Conditions and strategies for uniqueness of the solutions to cooperative localization and mapping problems using rigidity theory

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***Abstract*— In this paper, we use the rigidity theory to address two problems encountered in cooperative localization and mapping. First, we consider the problem of map merging in scenarios where a group of mobile agents explore an environment. We establish conditions for the agents to be able to exchange the environmental information that they have gathered in their own local coordinate frames. We relate these conditions to the sensing capabilities of the agents. Second, we study a scenario where a group of mobile agents in a network need to localize their positions. It is assumed that there are not enough measurements to achieve this task at any time instance. We propose a coordinated motion strategy that enables the agents to achieve this goal over a period of time. Numerical simulations are provided to demonstrate the results.**

I. Introduction

The problem of deploying sensor platforms for monitoring and exploration purposes has been studied in different scenarios over the last few years, e.g. habitat monitoring [1], forest fire detection [2], search and rescue [3], and exploration and map building [4].

Often the sensors do not have access to a global coordinate frame. This is the case when the sensors operate in settings where there is no reliable access to the global positioning system (GPS), as in contested [5] or dense urban environments [6], [7]. As a result, the spatial information gathered by each sensor platform is not readily useful to other sensors. Thus, as a first step to harness the potential of cooperation in such scenarios, the sensor platforms should be able to "translate" the information gathered by all the sensors to their own local coordinate frames. This is one of the main challenges of cooperative localization and mapping. The problem of cooperative localization and mapping has been the target of many recent studies, e.g. [8], [9], [10], [11], [12]. However, to the best of our knowledge, minimum requirements for the success of cooperative localization algorithms are not fully understood.

In this paper, we particularly focus on understanding the conditions under which each of the sensors can translate the information gathered by the other sensors to their own coordinate frame based on the type and the number of measurements collected by each of the agents. Moreover, we establish certain motion strategies that ensure the satisfaction of the aforementioned conditions. In other words, based on the sensing capabilities of the sensors, we determine the information type and the number of times that such information needs to be exchanged between the sensors, so that they can use each others' measurements in the future.

We draw from distance-based rigidity theory to establish the results presented in this paper. The application of rigid graph theory to distanced-based localization problems in sensor networks has been the focus of many recent studies [13], [14], [15], [16], [17]. Most of these studies focused on characterizing rigid graph properties that allow unique localization solutions for different networks based on the type of measurements that individual agents can collect [18], [19], [20], [13], [14], [16].

Here, we focus on the scenarios that the sensors can only measure their distances to other sensors or points of interest. First, we introduce the necessary preliminaries in Section II. In Section III, we study the case where multiple mobile platforms explore the environment and determine the necessary conditions so that each sensor pair can use the exchange information. Then in Section IV, we study what conditions need to be satisfied so that sensors can compute the position of all the other sensors in their own local coordinate frame by employing local motions. We propose a motion strategy for each sensing platform that ensure the satisfaction of these conditions. We demonstrate the applicability of the results proposed in this paper via numerical examples in Section V. Concluding remarks are presented in Section VI.

II. Preliminaries

A framework is a graph $\mathcal{G} = (\mathbf{V}(\mathcal{G}), \mathbf{E}(\mathcal{G}))$ with vertices in a set $\mathbf{V}(\mathcal{G}) = \{1, 2, \ldots, n\}$ and the edge set $\mathbf{E}(\mathcal{G})$ together with a map $\pi(i) : \mathbf{V}(\mathcal{G}) \to \mathbb{R}^2$. Denote such a framework by the ordered pair $(\mathcal{G}, \pi)$ with $\mathcal{G}$ being its *underlying graph*. For simplicity of notation denote $\pi(i)$ as π_i which is the coordinate vector associated with vertex $i \in \mathbf{V}(\mathcal{G})$. Suppose a set of positive real numbers (representing inter-vertex distances) $\mathcal{D} = \{d_{ij} : \{i, j\} \in \mathbf{E}(\mathcal{G})\}$ is defined. The framework is a realization if it results in $\|\pi_i - \pi_j\| = d_{ij}$ for any $\{i, j\} \in \mathbf{E}(\mathcal{G})$. The two frameworks $(\mathcal{G}, \pi)$ and $(\mathcal{G}, \bar{\pi})$ are equivalent if $\|\pi_i - \pi_j\| = \|\bar{\pi}_i - \bar{\pi}_j\|$ for any $\{i, j\} \in \mathbf{E}(\mathcal{G})$. The two frameworks $(\mathcal{G}, \pi)$ and $(\mathcal{G}, \bar{\pi})$ are congruent if $\|\pi_i - \pi_j\| = \|\bar{\pi}_i - \bar{\pi}_j\|$ for all pairs i and j whether or not $\{i, j\} \in \mathbf{E}(\mathcal{G})$. This is equivalent to saying that $(\mathcal{G}, \pi)$ can be obtained from $(\mathcal{G}, \bar{\pi})$ by an isometry of $\mathbb{R}^2$, i.e. a combination of translation, rotation, and reflection.

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This work is supported a McKenzie fellowship, an ETH fellowship, the Australian Research Council (LP130100605), and Rubicon Water Pty Ltd.

