself.

When $v_i > 0$ then agent i, $i \in \{2, 3\}$ moves toward the interior of the formation at some angle specified by $\alpha_i\gamma_i$. For example, if $\gamma_i = 1/2$ then with $v_i > 0$ the agent travels toward the interior of the formation along the bisection of α_i. If the speed of agent i is negative and $i \in \{2, 3\}$ then the agent travels toward the outside of the triangle.

The speed control input of agent 1 is defined as follows,

$$v_i = (r_{12} - r_{12}^*)c \tag{6}$$

where $c > 0$ is a constant. The heading β_1 is simply

$$\beta_1 = \phi_{12} \tag{7}$$

where now we highlight that agent 1 only attempts to maintain the desired distance r_{12}^* and not the desired angle α_1^*. This desired angle will subsequently be maintained as a consequence of the motion of agents 2 and 3.

Assumption 3. *The agents are not initially arranged such that $\alpha_1 = \pi$ and $\alpha_2 = \alpha_3 = 0$.*

If this assumption were violated then each agent would drive toward a neighbour agent until collision. Note other (initial) collinear arrangements with agent 1 not in between the other two agents are permitted.

The control law for agents 2 and 3 is relaxed as all we require is that the agents independently move toward the interior of the formation or toward the exterior of the formation dependent on the sign of their bearing constraint control error. We do not specify *a priori* an exact heading and indeed the agents can determine this independently at run-time given any criteria they desire as long as certain relaxed conditions are satisfied as shown above.

The control law for each agent is truly distributed and based only on the locally measured bearings and the locally measured range in the case of agent 1. We note that ϕ_{13} is not used and does not need to be sensed.

A. The Dynamics of the System

For simplicity, in this section we let $\gamma_i = 1/2$ for $i \in \{2, 3\}$. We note that this is simply done to ease the treatment and that analogous arguments to those presented subsequently hold when $0 < \gamma_i < 1$.

Consider first agent 1 with r_{12}^* specified. Then

$$\dot{r}_{12} = -(r_{12} - r_{12}^*)c - k(\alpha_2^* - \alpha_2)\cos(\frac{\alpha_2}{2}) \tag{8}$$

which is a superposition of agent 1's contribution $(r_{12}^* - r_{12})c$ to $\dot{r}_{12}$ and agent 2's contribution $-k(\alpha_2^* - \alpha_2)\cos(\alpha_2/2)$.

The formula for $\dot{\alpha}_i$, $i \in \{1,2,3\}$ will be derived now. Firstly, consider agent $i \in \{2,3\}$ with $v_i = (\alpha_i^* - \alpha_i)k$ and heading β_i defined as before (5) and note that $\mathcal{N}_i = \{(i+1),(i-1)\}$. If agents $i+1$ and $i-1$ are static, then

$$\begin{aligned} \dot{\alpha}_{i+1} &= -\frac{v_i}{r_{i(i+1)}}\sin(\frac{\alpha_i}{2}) \\ &= -\frac{k}{r_{i(i+1)}}\sin(\frac{\alpha_i}{2})(\alpha_i^* - \alpha_i) \end{aligned} \tag{9}$$

using the formula for the angular velocity in terms of the cross-radial component of the velocity of agent i. The sign is negative since if α_i increases, i.e. if $(\alpha_i^* - \alpha_i)k > 0$, then α_{i+1} decreases. Similarly

$$\dot{\alpha}_{i-1} = -\frac{k}{r_{i(i-1)}}\sin(\frac{\alpha_i}{2})(\alpha_i^* - \alpha_i) \tag{10}$$

In addition, $\dot{\alpha}_i$ is affected directly by $\alpha_i^* - \alpha_i$. Note that $\sum_i \dot{\alpha}_i = 0$. Thus, we have

$$\begin{aligned} \dot{\alpha}_i &= \frac{k(\alpha_i^* - \alpha_i)}{r_{i(i+1)}}\sin(\frac{\alpha_i}{2}) + \frac{k(\alpha_i^* - \alpha_i)}{r_{i(i-1)}}\sin(\frac{\alpha_i}{2}) \\ &= \frac{r_{i(i+1)} + r_{i(i-1)}}{r_{i(i+1)}r_{i(i-1)}}\sin(\frac{\alpha_i}{2})(\alpha_i^* - \alpha_i)k \end{aligned} \tag{11}$$

when agents $i+1$ and $i-1$ are static.

