Performance bounds for robust estimation using the H_∞ filter

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***Abstract*— A key concern in network observability is to quantify performance and robustness limitations for state estimation from noisy sensors in terms of its dynamical properties and sensor architecture. We develop performance bounds for the robust H_∞ filter, a generalization of the Kalman filter. Utilizing an eigenvalue bound on the observability Gramian, we derive a related eigenvalue bound on the estimation error covariance matrix from the generalized Riccati equation of the H_∞ filter. As a special case, we obtain estimation performance bounds on the Kalman Filter. The bounds reflect the cardinality of the network and sensor set, the stability of the network, and the number and specific set of states to be estimated. We illustrate our results with numerical analysis on a regular network showing how the bounds change with system parameters.**

I. Introduction

A key concern in network observability is to quantify performance and robustness limitations for state estimation from noisy sensors in terms of its dynamical properties and sensor architecture. Various algebraic and structural metrics for network observability and controllability have been proposed to quantify these properties. Such metrics can guide the design of network control architectures, by evaluating possible sensor and actuator configurations in the network. Examples include structural Kalman rank [1]–[5], Gramians [6]–[10] analysing optimal or robust performance [11]–[19]. Certain metrics allow for suboptimality guarantees for sensor and actuator placement algorithms (using concepts such as submodularity and supermodularity) [19]–[21]. Other metrics lacking these properties still show acceptable performance for most real-world systems under greedy algorithms [14] but do not guarantee performance. These point to fundamental difficulties in network controllability and observablity.

Recent research [14], [22] has focused on performance bounds of fixed-size sensor observability and actuator controllability using the recursion of the Kalman Filter and LQR optimization problem respectively and the corresponding Gramian. Other research focuses have expanded on the Kalman Filter to study properties such as resilience [18], [23], [24]. Application-based research on the robust approach to estimation under the H_∞ filter [25]–[27] has focused on exploiting specific structure and system properties and restricts the algorithms developed to guarantees of feasibility. We aim to close the gap to quantitative metrics evaluating robust estimation in parallel to our work on robust control. [1]

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We propose a generalized quantitative metric for the robust network observability with a fixed-size sensor set using a game framework of the H_∞ filter and the Observability Gramian. We define performance bounds as the lower bound on the largest eigenvalue of the error covariance matrix. For a feasible filter, this value is finite. The contributions of this paper are:

- We derive a bound on the minimum eigenvalue of the Observability Gramian based on the system parameters (Theorem 1), analogous to a bound derived for the controllability Gramian in [6].
- We then derive the performance bounds on the error covariance matrix from the Riccati equation of the H_∞ filter via a dynamic game framework (Theorem 2). As a special case, we obtain estimation performance bounds on the Kalman Filter.
- We illustrate the results with numerical experiments. We highlight the significance of the scale of the network and the diminishing returns of increasing the number of sensors, the importance of positions of sensors, the choice and relative weights of states to be estimated and the value of information on the disturbances to the system. This is covered in Section IV.

Notation: We define $\mathbb{S}^n_{++} \subset \mathbb{S}^n_{+} \subset \mathbb{R}^{n\times n}$ to represent the set of positive definite and positive semidefinite matrices respectively. The identity matrix in $\mathbb{R}^{n\times n}$ is represented by $\mathbf{I}_{n\times n}$. For a matrix A, its transpose is $A^\top$. We have $\|x\|^2_A = x^\top Ax$. The set of eigenvalues or the spectrum of matrix A is denoted by $\mathrm{spec}(A)$. The condition number of matrix A is denoted by $\mathrm{cond}(A)$.