Definition 1 (Rigidity): A network $(\mathcal{G}, \pi)$ is *rigid* if there exists a sufficiently small positive ϵ such that if $(\mathcal{G}, \bar{\pi})$ is equivalent to $(\mathcal{G}, \pi)$ and $\|\pi_i - \bar{\pi}_i\| \leq \epsilon$ for all $i \in \mathbf{V}(\mathcal{G})$ then $(\mathcal{G}, \bar{\pi})$ is congruent to $(\mathcal{G}, \pi)$. Intuitively, a rigid network is one that cannot flex.

There exist rigid networks $(\mathcal{G}, \pi)$ and $(\mathcal{G}, \bar{\pi})$ which are equivalent but not congruent.

Definition 2 (Global Rigidity): A network $(\mathcal{G}, \pi)$ is *globally rigid* if every network $(\mathcal{G}, \bar{\pi})$ which is equivalent to $(\mathcal{G}, \pi)$ is also congruent to $(\mathcal{G}, \pi)$.

Generally, rigidity and global rigidity are generic properties of networks. This means that the rigidity (global rigidity) of a generic realization of a graph $\mathcal{G}$ depends (almost) only on the graph $\mathcal{G}$ and not the particular realization.

It should be noted that a framework is an abstraction of any network with vertices in some coordinate frame. However, even given the graph and distance set of a globally rigid framework, there is not enough information to position the framework absolutely in $\mathbb{R}^2$. In fact, the framework can only be positioned to within a translation, rotation, or reflection. Eliminating this non-uniqueness requires further knowledge, typically the positions of at least three vertices. The problem of assigning positions to the vertices of a network that satisfy some given distance constraints is termed the *localization problem* and formally is defined below.

Definition 3 (Distance Based Network Localization): A network with the underlying graph $\mathcal{G}$ is said to be localizable if π_i, $\forall i \in \mathbf{V}(\mathcal{G})$ can be determined uniquely given the positions of some of the sensors π_j, $j \in \mathbf{V}(\mathcal{G})'$, $\mathbf{V}(\mathcal{G})' \subset \mathbf{V}(\mathcal{G})$, and the set of distances d_{ij}.

In the rest of this paper, we relate the problems that arise in cooperative localization and mapping scenarios to the distance based network localization problem defined above. This enables us to provide conditions on when one can solve these problems.

III. Cooperative Mapping

Consider N mobile agents indexed by the set $\mathcal{N} = \{1, \ldots, N\}$ where the position of each agent i in a global coordinate frame at some time t is represented by $p_i(t)$. Moreover, assume there are m stationary environmental features, termed *landmarks*, at positions $\ell_{i'} \in \mathbb{R}^2$, $i' \in \{1, \ldots, m\}$. It is assumed that the each agent i does not have access either to the global values $\ell_{i'}$, $i' \in \{1, \ldots, m\}$ or its position $p_i(t)$. However, at any given time it is assumed that each agent i has access to its position in its own local coordinate frame. Denote this position as $p_{i,i}(t)$. The goal of each agent is to generate the map of the environment in its own coordinate frame. This entails agent i determining the position of the landmarks in $\{1, \ldots, m\}$ in its own coordinate frame. We denote the position of landmark i' in the coordinate frame of agent i by $\ell_{i,\mu_i(i')}$ where $\mu_i(i')$ is the label chosen for landmark i' by agent i. We assume that the $\mu_i(i') \neq \mu_j(i')$, i.e. the agents do not have access to a global labeling system for the landmarks. Moreover, it is assumed that each landmark i' is discovered by agent i if $\|p_i(t) - \ell_{i'}\| \leq r_d$ where $r_d > 0$ is termed the *detection radius*. Here, we note that in this paper we do not concern ourselves with the exact method that is employed to estimate the position of the landmark in the coordinate frame of the agent. The only important point is that such estimation is possible. For example, one can refer to [21], [22] for more information on methods and techniques for estimating the position of a landmark using distance information only.

Let $\mathcal{A}_i(t)$ be the set of all $\ell_{i,\mu_i(i')}$ such that landmark i' has been discovered by agent i until time t. Moreover, if at some time t, $\|p_i(t) - p_j(t)\| \leq r_d$ then the following two actions occur;

(i) Agents i and j add the tuples $(t, p_{i,i}(t), \|p_i(t) - p_j(t)\|)$ and $(t, p_{j,j}(t), \|p_i(t) - p_j(t)\|)$ to local sets $\mathcal{B}_{ij}(t)$ and $\mathcal{B}_{ji}(t)$, respectively.

(ii) Agent i transmits $\mathcal{A}_i(t)$ and $\mathcal{B}_{ij}(t)$ to j and vice versa.

Note that at any given time $|\mathcal{A}_i(t)|$ is the number of the landmarks discovered by agent i and $|\mathcal{B}_{ij}(t)| = |\mathcal{B}_{ji}(t)|$ is the number of the times that agents i and j have encountered and exchanged their local maps. In the first problem considered in this paper we are interested in understanding the conditions that allow agent i to learn about the landmarks that it has not yet visited through exchanges of the type described above with other agents. We have the following problem.