Now hold agents $i \in \{2,3\}$ stationary and note that the motion of agent 1 is along the bearing ϕ_{12} with a speed $(r_{12} - r_{12}^*)$. If $r_{12} > r_{12}^*$ then agent 1 is moving toward agent 2 and vice-versa if $r_{12} < r_{12}^*$. The motion of agent 1 also affects α_1 and α_3 but not α_2. We then have

$$\dot{\alpha}_3 = -\frac{c}{r_{13}}\sin(\alpha_1)(r_{12} - r_{12}^*) \tag{12}$$

Immediately we have

$$\dot{\alpha}_1 = \frac{c}{r_{13}}\sin(\alpha_1)(r_{12} - r_{12}^*) \tag{13}$$

with agents $i \in \{2,3\}$ stationary. Now, when all agents move with a motion governed by their individual control laws we have

$$\begin{aligned} \dot{\alpha}_1 &= \frac{c}{r_{13}}\sin(\alpha_1)(r_{12} - r_{12}^*) - \frac{k}{r_{13}}\sin(\frac{\alpha_3}{2})(\alpha_3^* - \alpha_3) \\ &\quad -\frac{k}{r_{12}}\sin(\frac{\alpha_2}{2})(\alpha_2^* - \alpha_2) \qquad (14) \\ \dot{\alpha}_2 &= \frac{r_{12} + r_{23}}{r_{12}r_{23}}\sin(\frac{\alpha_2}{2})(\alpha_2^* - \alpha_2)k \\ &\quad -\frac{1}{r_{23}}\sin(\frac{\alpha_3}{2})(\alpha_3^* - \alpha_3)k \qquad (15) \\ \dot{\alpha}_3 &= \frac{r_{13} + r_{23}}{r_{13}r_{23}}\sin(\frac{\alpha_3}{2})(\alpha_3^* - \alpha_3)k \\ &\quad -\frac{c}{r_{13}}\sin(\alpha_1)(r_{12} - r_{12}^*) - \frac{k}{r_{23}}\sin(\frac{\alpha_2}{2})(\alpha_2^* - \alpha_2) \qquad (16) \end{aligned}$$

Now for future notational brevity let

$$f_{ij} = \frac{k}{r_{ij}}\sin(\frac{\alpha_j}{2}) \tag{17}$$

and

$$g_i = \frac{r_{i(i+1)} + r_{i(i-1)}}{r_{i(i+1)}r_{i(i-1)}}\sin(\frac{\alpha_i}{2})k \tag{18}$$

and

$$h_1 = \frac{c}{r_{13}}\sin(\alpha_1) \tag{19}$$

where we note $g_i \geq 0$ with $i \in \{2,3\}$, $h_1 \geq 0$ and $f_{ij} \geq 0$ for all $i \in \{1,2,3\}$ and $j \in \{2,3\}$ when $\alpha_i \in [0,\pi]$, $\forall i$. Note that $g_i = f_{ji} + f_{ki}$ for distinct $i,j,k \in \{1,2,3\}$.