II. Robust State Estimation in Networks

We begin by describing the network and filter structure, the estimation cost function and key system parameters. The network dynamics and measurements are modeled by a time-invariant discrete-time linear system evolving on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as

$$x_{k+1} = Fx_k + w_k \quad y_k = H_{\mathcal{K}} x_k + v_k \quad z_k = Lx_k \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $y_k \in \mathbb{R}^m$ is the measurement vector, and $z_k \in \mathbb{R}^l$ is a vector of quantities to be estimated. Each state is associated with a node in the graph $\mathcal{G}$ with the matrix F denoting the dynamics of the network derived from the weighted adjacency matrix of the graph, and such that a set of sensors that measure node state values can be placed in the network. Thus, the set of available sensors $\mathcal{H} = \{e_1, ..., e_M\}$ is associated with $M \le n$ canonical row vectors of $\mathbf{R}^n$ (i.e., $e_i \in \mathbf{R}^{1\times n}$). The rows of the sensor measurement matrix $H_{\mathcal{K}} = [e_1^\top \dots e_{|\mathcal{K}|}^\top]^\top \in \mathbb{R}^{|\mathcal{K}|\times n}$ are formed by a subset of sensors $\mathcal{K} \subseteq \mathcal{H}$. The process noise and the measurement noise are denoted by $w_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^{|\mathcal{K}|}$, respectively.

a) Robust Estimation via a Dynamic Game: We consider a problem of robust state estimation, where z_k is to be estimated from a sequence of output measurements, and the initial state, process noise, and measurement noise are treated as adversarial disturbances. The cost function to be optimized is given by

$$J = -\theta^2\|x_0 - \hat{x}_0\|^2_{P_0^{-1}} + \sum_{k=0}^{N-1}\left(\|z_k - \hat{z}_k\|^2_{S_k} -\theta^2\left(\|w_k\|^2_{Q_k^{-1}} + \|v_k\|^2_{R_k^{-1}}\right)\right).$$

where $S_k \in \mathbb{S}^l_+$ is a weight on the state estimation error, and here $P_0 \in \mathbb{S}^n_{++}$, $Q_k \in \mathbb{S}^n_{++}$, and $R_k \in \mathbb{S}^{|\mathcal{K}|}_{++}$ are interpreted as penalties on the adversarial initial state, process noise, and sensor noise, respectively. There are no bounds on the external disturbances but they are penalized in the cost function. The parameter $\theta > 0$ designates this penalty. For given dynamics parameters, there exists a value of θ defining a minimum penalty on the external disturbances below which estimation error covariance is unbounded.

The optimal robust cost is defined as $J^* = \min_{\hat{z}} \max_{w_k,v_k,x_0} J$, i.e., we seek to minimize worst case estimation error. Using (1) and defining $\overline{S} = L^\top SL$, the optimal cost function can be written as $J^* = \min_{\hat{x}} \max_{w_k,y_k,x_0} J$ with

$$J = -\theta^2\|x_0-\hat{x}_0\|^2_{P_0^{-1}} + \sum_{k=0}^{N-1}\Bigg[\|x_k - \hat{x}_k\|^2_{\overline{S}} - \theta^2\left(\|w_k\|^2_{Q^{-1}} + \|y_k - H_{\mathcal{K}}x_k\|^2_{R^{-1}}\right)\Bigg]. \quad (2)$$

The solution to this $\mathcal{H}_\infty$ robust filtering problem can be computed using a constrained Lagrangian optimization approach. Let P_k be the estimation error covariance and K_k be the estimation gain at time step k. The optimal solution of the H_∞ filter with the above cost function is recursively given by

$$\begin{aligned} K_k &= P_k\left[\mathbf{I} - \theta^{-2}\overline{S}P_k + H_{\mathcal{K}}^\top R^{-1}H_{\mathcal{K}}P_k\right]^{-1} H_{\mathcal{K}}^\top R^{-1} \\ \hat{x}_{k+1} &= F\hat{x}_k + FK_k(y_k - H_{\mathcal{K}}\hat{x}_k) \\ P_{k+1} &= F\left[P_k^{-1} - \theta^{-2}\overline{S} + H_{\mathcal{K}}^\top R^{-1}H_{\mathcal{K}}\right]^{-1}F^\top + Q, \end{aligned} \quad (3)$$

provided that $P_k^{-1} - \theta^{-2}\overline{S} + H_{\mathcal{K}}^\top R^{-1}H_{\mathcal{K}} \succ 0$.