Problem 1: Let $\|p_i(t) - p_j(t)\| \leq r_d$ at some time t. What are the conditions so that $\ell_{i,\mu_i(i')}$ can be computed and added to $\mathcal{A}_i(t)$ for all i' where $\ell_{j,\mu_j(i')} \in \mathcal{A}_j(t)$?

First we make the following assumptions.

Assumption 1: The agents have access to a global time reference[1].

Assumption 2: At any given time t, each agent i has access to its position in its own local coordinate frame. In other words, $p_{i,i}(t)$ is known for all t.

To be able to address the problem we define the following graphs and frameworks for each agent i. Let $\mathcal{X}_i(t)$ be the complete graph over the vertex set $\{i' : \ell_{i,\mu_i(i')} \in \mathcal{A}_i(t)\}$, and $\mathcal{D}_i(t)$ be a set of distances associated with the edges of $\mathcal{X}_i(t)$, i.e.

$$\mathcal{D}_i = \{\|\ell_{i,\mu(i')} - \ell_{i,\mu(j')}\| : \ell_{i,\mu(i')}, \ell_{i,\mu(j')} \in \mathcal{A}_i(t)\}.$$

Similarly, let $\mathcal{H}_{ij}(t)$ be a graph with the vertex set

$$\mathbf{V}(\mathcal{H}_{ij}(t)) = \{i_\tau : \forall \tau \in \mathcal{T}_{i,ij}\} \cup \{j_\tau : \forall \tau \in \mathcal{T}_{j,ij}\},$$

where

$$\mathcal{T}_{i,ij} = \{\tau : \exists (\tau, p, d) \in \mathcal{B}_{ij}(t)\}$$

and

$$\mathcal{T}_{j,ij} = \{\tau : \exists (\tau, p, d) \in \mathcal{B}_{ji}(t)\}.$$

Note that because of Assumption 1 $\mathcal{T}_{i,ij} = \mathcal{T}_{j,ij}$. Thus we can define the edge set as

$$\mathbf{E}(\mathcal{H}_{ij}(t)) = \{\{i_\tau, j_\tau\} : \tau \in \mathcal{T}_{i,ij}\}.$$

Moreover, let the set $\mathcal{D}_{ij}(t)$ (which is equivalent to $\mathcal{D}_{ji}(t)$) be the set of distance measurements collected at each encounter between i and j. These distances correspond to the edges of $\mathcal{H}_{ij}(t)$.

[1] This assumption is not crucial, but it greatly simplifies the notation and thus is adopted.

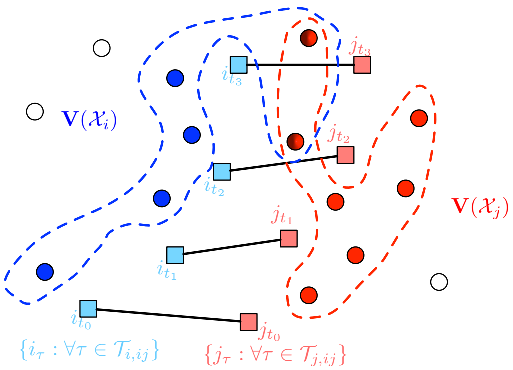


Fig. 1: The blue and red circles denote the set of landmarks discovered by i and j respectively. The squares correspond to the cases where $\|p_i(t) - p_j(t)\| \leq r_d$. The solid lines correspond to the distance measurements, i.e. $\mathbf{E}(\mathcal{H}_{ij}(t))$.

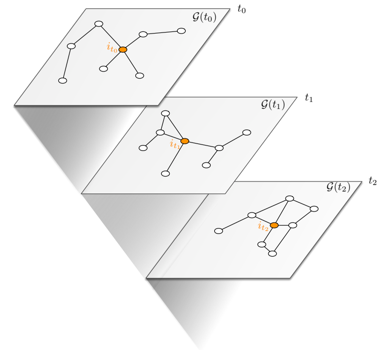


Fig. 2: An example for $\mathcal{G}(t)$ for $t = t_0, t_1, t_2$. Note that node i_{t_k} is the vertex label for agent i at time t_k.

Assuming that the agents i and j have an encounter at some time t, i.e. $\|p_i(t) - p_j(t)\| \leq r_d$, After conducting the distance measurements and the information exchange of the type outlined above, both agents will have access to a graph $\mathcal{G}_{ij}(t)$, where

$$\mathbf{V}(\mathcal{G}_{ij}(t)) = \mathbf{V}(\mathcal{X}_i(t)) \cup \mathbf{V}(\mathcal{X}_j(t)) \cup \mathbf{V}(\mathcal{H}_{ij}(t)), \tag{1}$$

$$\begin{aligned}\mathbf{E}(\mathcal{G}_{ij}(t)) = {} & \mathbf{E}(\mathcal{X}_i(t)) \cup \mathbf{E}(\mathcal{X}_j(t)) \cup \mathbf{E}(\mathcal{H}_{ij}(t)) \\ & \cup \{\{l, i_\tau\} : l \in \mathbf{V}(\mathcal{X}_i(t)), \tau \in \mathcal{T}_{i,ij}\} \\ & \cup \{\{l, j_\tau\} : l \in \mathbf{V}(\mathcal{X}_j(t)), \tau \in \mathcal{T}_{i,ij}\}.\end{aligned} \tag{2}$$

An example for the vertex sets described above is presented in Fig. 1.