The following system of differential equations is obtained

$$\dot{\alpha} = \begin{bmatrix} h_1 & f_{12} & f_{13} \\ 0 & -g_2 & f_{23} \\ -h_1 & f_{32} & -g_3 \end{bmatrix} \left(\begin{bmatrix} r_{12} \\ \alpha_2 \\ \alpha_3 \end{bmatrix} - \begin{bmatrix} r_{12}^* \\ \alpha_2^* \\ \alpha_3^* \end{bmatrix} \right) \tag{20}$$

where

$$\alpha = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}^{\mathrm{T}} \tag{21}$$

is defined on a 2-simplex in α-space, denoted by $\mathcal{M}_\alpha$, with vertices $\alpha = [\pi\ 0\ 0]^\top$, $\alpha = [0\ \pi\ 0]^\top$ and $\alpha = [0\ 0\ \pi]^\top$. The other differential equation of interest concerns $\dot{r}_{12}$ and is given by (8). Define the following control errors

$$\begin{aligned} e_1 &= r_{12} - r_{12}^* &(22) \\ e_2 &= \alpha_2 - \alpha_2^* &(23) \\ e_3 &= \alpha_3 - \alpha_3^* &(24) \end{aligned}$$

with $\mathbf{e} = [e_1\ e_2\ e_3]^\top$. Note that $\mathbf{e} \to 0$ implies the formation reaches its desired shape since $e_2 \to 0$ and $e_3 \to 0$ implies $\alpha_1 \to \alpha_1^*$. In addition, $\mathbf{e} \to 0$ implies the formation comes to rest since $\mathbf{e} \to 0$ implies $v_i \to 0$ for all $i \in \{1,2,3\}$. The dynamics of $\mathbf{e} = [e_1\ e_2\ e_3]^\top$ are

$$\dot{\mathbf{e}} = \begin{bmatrix} -1 & k\cos(\frac{\alpha_2}{2}) & 0 \\ 0 & -g_2 & f_{23} \\ -h_1 & f_{32} & -g_3 \end{bmatrix} \mathbf{e} = \mathbf{F}(\mathbf{e})\mathbf{e} \tag{25}$$

where $\cos(\frac{\alpha_2}{2}) \geq 0$ when $\alpha_2 \in [0,\pi]$.

From the Peano Existence Theorem we know that solutions of (25) exist locally, given any initial condition $\mathbf{e}(t_0)$ with Assumption 1 holding, on some time interval $[t_0, t_1]$ where $t_1 = t_1(\mathbf{e}(t_0))$.

Lemma 1. *Suppose Assumptions 1, 2 and 3 hold. Suppose $r_{13} \geq c \cdot k$. Then the eignevalues of*

$$\mathbf{F}(\mathbf{e}) = \begin{bmatrix} -1 & k\cos(\frac{\alpha_2}{2}) & 0 \\ 0 & -g_2 & f_{23} \\ -h_1 & f_{32} & -g_3 \end{bmatrix} \tag{26}$$

have strictly negative real parts if and only if $\alpha_i \notin \{0,\pi\}$, $\forall i \in \mathcal{V}$ with $\alpha_1 + \alpha_2 + \alpha_3 = \pi$.

Proof: The characteristic polynomial of $\mathbf{F}(\mathbf{e})$ is given by

$$a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 \tag{27}$$

where we abbreviate the coefficient of λ^i by a_i. Note that $a_3 = 1$ and

$$a_2 = -\mathrm{tr}\,(\mathbf{F}(\mathbf{e})) \tag{28}$$
$$a_0 = -\det\,(\mathbf{F}(\mathbf{e})) \tag{29}$$

and some simple computations give

$$a_1 = g_2 + g_3 + f_{12}f_{13} + f_{12}f_{23} + f_{13}f_{32} \tag{30}$$

Routh-Hurwitz states that the roots of this polynomial will have strictly negative real parts if and only if $a_i > 0$ for all $i \in \{0, \ldots, 3\}$ and $a_2 a_1 > a_3 a_0$.