The given system has time invariant dynamics F, a fixed sensor set $\mathcal{K}$, constant interest matrix L, estimation weight matrix S, state cost Q and measurement cost R. If $P^{-1} - \theta^{-2}\overline{S} + H_{\mathcal{K}}^\top R^{-1}H_{\mathcal{K}} \succ 0$ and the error covariance converges, the infinite horizon error covariance P satisfies

$$P = F\left[P^{-1} - \theta^{-2}\overline{S} + H_{\mathcal{K}}^\top R^{-1}H_{\mathcal{K}}\right]^{-1}F^\top + Q. \quad (4)$$

b) Observability Gramian: We define $\mathcal{O}_{K,T}$ to be the Observability Gramian for a sensor set $K \subseteq \mathcal{H}$ with the measurement matrix $H_{\mathcal{K}}$ over a time horizon T. We denote the eigenvector decomposition of the dynamics matrix by

$$V^{-1}FV = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} = \Lambda; \quad H_{\mathcal{K}}V = \begin{bmatrix} H_{\mathcal{K}1} & H_{\mathcal{K}2} \end{bmatrix} = \mathcal{C}.$$

For a given value of $\mu \in \mathbb{R}_{\geq 0}$ and $n_\mu := |\{\lambda : \lambda \in \operatorname{spec}(F), |\lambda| \leq \mu\}|$, $F_1 \in \mathbb{R}^{n_\mu \times n_\mu}$ and F_2 are both symmetric diagonal matrices with $\operatorname{spec}(F_1) = \{\lambda : \lambda \in \operatorname{spec}(F), |\lambda| \leq \mu\}$. Similarly, $H_{\mathcal{K}1}$ and $H_{\mathcal{K}2}$ are partitions of $H_{\mathcal{K}}$ of suitable size. The eigen-decomposition of the Observability Gramian $\tilde{\mathcal{O}}_{K,T}$ is given by

$$\mathcal{O}_{K,T} = \sum_{\tau=0}^{T-1}(F^\top)^\tau H_{\mathcal{K}}^\top H_{\mathcal{K}}F^\tau = V^\top \underbrace{\sum_{\tau=0}^{T-1}\Lambda^\tau \mathcal{C}^\top \mathcal{C}\Lambda^\tau}_{\tilde{\mathcal{O}}_{K,T}} V. \quad (5)$$

In this paper, we focus on unstable open-loop dynamics with $|\lambda_{\max}(F)| > 1$. This requires a non-trivial solution to the finite horizon steady state average-cost filtering problem (since for stable systems, any finite state estimate yields finite steady state error), and leads to performance limits for robust estimation in networks.

III. Performance Bounds of Robust Estimation

In this section, we develop a bound on the largest eigenvalue of the error covariance of the H_∞ filter. To prove the result, we first establish bounds on the smallest eigenvalue of the Observability Gramian.

A. Bounds on smallest eigenvalue of Observability Gramian

We begin with the following result that bounds the smallest eigenvalue of the Observability Gramian in terms of the system dynamics and the sensor set. This theorem is analogous to Theorem 3.1 in [6] on the Controllability Gramian.

Theorem 1: Consider an undirected network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}| = n$, weighted adjacency matrix F with eigenvector matrix V and observer set K sensors of sensor set $\mathcal{H}$. Let $\lambda_{\min}(F) < 1$ and for any $\mu \in [\lambda_{\min}(F), 1)$, $n_\mu \triangleq |\{\lambda : \lambda \in \operatorname{spec}(F), |\lambda| \leq \mu\}|$. Let $\alpha_V \triangleq \operatorname{cond}^2(V)\frac{\|V\|_2^2}{\|V^{-2}\|_2^2}$. For all $T \in \mathbb{N}_{>0}$, it holds for the Observability Gramian $\mathcal{O}$ that

$$\lambda_{\min}(\mathcal{O}_{K,T}) \leq \alpha_V \frac{\mu^{2(\lceil \frac{n_\mu}{|\mathcal{K}|}\rceil - 1)}}{1-\mu^2}. \quad (6)$$

Proof: The proof follows Theorem 3.1 in [6]. The difference between the Controllability Gramian in the reference and the Observability Gramian used here is accounted for the in the scaling factor α_V. ■

This result is used to derive performance bounds on robust estimation for the H_∞ filter.