Additionally, the agents have access to a distance set $\mathcal{D}_{ij}(t)$ with all the distances associated with the edges in $\mathbf{E}(\mathcal{G}_{ij}(t))$. It can be observed that agent i knows all the positions associated with the vertices in

$$\mathcal{V}'_{ij}(t) = \mathbf{V}(\mathcal{X}_i(t)) \cup \{i_\tau : \forall \tau \in \mathcal{T}_{i,ij}\} \tag{3}$$

in its own coordinate frame. Similarly, agent j has access to the positions of the vertices in $\mathcal{V}'_{ij}(t)$ where

$$\mathcal{V}'_{ji}(t) = \mathbf{V}(\mathcal{X}_j(t)) \cup \{j_\tau : \forall \tau \in \mathcal{T}_{i,ij}\}. \tag{4}$$

At any given time if agent i is capable of solving the localization problem associated with a network with the underlying graph $\mathcal{G}_{ij}(t)$ and distance set $\mathcal{D}_{ij}(t)$ then it can find the position of all the landmarks that agent j has visited. Thus, addressing Problem 1 is equivalent to finding the conditions for unique localization of a network with the underlying graph $\mathcal{G}_{ij}(t)$ and distance set $\mathcal{D}_{ij}(t)$ in the local coordinate frames of agents i and j. We have the following result.

Proposition 1: At time t, for all i' such that $\ell_{j,\mu_j(i')} \in \mathcal{A}_j(t)$, $\ell_{i,\mu_i(i')}$, can be computed and added to $\mathcal{A}_i(t)$ if $|\mathcal{B}_{ij}(t)| \geq 4$ and $\nexists \{(t,p,d), (\bar{t},p,\bar{d})\} \subseteq \mathcal{B}_{ij}(t)$ with $t \neq \bar{t}$.

Proof: Consider a network with the underlying graph $\mathcal{G}_{ij}(t)$ and distance set $\mathcal{D}_{ij}(t)$. It is uniquely localizable in the coordinate frame of agent i if it is globally rigid and $|\mathcal{V}'_{ji}(t)| \geq 3$. It can be seen that $\mathcal{G}_{ij}(t)$ is composed of two complete graphs on $\mathcal{V}'_{ji}(t)$ and $\mathcal{V}'_{ij}(t)$ given in (3) and (4) along with the edges in $\mathbf{E}(\mathcal{H}_{ij}(t))$. The complete graphs on $\mathcal{V}'_{ji}(t)$ and $\mathcal{V}'_{ij}(t)$ correspond to globally rigid networks. Furthermore, we know from [23] that the resulting network obtained from connecting two globally rigid networks is globally rigid if there are at least four edges that do not share any vertices connect them to each other. Thus, for $\mathcal{G}_{ij}(t)$ to be associated with a globally rigid network, first, $|\mathbf{E}(\mathcal{H}_{ij}(t))| \geq 4$. Second, they should not be incident on a common vertex or equivalently on two or more co-located vertices in $\mathbf{V}(\mathcal{H}_{ij}(t))$. In other words, there must not be two (t,p,d) and $(\bar{t},p,\bar{d})$ with $t \neq \bar{t}$ such that both (t,p,d) and $(\bar{t},p,\bar{d})$ are in $\mathcal{B}_{ij}(t)$. ■

IV. Cooperative Localization

In this section, we consider the scenario where a group of mobile agents, $\mathcal{N} = \{1, \ldots, N\}$, form a network that is not globally rigid, and as a result their positions cannot be calculated uniquely in any coordinate frame. Similar to Section III, each agent i can measure its distance to agent j if their distance from each other is less than r_d. Assuming that each agent has access to all the distance measurements in the network, the goal of the agents is to uniquely localize the network in their own coordinate frame. This problem is stated below.

Problem 2: Let $\mathcal{G}(t) = (\mathcal{N}(t), \mathbf{E}(\mathcal{G}(t)))$ be a graph at time t with vertex set $\mathcal{N}(t) = \{1_t, 2_t \ldots, N_t\}$ and edge set $\mathbf{E}(\mathcal{G}(t))$ where $\mathbf{E}(\mathcal{G}(t)) = \{\{i,j\} : \|p_i(t) - p_j(t)\| \leq r_d\}$ and $t = t_0, t_1, \ldots, t_\kappa$. Note that $p_i(t)$ corresponds to the vertex labeled i_t in graph $\mathcal{G}(t)$[2]. An example of such graphs for $\kappa = 2$ is depicted in Fig. 2. Let $\mathcal{D}(t)$ be the set of all distances associated with the edges in $\mathbf{E}(\mathcal{G}(t))$. Moreover, assume $\mathcal{G}(t_0)$ is connected and Assumptions 1 and 2 hold. Answers to the following questions are desired.

1) What are the conditions that need to be satisfied for $p_i(t)$, $i \in \mathcal{N}$, and κ such that each agent i can calculate $p_{i,j}(t)$, $\forall j \in \mathcal{N}$ and $t \in \{t_0, t_1, \ldots, t_\kappa\}$, given $\mathcal{G} = \bigcup_{k=0}^{\kappa} \mathcal{G}(t_k)$ and $\mathcal{D} = \bigcup_{k=0}^{\kappa} \mathcal{D}(t_k)$?