We then have

$$\begin{aligned} \det\,(\mathbf{F}(\mathbf{e})) &= -g_2 g_3 - h_1 f_{23}\cos(\frac{\alpha_2}{2}) + f_{23}f_{32} \\ &= -(f_{12}+f_{32})(f_{13}+f_{23}) \\ &\quad -h_1 f_{23}\cos(\frac{\alpha_2}{2})k + f_{23}f_{32} \\ &= -\left(f_{12}f_{13} + f_{12}f_{23} + f_{13}f_{32}\right) \\ &\quad -h_1 f_{23}\cos(\tfrac{\alpha_2}{2})k \\ &< 0 \end{aligned} \tag{31}$$

whenever $\alpha_i \notin \{0, \pi\}$, $\forall i \in \mathcal{V}$. Similarly, $\mathrm{tr}\,(\mathbf{F}(\mathbf{e})) = -[1 + g_2 + g_3] < 0$. Moreover, $a_1 > 0$ whenever $\alpha_i \notin \{0, \pi\}$, $\forall i \in \mathcal{V}$ and $a_3 = 1 > 0$. Thus, $a_i > 0$ for all i and it remains to show that $a_2 a_1 > a_3 a_0$. Therefore,

$$\begin{aligned} a_2 a_1 &= (1 + g_2 + g_3)(g_2 + g_3 + f_{12}f_{13} + f_{12}f_{23} + f_{13}f_{32}) \\ &\geq (g_2 + g_3 + f_{12}f_{13} + f_{12}f_{23} + f_{13}f_{32}) \\ &= g_2 + g_3 + a_0 - h_1 f_{23}\cos(\tfrac{\alpha_2}{2})k \\ &\geq f_{23} + a_0 - h_1 f_{23}\cos(\tfrac{\alpha_2}{2})k \\ &\geq a_0 \end{aligned} \tag{32}$$

with strict inequality whenever $\alpha_i \notin \{0, \pi\}$, $\forall i \in \mathcal{V}$. In the last step we used the fact that when $r_{13} \geq c \cdot k$ it follows that $h_1 f_{23}\cos(\frac{\alpha_2}{2})k \leq f_{23}$ as required. Thus, with $r_{13} \geq c \cdot k$ it follows from Routh-Hurwitz that $\mathbf{F}(\mathbf{e})$ has eigenvalues with strictly negative real parts if and only if $\alpha_i \notin \{0, \pi\}$, $\forall i$. $\square$

The following result concerns the equilibrium points of (25).

Theorem 1. *Suppose Assumptions 1, 2 and 3 hold. Then any point for which $\mathbf{e} = \mathbf{0}$ is an equilibrium point. Conversely, consider any equilibrium point at which $r_{13} \geq c \cdot k$ and $\alpha_1 \neq \pi$. Suppose further that $\alpha_1^* \neq \pi$. Then at this equilibrium point $\mathbf{e} = \mathbf{0}$.*

Proof: Sufficiency $\dot{\mathbf{e}} = \mathbf{0} \Leftarrow \mathbf{e} = \mathbf{0}$ is obvious and holds regardless of α_i. Thus, we focus on necessity $\dot{\mathbf{e}} = \mathbf{0} \Rightarrow \mathbf{e} = \mathbf{0}$ when $\alpha_1 \neq \pi$.

Firstly, when $r_{13} \geq c \cdot k$ then from Lemma 1, $\mathbf{F}(\mathbf{e})$ is non-singular if and only if $\alpha_i \notin \{0, \pi\}$. Thus, $\dot{\mathbf{e}} = \mathbf{0} \Rightarrow \mathbf{e} = \mathbf{0}$ if $\alpha_i \notin \{0, \pi\}$, $\forall i \in \mathcal{V}$.

Now there are three cases captured by $\alpha_i \notin \{0, \pi\}$ characterised by $\alpha_j = \pi$ for each $j \in \{1, 2, 3\}$. We have excluded one where $\alpha_1 = \pi$ in the Theorem statement. We consider the other two cases at once where $\alpha_1 = 0$ and $\alpha_i \in \{0, \pi\}$ for $i \in \{2, 3\}$ with $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. Assumption 3 is not violated. However, $\mathbf{F}(\mathbf{e})$ is clearly singular. Moreover,

$$\dot{\alpha} = \left\{ \begin{array}{ll} \begin{bmatrix} 0 & f_{12} & 0 \\ 0 & -g_2 & 0 \\ 0 & f_{32} & 0 \end{bmatrix} \mathbf{e}, & \text{if } \alpha_2 = \pi \\ \begin{bmatrix} 0 & 0 & f_{13} \\ 0 & 0 & f_{23} \\ 0 & 0 & -g_3 \end{bmatrix} \mathbf{e}, & \text{if } \alpha_3 = \pi \end{array} \right. \tag{33}$$