B. Performance bounds of robust estimation

The following result provides a performance bound on the H_∞ filter in terms of the maximum eigenvalue of the estimation error covariance matrix.

Theorem 2: Consider a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with dynamics matrix F, state measurement matrix $H_{\mathcal{K}}$ of sensor set $K \subseteq \mathcal{H}$ available sensors, process noise w_k, measurement noise v_k and cost bound θ. Suppose that F is Schur unstable and let $\lambda_{\max}(F) > 1$ denote the eigenvalue of F with maximum magnitude. Suppose further that F is diagonalizable by the eigenvector matrix V and for any $\eta \in (1, \lambda_{\max}(F)]$, define $n_\eta = |\{\lambda : \lambda \in \operatorname{spec}(F), |\lambda| \geq \eta\}|$. Then, for P_∞, the error covariance of the H_∞ filter calculated from the infinite-horizon algebraic Riccati equation (4), we have For all $\eta \in (1, \lambda_{\max}(F)]$, it holds

$$\lambda_{\max}(P_\infty) \geq \frac{1}{\alpha_V} \frac{\eta^2 - 1}{\left[\eta^{-2\lceil \frac{n_\eta}{|\mathcal{K}|} \rceil} - \theta^{-2}\left(\eta^{-2\lceil \frac{n_\eta}{|\mathcal{K}_L|}\rceil} + \frac{\eta^2-1}{\alpha_V}\Lambda_{\overline{S}}\right)\right]}. \tag{7}$$

where $|\mathcal{K}_L|$ is the number of states in the interest vector, $\alpha_V = \text{cond}^2(V)\frac{\|V\|_2^2}{\|V^{-2}\|_2^2}$ and $\Lambda_{\overline{S}} \triangleq \lambda_{\min}(\overline{S})$ for $z = Lx$ for diagonal matrix L.

Proof: Consider the system dynamics with H_∞ filter described in (1). Without loss of generality, we assume that symmetric measurement noise covariance $R = \mathbf{I}$ and estimation weights $S = \mathbf{I}$ as the values can be absorbed into the $H_\mathcal{K}$ matrix and L matrix respectively. We assume a small Q cost matrix and that $P_k >> Q$ as time-steps increase. Inverting the Riccati equation (3), we get

$$P_{k+1}^{-1} = F^{-\top}\left[P_k^{-1} - \theta^{-2}\overline{S} + H_\mathcal{K}^\top H_\mathcal{K}\right]^{-1} F^{-1}.$$

We define $X_k \triangleq P_k^{-1} - \theta^{-2}\overline{S} + H_\mathcal{K}^\top H_\mathcal{K}$. From the initial values at time $k = 0$, we have $X_0 = P_0^{-1} - \theta^{-2}\overline{S} + H_\mathcal{K}^\top H_\mathcal{K}$. Using the same equation, we also define a new recursion $X_{k+1} = F^{-\top} X_k F^{-1} - \theta^{-2}\overline{S} + H_\mathcal{K}^\top H_\mathcal{K}$. We now get the N-step value as

$$\begin{aligned} X_N = &\underbrace{\sum_{k=0}^{N}\left[(F^{-\top})^k H_\mathcal{K}^\top H_\mathcal{K}(F^{-1})^k\right]}_{\overline{X}_N} \\ &- \theta^{-2}\underbrace{\sum_{k=0}^{N}\left[(F^{-\top})^k \overline{S}(F^{-1})^k\right]}_{\tilde{X}_N} + \underbrace{(F^{-\top})^N P_0^{-1}(F^{-1})^N}_{\tilde{P}} \\ = &\overline{X}_N - \theta^{-2}\tilde{X}_N + \tilde{P}. \end{aligned}$$

Using Weyl's Inequality for matrices, we have eigenvalue bounds on the matrices as

$$\begin{aligned} \lambda_{\min}(X_N) &\leq \lambda_{\min}(\overline{X}_N) + \lambda_{\max}(-\theta^{-2}\tilde{X}_N + \tilde{P}) \\ &\leq \lambda_{\min}(\overline{X}_N) - \theta^{-2}\lambda_{\min}(\tilde{X}_N) + \lambda_{\max}(\tilde{P}). \end{aligned} \tag{8}$$

Under the constraints of the Riccati equation, we have a feasible sensor set. So $X_N \succ 0 \implies \lambda_{\min}(X_N) > 0$.