[2] For simplicity, we use $\mathcal{N}$ instead of $\mathcal{N}(t)$ when it is clear that we are interested in the state of each agent i at time t.

2) What is a motion strategy for i such that these conditions are satisfied?

Answering the first question in Problem 2 is straightforward. Note that the question of finding $p_{i,j}(t)$, $\forall j \in \mathcal{N}$ and $t \in \{t_0, t_1, \dots, t_\kappa\}$ given $\mathcal{G}$ and $\mathcal{D}$ is the same as the question of unique localizability of a network with $\mathcal{G}$ as its underlying graph, $\mathcal{D}$ as the distance set associated with its edges, and the positions of the vertices in some anchor set $\mathcal{V}_i'$ in the coordinate frame of agent i with $|\mathcal{V}_i'| \geq 3$. It is assumed that $p_{i,i}(t)$ is known for all $t \in \{t_0, t_1, \dots, t_\kappa\}$. Thus, agent i has access to the positions of the vertices in the set $\mathcal{V}_i' = \{p_{i,i}(t_0), \dots, p_{i,i}(t_\kappa)\}$. Hence, $p_{i,j}(t)$, $\forall j \in \mathcal{N}$ and $t \in \{t_0, t_1, \dots, t_\kappa\}$ can be calculated uniquely if the network with $\mathcal{G}$ as its underlying graph and the distance set $\mathcal{D}$ along with known positions of the vertices in $\mathcal{V}_i'$ is uniquely localizable. This is equivalent to the network being globally rigid and $\kappa \geq 2$. However, this in itself does not shed much light on how one can guarantee that these conditions hold. To this aim we consider two special cases. First, we consider a scenario where $\mathcal{G}(t_0)$ corresponds to a globally rigid graph, and only agent i is capable of motion. Let $\mathcal{N}_i(t) = \{j : \|p_i(t) - p_j\| \leq r_d\}$. Note that this set is the same as the set of agents j that share an edge with agent i at time t. We have the following result.

Proposition 2: Assume that the following conditions hold for $t = t_0, \dots, t_\kappa$.

1) The network with the underlying graph $\mathcal{G}(t_0)$ at time t_0 is globally rigid.
2) The neighbors of agent i remain constant, i.e. $\mathcal{N}_i(t_0) \subseteq \mathcal{N}_i(t_\kappa)$, $k = 1, \dots, \kappa$.
3) There are no t and $\bar{t}$ such that $\|p_i(t) - p_j\| = 0$ or $\|p_{i,i}(t) - p_{i,i}(\bar{t})\| = 0$.
4) $\kappa \geq 2$.

Then, agent i can calculate $p_{i,j}(t)$, $\forall j \in \mathcal{N}$ and $t \in \{t_0, t_1, \dots, t_\kappa\}$, given $\mathcal{G}(t_0)$ and $\mathcal{D} = \mathcal{D}(t_0) \cup \{\|p_i(t_k) - p_j\| : j \in \mathcal{N}_i, k = 1, \dots, \kappa\}$.

Proof: We first consider the network at t_0. Even though it is globally rigid, but due to the fact that there agent i only has access to one position, i.e. $p_{i,i}(t_0)$, the network is not localizable in the coordinate frame of agent i. At time t_1, agent i moves to a new position $p_{i,i}(t_1)$ such that its neighbor set contains its neighbors at time t_0, i.e. $\mathcal{N}_i(t_0) \subseteq \mathcal{N}_i(t_1)$. Thus, it has access to $\{\|p_i(t_1) - p_j\| : j \in \mathcal{N}_i(t_0)\} \cup \{\|p_i(t_1) - p_j\| : j \in \mathcal{N}_i(t_1) \backslash \mathcal{N}_i(t_0)\}$ as well as $\|p_{i,i}(t_0) - p_{i,i}(t_1)\|$. This information can be abstracted as a network with an underlying graph $\overline{\mathcal{G}}$ with $\mathbf{V}(\overline{\mathcal{G}}) = \mathcal{N} \cup \{i_{t_1}\}$ and $\mathbf{E}(\overline{\mathcal{G}}) = \mathbf{E}(\mathcal{G}) \cup \{\{i_{t_1}, j\} : j \in \mathcal{N}_i(t_1)\}$. It is easy to see that this network is globally rigid as well. However, agent i still has access to only two positions in this network, i.e. $p_{i,i}(t_0)$ and $p_{i,i}(t_1)$. Similarly, repeating this for the case where agent i has moved to a $p_{i,i}(t_2)$ results in a network that is localizable in the coordinate frame of i. This continues to hold for all $\kappa > 2$ as well. This completes the proof. ■

Before continuing any further we propose a motion strategy for agent i such that the conditions of Proposition 2 are satisfied. By construction we have the following result.

Proposition 3: If for all t_k, $k = 0, \dots, \kappa$, agent i is at the position obtained from the application of Algorithm 1, then $\mathcal{N}_i(t_0) \subseteq \mathcal{N}_i(t_k)$, $k = 1, \dots, \kappa$.