Note the fact $\dot{\mathbf{e}} \neq 0 \Leftrightarrow \dot{\alpha} \neq 0$. Suppose $e_i \neq 0$ when $\alpha_i = \pi$ for $i \in \{2, 3\}$. Then $\dot{\alpha}_i \neq 0$ for all $i \in \mathcal{V}$ and $\dot{\mathbf{e}} \neq 0 \Leftrightarrow \dot{\alpha} \neq 0$ and thus $\dot{\mathbf{e}} \neq 0$. It thus follows that $\mathbf{e} \neq \mathbf{0} \Rightarrow \dot{\mathbf{e}} \neq \mathbf{0}$ or $\dot{\mathbf{e}} = \mathbf{0} \Rightarrow \mathbf{e} = \mathbf{0}$ if $\alpha_1 \neq \pi$. $\square$

If $\alpha_1 = \pi$ then each agent will drive toward a neighbour agent until collision (except in the trivial case where $\alpha_1^* = \pi$ in which case $\mathbf{e} = \mathbf{0}$). Thus, when $\alpha_1 = \pi$ and $\alpha_1^* \neq \pi$ it follows $\dot{\mathbf{e}} = \mathbf{0}$ and $\mathbf{e} \neq \mathbf{0}$; i.e. the formation will remain collinear for some time (until collision). Hence, $\alpha_1 \neq \pi$ is trivially a necessary condition for $\dot{\mathbf{e}} = 0 \Leftrightarrow \mathbf{e} = \mathbf{0}$. Given Assumption 3 we show later that if $\alpha_1^* \neq \pi$ then $\alpha_1 \neq \pi$ is never achieved and thus $\dot{\mathbf{e}} = 0 \Leftrightarrow \mathbf{e} = \mathbf{0}$ holds in any case.

The next result is the main result of this paper.

Theorem 2. *Suppose Assumptions 1, 2, 3 hold. Suppose further that $r_{13}^* \geq c \cdot k$. If $\alpha_i^* \in (0, \pi)$ then there exists a neighbourhood $\mathcal{U}$ of $\mathbf{e} = \mathbf{0}$ within which solutions to (25) exist for all $t \in [0, \infty)$ and within which solutions converge asymptotically and at an exponential rate to zero $\mathbf{e} = \mathbf{0}$.*

Note that r_{12}^* and α_i^* for all $i \in \mathcal{V}$ are specified according to Assumptions 1 and 2. However, it follows that r_{ij}^* for all $i \neq j$ are defined as a consequence.

Proof: We have

$$\dot{\mathbf{e}} = \mathbf{F}(\mathbf{e})\mathbf{e} \tag{34}$$

and we know $\mathbf{F}(\mathbf{e})$ is nonsingular whenever $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$. Linearization of (34) about the point $\mathbf{e} = 0$ leads to

$$\dot{\mathbf{e}} = \mathbf{A}(\alpha^*)\mathbf{e} \tag{35}$$

where $\mathbf{A}(\alpha^*)$ is a constant matrix and denotes the gradient of $\mathbf{F}(\mathbf{e})\mathbf{e}$ with respect to $\mathbf{e}$ and evaluated at $\mathbf{e} = 0$. Note that because of the particular form of (34) we have $\mathbf{A}(\alpha^*) = \mathbf{F}(\mathbf{e})|_{\alpha_i = \alpha_i^*, r_{12} = r_{12}^*}$. Now it follows that $\mathbf{A}(\alpha^*)$ is stable, i.e. $\mathbf{A}(\alpha^*)$ has strictly negative real eigenvalues, for all $\alpha_i^* \in (0, \pi)$. This follows from the proof of Lemma 1. Now within a neighborhood of the origin $\mathcal{U}$ it follows from the Hartman-Grobman theorem that solutions of (25) exist on $t \in [0, \infty)$ and converge at an exponential rate when $\alpha_i^* \in (0, \pi)$. $\square$