Consider a discrete-time LTI system with the system-measurement pair $(F^{-1}, H_\mathcal{K})$. The Observability Gramian of this system over an N-step horizon is

$$\overline{X}_N = \sum_{k=0}^{N}\left[(F^{-\top})^k H_\mathcal{K}^\top H_\mathcal{K}(F^{-1})^k\right].$$

If $\lambda_{\max}(F) > 1$, then $\lambda_{\min}(F^{-1}) < 1$. Let V be the eigenvector matrix of F (and F^{-1}). For $\mu \in (\lambda_{\min}(F^{-1}), 1]$, we define $n_\mu \triangleq |\{\lambda : \lambda \in \text{spec}(F^{-1}), |\lambda| < \mu\}|$. For the sensor set K with cardinality $|\mathcal{K}|$, we use (6) to state

$$\lambda_{\min}(\overline{X}_N) \leq \alpha_V \frac{\mu^{2\left(\lceil \frac{n_\mu}{|\mathcal{K}|}\rceil - 1\right)}}{1-\mu^2}. \tag{9}$$

If this system is N-step observable, then $\overline{X}_N \succ 0$.

We similarly interpret

$$\tilde{X}_N = \sum_{k=0}^{N}\left[(F^{-\top})^k \overline{S}(F^{-1})^k\right] = \sum_{k=0}^{N}\left[(F^{-\top})^k L^\top L(F^{-1})^k\right]$$

as the N-step Observability Gramian for the system-measurement pair (F^{-1}, L). Applying (6), with $|\mathcal{K}_L|$ to be the number of states measured by L, we get the eigenvalue relationship

$$\lambda_{\min}(\tilde{X}_N) \leq \alpha_V \frac{\mu^{2\left(\lceil \frac{n_\mu}{|\mathcal{K}_L|}\rceil - 1\right)}}{1-\mu^2}. \tag{10}$$

Substituting the relations (9) and (10) in the eigenvalue bounds from Weyl's inequality (8) we get

$$\begin{aligned} &\lambda_{\min}(X_N) \\ &\quad \leq \alpha_V\left[\frac{\mu^{2\left(\lceil \frac{n_\mu}{|\mathcal{K}|}\rceil - 1\right)}}{1-\mu^2} - \theta^{-2}\frac{\mu^{2\left(\lceil \frac{n_\mu}{|\mathcal{K}_L|}\rceil - 1\right)}}{1-\mu^2}\right] + \lambda_{\max}(\tilde{P}). \end{aligned}$$

If $1 < \lambda_{\min}(F)$, then $\lambda(\tilde{P}) \to 0$ over time. This can be interpreted as the case where the observability depends purely on the feasibility of a sensor set. If $1 > \lambda_{\min}(F)$, $(F^{-1})^N$ has significant unstable eigenvalues along some eigenvectors. Sensors are not needed for these stable nodes of F which converge asymptotically to the origin over time and the model can be reduced to only consider the unstable dynamics.

From the definition of X_k, the constraint on the Riccati equation in (3) and as $P_k \succ 0$ by definition, we have

$$\begin{aligned} &\lambda_{\min}(P_\infty^{-1}) \leq \lambda_{\min}(X_\infty) - \theta^2\lambda_{\min}(\overline{S}) \\ &\frac{1}{\lambda_{\max}(P_\infty)} \leq \alpha_V\left[\frac{\mu^{2\left(\lceil \frac{n_\mu}{|\mathcal{K}|}\rceil - 1\right)}}{1-\mu^2}\right. \\ &\qquad\qquad \left. -\theta^{-2}\frac{\mu^{2\left(\lceil \frac{n_\mu}{|\mathcal{K}_L|}\rceil - 1\right)} + \frac{1-\mu^2}{\alpha_V}\lambda_{\min}(\overline{S})}{1-\mu^2}\right]. \end{aligned}$$