Algorithm 1 Agent i motion strategy in a network of stationary agents to achieve unique localizability.

Require: $\kappa \geq 2$, $p_{i,i}(t_0)$
for $k = 0, \dots, \kappa - 1$ **do**
  $\delta_i(t_k) \leftarrow \min_{j \in \mathcal{N}_i(t_0)} \; r_d - \|p_i(t_k) - p_j\|$
  Pick $p_{i,i}(t_{k+1})$ uniformly from the set $\{p : \|p_{i,i}(t_k) - p\| \leq \delta_i(t_{k-1})\}$
  Move to $p_{i,i}(t_{k+1})$ at t_{k+1}
end for

Hence, if agent i is at the position obtained from the application of Algorithm 1 and $\kappa \geq 2$ and $\delta_i(t_0) \neq 0$, then the conditions of Proposition 2 are satisfied with probability one. Furthermore, after κ steps agent i has access to enough information to estimate the positions of all the agents in the network in its own local coordinate frame.

Next we look into the case where $\mathcal{G}(t_0)$ is a path graph. Similar to the previous case we have the following result. We have the following result.

Proposition 4: Assume that the following conditions hold for $t = t_0, \dots, t_\kappa$.

1) The network with the underlying graph $\mathcal{G}(t_0)$ at time t_0 is a path graph.
2) The set of neighbors of each agent i at any time t_k contains the initial neighbors of each agent i, i.e. $\mathcal{N}_i(t_0) \subseteq \mathcal{N}_i(t_k)$, $\forall i \in \mathcal{N}$, $k = 1, \dots, \kappa$.
3) There is no t, $\bar{t}$, i, or j such that $\|p_i(t) - p_j(t)\| = 0$.
4) $\kappa \geq 3$

Then, each agent i can calculate $p_{i,j}(t)$, $\forall j \in \mathcal{N}$ and $t \in \{t_0, t_1, \dots, t_\kappa\}$, given

$$\mathcal{D} = \bigcup_{k=0}^{\kappa} \mathcal{D}(t_k) \cup \{\|p_j(t_{\bar{k}}) - p_j(t_k)\| : j \in \mathcal{N}, \bar{k}, k = 0, \dots, \kappa\},$$

and $p_{i,i}(t_0), \dots, p_{i,i}(t_\kappa)$.

Proof: The proof follows from augmenting the initial network by the information that becomes available at any given time, and consequently showing that after 4 steps, (similar to the proof of Proposition 1), the network becomes globally rigid. ■

As before, we propose a motion strategy as outlined in Algorithm 2 for each agent i so that the $\mathcal{N}_i(t_0) \subseteq \mathcal{N}_i(t_k)$, $\forall i \in \mathcal{N}$, $k = 1, \dots, \kappa$. In fact, if the motion of the agents

Algorithm 2 Agent i motion strategy in a network of mobile agents to ensure unique localizability.

Require: $\kappa \geq 2$, $p_{i,i}(t_0)$
for $k = 2, \dots, \kappa - 1$ **do**
  $\delta_i(t_k) \leftarrow \min_{j \in \mathcal{N}_i(t_0)} \; r_d - \|p_i(t_k) - p_j(t_k)\|$
  Pick $p_{i,i}(t_{k+1})$ uniformly from the set $\{p : \|p_{i,i}(t_k) - p\| \leq \delta_i(t_k)/2\}$
  Move to $p_{i,i}(t_{k+1})$ at t_{k+1}
end for

is governed by Algorithm 2, then we have the following corollary.

Corollary 1: Let the motion of each agent i be governed by Algorithm 2. Moreover, assume that $\mathcal{G}(t_0)$ is connected, $\kappa \geq 3$, and $\delta_i(t_0) \neq 0$. Then, each agent i can calculate $p_{i,j}(t)$, $\forall j \in \mathcal{N}$ and $t \in \{t_0, t_1, \ldots, t_\kappa\}$ with probability one, given

$$\mathcal{D} = \bigcup_{k=0}^{\kappa} \mathcal{D}(t_k) \cup \{\|p_j(t_{\bar{k}}) - p_j(t_k)\| : j \in \mathcal{N}, \bar{k}, k = 0, \ldots, \kappa\},$$

and $p_{i,i}(t_0), \ldots, p_{i,i}(t_\kappa)$.

So far we have not explicitly commented on the constraints that govern the motion of each agent i. In fact, Algorithms 1 and 2 rely on the assumption that agent i can reach $p_{i,i}(t_{k+1})$, $t \in \{t_0, t_1, \ldots, t_{\kappa-1}\}$, starting from $p_{i,i}(t_k)$. In what comes next, we consider the case where not all the points from the set $\{p : \|p_{i,i}(t_k) - p\| \leq \delta_i(t_k)\}$ as described in Algorithm 1 (and similarly the set $\{p : \|p_{i,i}(t_k) - p\| \leq \delta_i(t_k)/2\}$ as described in Algorithm 2) are reachable from $p_{i,i}(t_k)$ due to constraints on the motion of agent i. We consider the motion of agent i between t_k and t_{k+1} to be governed by

$$\dot{x}_{i,i}(\tau) = f(x_{i,i}(\tau), u_i(\tau)) \tag{5}$$

where $t_k \leq \tau \leq t_{k+1}$, $x_{i,i}(\tau) = [p_{i,i}(\tau)^\top,\ v_{i,i}(\tau)^\top]^\top \in \mathbb{X}$ is the state of agent i in its coordinate frame with $p_{i,i}(\tau)$ and $v_{i,i}(\tau)$ being its position and velocity in its coordinate frame, $\mathbb{X}$ is the set of feasible states, and $u_i(\tau) \in \mathbb{U}$ is the control input with $\mathbb{U}$ being the set of feasible inputs. We first have the following definition.