When the desired formation is a line then linearization fails and $\mathbf{A}(\alpha^*)$ is singular. However, we conjecture that at least local convergence exists when $\alpha_i^* \in \{0, \pi\}$ given Assumptions 1, 2 and 3 but we do not explore this case further.

IV. Examples of Triangular Formation Control with the Relaxed Control Law

The proof of stability was given when $\gamma_i = 1/2$ for $i \in \{2, 3\}$. Analogous statements can be made (with some additional cumbersome notation) concerning stability with $0 < \gamma_i < 1$. We omit these statements for brevity but in the illustrative examples in this section we allow $0 < \gamma_i < 1$ for $i \in \{2, 3\}$.

A. *Triangle to Triangle Formation with Time-Varying Random γ_i Values*

The first example shows how the formation converges to a desired triangle given a random initial triangle configuration. The desired formation is an equilateral triangle with $r^*_{12} = 25$. We randomly change γ_i, $\forall i$ every ε seconds for some small $\varepsilon > 0$. Each γ_i is randomly chosen to be within $(0, 1)$ with a uniform distribution. The formation motion is illustrated in Figure 1.

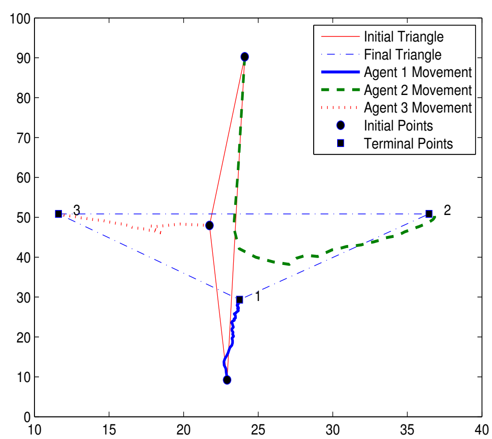

Fig. 1. The motion of the formation with a desired terminal equilateral angle constraint and $r^*_{21} = 25$. We randomly change γ_i, $\forall i$ every ε seconds for some small $\varepsilon > 0$..

The initial position of the three agents are randomly distributed in $\mathcal{M}_\alpha$ and Figure 1 illustrates the trajectories of the formation as it converges to the desired shape.

The convergence of the angle errors to zero and the distance error between agent 2 and 1 is shown in Figure 2.

The jittery behaviour is a result of the randomly varying γ_i values. Indeed, the agents are varying their headings quite drastically which illustrates the relaxed nature of the control law and its robustness. In the next example we will hold the γ_i values at some arbitrary constant value to illustrate the difference in the behaviour of the formation.

B. *Triangle to Triangle Formation with Fixed Arbitrary γ_i Values*

The second example is similar to the first and shows how the formation converges to a desired triangle given a random initial triangle configuration. The desired formation is an equilateral triangle with $r^*_{12} = 25$. We set $\gamma_i = 0.1$ for $i \in \{2, 3\}$. The formation motion is illustrated in Figure 3.

The initial position of the three agents are randomly distributed in $\mathcal{M}_\alpha$ and Figure 3 illustrates the trajectories of the formation as it converges to the desired shape.