We invert this expression, substitute $\Lambda_{\overline{S}} \triangleq \lambda_{\min}(\overline{S})$ and for $\eta = \frac{1}{\mu}$, we define $n_\eta \triangleq |\{\lambda : \lambda \in \text{spec}(F), |\lambda| \geq \eta\}|$ to get (7). ■

As $\theta \to \infty$, we obtain an analogous performance bound for the Kalman filter.

Corollary 1: The Kalman Filter is a special case of the H_∞ filter where the parameter $\theta \to \infty$. In this case, the bound on the error covariance is given by

$$\lambda_{\max}(P_\infty) \geq \frac{1}{\alpha_V}\frac{1-\mu^2}{\mu^{2\lceil \frac{n_\mu}{|\mathcal{K}|}\rceil}}. \tag{11}$$

Discussion: An Observability Gramian with strictly positive eigenvalues indicates a feasible filter for estimation. The magnitude of these eigenvalues indicates the energy or effort required for estimation. The result bounds the minimum effort required for estimation along the eigenvector that is most difficult to estimate.

We see the dependence of the error covariance matrix on the dynamics of the system through the parameters η and n_η. For η close enough to 1, n_η is the number of unstable eigenvalues of the dynamics. We see that the performance bound increases exponentially with n_η for a fixed size sensor set. Similarly, we would require a proportional increase in the number of sensors $|\mathcal{K}|$ to maintain a constraint on the worst-case estimation performance. We see a similar observation for the performance bounds in terms of the matrix L in $|\mathcal{K}_L|$ and the sensor selection set $|\mathcal{K}|$. We also see that reducing θ, which corresponds to increasing the effects of the adversarial disturbances (or equivalently, increasing model error), increases the performance lower bound, indicating increasing difficulty in state estimation.

In this paper, the control system is described for a network where the underlying adjacency matrix of the graph defines the dynamics of the system. This allows edge weights to be real-valued, not strictly non-negative. We may also define stability and connectivity in terms of network characteristics. The convenience of a network or graph based system helps visualize actuator placement as a selection of nodes for control inputs. In this work, the network topology is brought into consideration through the dynamics matrix and its corresponding eigenvector matrix. While our analysis here merely uses the framework of the network, a valuable research direction lies in studying the effect of network parameters on actuator set selection.

Accounting for the relaxations under Weyl's inequality and the assumptions on the initial state estimation weights, these performance bounds can be conservative. Further considerations on the state space dynamics and sensor set could improve the bound. In applications, the robust estimation metric can guide the design of sensor architectures in networks. In the following section, we explore how various quantities affect robust estimation performance.

IV. Numerical Analysis

In this section, we illustrate the relationship between the system parameters and the performance bounds of the sensor set. The parameters are the cardinality of the sensor set, the size and topology of the network and the bounds on the cost function. Sensors are placed by a greedy algorithm without replacement (order of complexity: $\mathcal{O}(\frac{n!}{|K|!})$), seeking to minimize the trace of the error covariance matrix. We test the parameter variations on a standard path graph with 21 nodes given by A and the scaling parameter ρ to adjust its eigenvalues and hence open-loop autonomous stability.

$$A = \frac{\rho}{3}\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 & 1 \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix},$$

Comparison of H_∞ and Kalman Filter: We begin by comparing the H_∞ and Kalman filter for a fixed network with change in number of sensors. Both filters operate under the same cost matrices to estimate all states with equal weight. The results are given in Fig. 1. We see from the top graph that the maximum bounds on the error is reduced under a larger sensor set as described in the bottom graph for a fixed value of θ. We see that the H_∞ filter results in higher error as it is robust but not optimal. With a single sensor, its error covariance is significantly higher than the Kalman filter but difference reduces as the number of sensors increases.