Definition 4 (Reachable Position Set): The reachable position set $\mathcal{S}(x_{i,i}(t_k), t_{k+1})$ of the agent i governed by (5) from the initial state $x_{i,i}(t_k)$, is the set of all positions that are reachable along a trajectory satisfying (5) from the agent's position and velocity at t_k, in other words:

$$\begin{aligned}\mathcal{S}(x_{i,i}(t_k), t_{k+1}) = \{p(t_{k+1}) : \ & \dot{x}(\tau) = f(x_{i,i}(\tau), u(\tau)),\\ & x(\tau) = [p(\tau)^\top, v(\tau)^\top]^\top,\\ & x(t_k) = x_{i,i}(t_k), x(\tau) \in \mathbb{X}\\ & \exists u(\tau) \in \mathbb{U}, t_k \leq \tau \leq t_{k+1}\}.\end{aligned} \tag{6}$$

Given this definition, Algorithms 1 and 2 can be rewritten to reflect the constraints on each of the agents' motion. Specifically, in line 3 of Algorithms 1 and 2, $p_{i,i}(t_k)$ should be picked uniformly from the sets

$$\{p : \|p_{i,i}(t_k) - p\| \leq \delta_i(t_k)\} \cap \mathcal{S}(x_{i,i}(t_k), t_{k+1})$$

and

$$\{p : \|p_{i,i}(t_k) - p\| \leq \delta_i(t_k)/2\} \cap \mathcal{S}(x_{i,i}(t_k), t_{k+1}),$$

respectively.

We observe that computing $\mathcal{S}(x_{i,i}(t_k), t_{k+1})$ in (6) is computationally cumbersome and might not be practical for all systems. We conclude this section by commenting on a special case where $\mathcal{S}(x_{i,i}(t_k), t_{k+1})$ can be calculated efficiently. This is the case where motion of each agent i governed by (5) corresponds to that of a small fixed-wing unmanned aerial vehicle (UAV). In other words, agent i can only travel with a constant speed, $\bar{v}$, and has a minimum turning radius, r_{min}. We have the following result for this scenario.

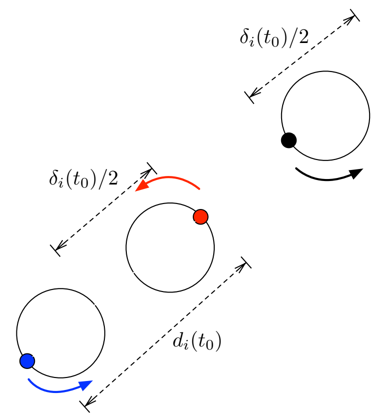


Fig. 3: An example where $4r_{min} \leq \min(d_i(t_0), \delta_i(t_0))$ holds for 3 agents with constant speed and minimum turning radius constraints.

Proposition 5: Consider the case where each agent $i \in \mathcal{N}$ is a nonholonomic vehicle with constant speed $\bar{v}$ and minimum turning radius r_{min}. Moreover, assume $4r_{min} \leq \min(d_i(t_0), \delta_i(t_0))$ for all $i \in \mathcal{N}$ where $\delta_i(t_0) = \min_{j \in \mathcal{N}_i(t_0)} r_d - \|p_i(t_0) - p_j(t_0)\|$, $d_i(t_0) = \min_{j \in \mathcal{N}_i(t_0)} \|p_i(t_0) - p_j(t_0)\|$, $\kappa \geq 2$, and $\mathcal{G}(t_0)$ is connected. If each agent travels along a circle of radius r_d in clockwise or counter-clockwise direction from their initial position $p_{i,i}(t_0)$ and collect measurements at $t_1, \ldots, t_\kappa$ Then, each agent i can calculate $p_{i,j}(t)$, $\forall j \in \mathcal{N}$ and $t \in \{t_0, t_1, \ldots, t_\kappa\}$, given

$$\mathcal{D} = \bigcup_{k=0}^{\kappa} \mathcal{D}(t_k) \cup \{\|p_j(t_{\bar{k}}) - p_j(t_k)\| : j \in \mathcal{N}, \bar{k}, k = 0, \ldots, \kappa\},$$

and $p_{i,i}(t_0), \ldots, p_{i,i}(t_\kappa)$.