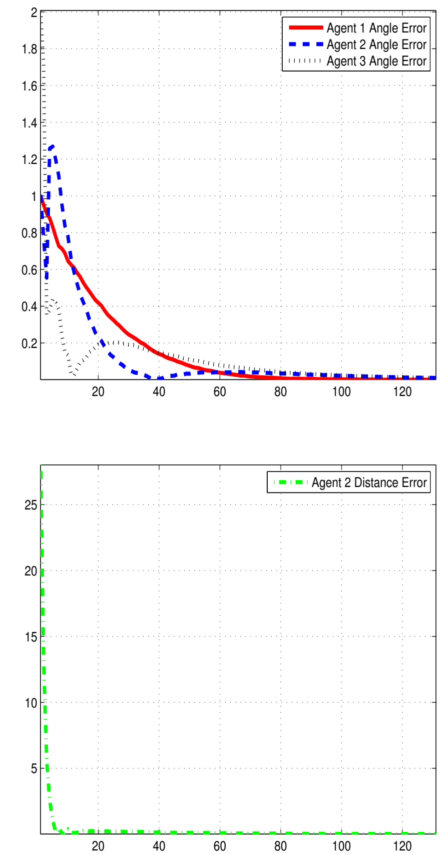

Fig. 2. The error convergence for the first example with randomly changing γ_i values.

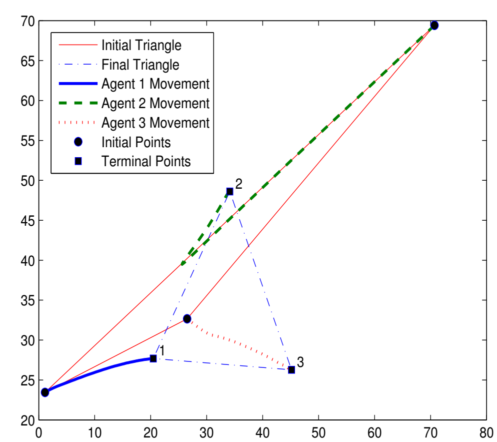

Fig. 3. The motion of the formation with a desired terminal equilateral angle constraint and $r^*_{21} = 25$. We set $\gamma_i = 0.1$ for $i \in \{1, 2\}$ and $\gamma_3 = 0.7$.

The convergence of the angle errors to zero and the distance error between agent 2 and 1 is shown in Figure 4.

The motion is smoother in this case since the heading

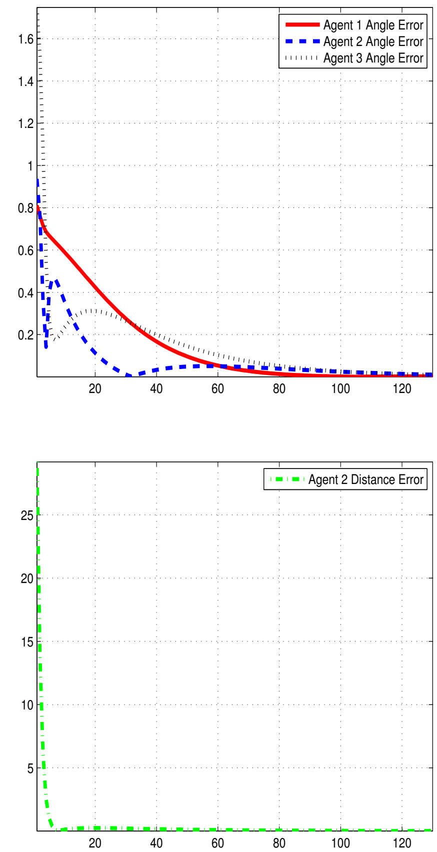

Fig. 4. The error convergence for the first example with $\gamma_i = 0.1$ for $i \in \{2, 3\}$.

factors γ_i are not constant jumping between random values. However, in both cases the formation converges illustrating the novelty of the relaxed nature of the control scheme proposed in this work.

V. Conclusion

A distributed control law for triangular formation control with a mixture of bearing and range measurements and relative pair-wise inter-agent angle constraints and a single range constraint is introduced. A convergence result is established which ensures the desired formation configuration is locally asymptotically stable. We noted that there is only one equilibrium point in the error space and thus if one were to show there were no complicated error dynamics (e.g. periodic orbits etc.) than global stability would follow. We do not explore this concept further in this paper.

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