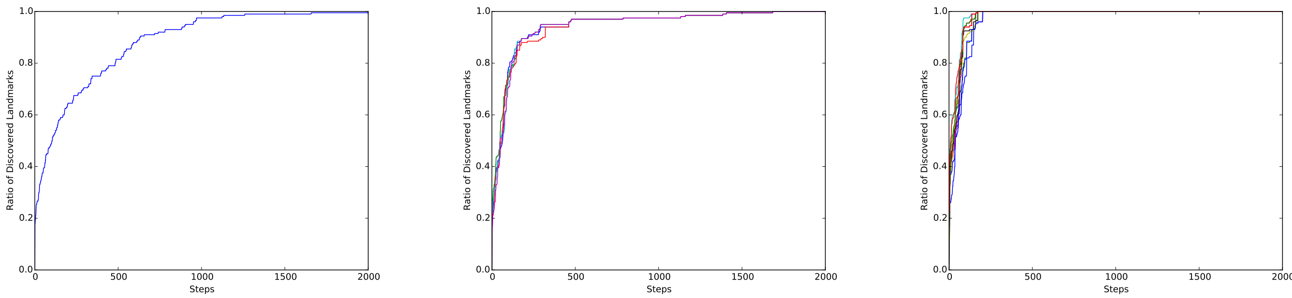
Proof: Note that, since $4r_{min} \leq \delta_i(t_0)$, $\mathcal{N}_i(t_0) \subseteq \mathcal{N}_i(t_k)$, $\forall i \in \mathcal{N}$, $k = 1, \ldots, \kappa$. Furthermore, since generically $\frac{2\pi r_{min}}{\bar{v}(t_k - t_{k+1})}$ is not an integer and $4r_{min} \leq d_i(t_0)$, there are no $\bar{t}, t \in \{t_0, \ldots, t_\kappa\}$ or $i, j \in \mathcal{N}$ such that $\|p_i(t) - p_j(\bar{t})\| = 0$. Then, similar to Proposition 4, after κ steps, the information available to each agent i will be enough to find all $p_{i,j}(t)$, $\forall j \in \mathcal{N}$ and $t \in \{t_0, t_1, \ldots, t_\kappa\}$. ■

A scenario with 3 agents where the condition $4r_{min} \leq \min(d_i(t_0), \delta_i(t_0))$ is satisfied is depicted in Fig. 3.

V. Simulations

In the first scenario, we consider the case where the agents explore an unknown environment with $m = 20$ landmarks as outlined in Section III. We assume that $r_d = 5$ and the agents operate in a 20×20 environment. We assume that the agents' motion is determined randomly, i.e. at each step t_k each

Preprint submitted to 54th IEEE Conference on Decision and Control.
Received March 24, 2015.



(a) $N = 1$ (b) $N = 5$ (c) $N = 10$

Fig. 4: The ratio of discovered landmarks by each of the agents versus time steps for 3 scenarios.

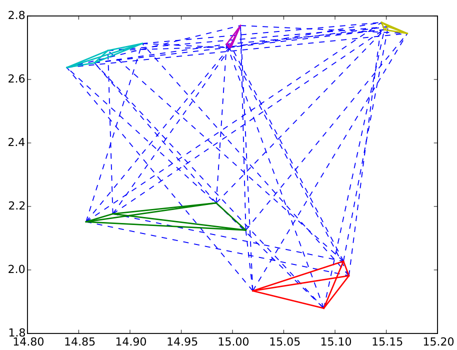


Fig. 5: The positions of each of the agents in 4 time steps. The solid lines correspond to the displacement of each of the agents in different time steps and the dashed lines correspond to the inter-agent distance measurements collected at each time.

agent i moves in the direction of Δp_i where Δp_i is selected uniformly from the set $\{p : \|p_{i,i} - p\| \leq 2\}$. We compare the rate of landmarks discovery for three cases where $N = 1$, $N = 5$ and $N = 10$ agents explore the environment and share their discovered landmarks when the conditions of Proposition 1 are satisfied. The simulations are repeated ten times for different landmark positoins and agents trajectories and the results are depicted in Fig. 4. We observe that when more agents are used to cooperatively map the environment, *every* agent discovers the position of all landmarks much faster than with a single agent. In the second scenario, we consider the case where $N = 5$ agents with single integrator dynamics apply Algorithm 2 for $\kappa = 4$ to be able to satisfy the conditions of Proposition 4. The position of the agents after 4 steps and all the collected measurements are depicted in Fig. 5. It can be easily checked that the resulting graph is globally rigid and as a result, uniquely localizable.

VI. Conclusions

In this paper we considered different situations where a group of agents aim to cooperatively map and localize via collecting distance measurements to each other or landmarks. Through the application of the rigidity theory, we established the conditions under which it is possible for the agents to gain access to the data collected by other agents and we related these conditions to the number of measurements exchanged between each agent pair. We demonstrated that there exists a motion strategy that enables the agents to satisfy the above-said conditions. We note that even though the results that we proposed here are for the case that the agents have only access to distance measurements, similar results can be obtained for the case where the agents collect only bearing measurements. A future research direction is to extend the results addressed here to other measurements, such as bearing. Another important future direction is to develop a numerically efficient algorithm to solve the localization problems obtained from agents carrying out the maneuvers of the kind outlined in Algorithms 1 and 2 and collecting the necessary measurements. Another research direction is to consider executing these motions with some cost associated with the agents' operations in the presence of disturbances.

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Preprint submitted to 54th IEEE Conference on Decision and Control.
Received March 24, 2015.