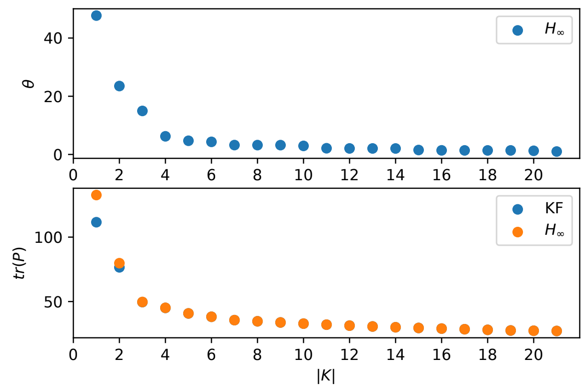

Fig. 1. The top plot shows the variation in the minimum value of the penalty parameter θ with increase in size of sensor set for the H_∞ filter. The bottom plot shows the variation in the error covariance plot for the Kalman Filter and the H_∞ filter evaluated at a fixed value of θ.

Effect of network size: We compare the effect of network size under limitation of a fixed sensor set on the smallest bounds on the observer error function. We consider a path graph of $n = \{5, 10, \ldots, 40\}$ nodes. A fixed sensor set of $|\mathcal{K}| = 3$ positioned in the middle of the network is set to identify the states of the terminal nodes of the path network, given by $z_k = \begin{bmatrix}(x_k)_1 & (x_k)_n\end{bmatrix}^\top$. The results are plotted in Fig. 2. From the top plot, we can see that the minimum penalty increases with network size. This shows that for a growing network with a fixed sensor set, external disturbances have increased influence. From the bottom graph, we can see that there is an exponential increase in the error trace(P) with increasing number of unstable nodes, which is related to network size, for the fixed sensor set at a fixed penalty parameter.

Effect of relative importance of interest states: In this analysis, we fix the network model and randomly select a subset of states to estimate. Then we compare the change in error bound parameter θ and costs of partial estimation of states by gradually increasing the number of states to be estimated. The results are plotted in Fig. 3. We use $|S|$ to denote the number of states we are interested in estimating, equivalently the number of equally weighted, non-zero diagonal elements of the S matrix. From the top graph, we see that increasing the number of states to be estimated increases the minimum penalty placed on the disturbances affecting error bounds on the system. For a fixed sensor set, it is more accurate to measure states close to and at the position of the sensors.

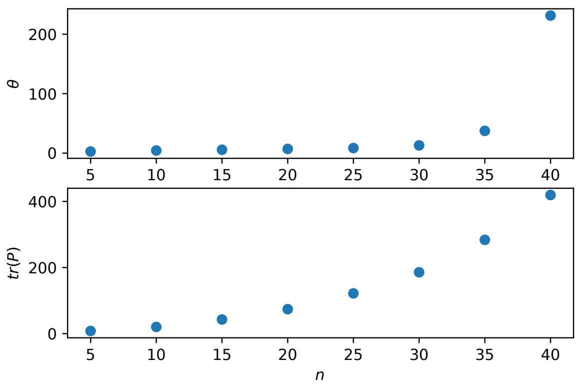

Fig. 2. The top graph shows the variation in the minimum value of the penalty parameter θ with the size of an unstable path network of size n for a limited sensor set size. The bottom graph shows the change in the error covariance matrix at a fixed value of θ.

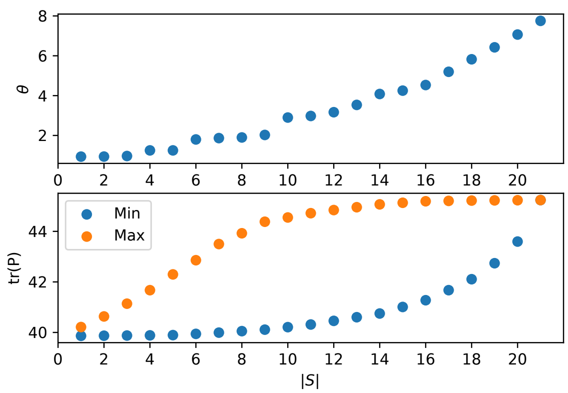

Fig. 3. The top graph shows the variation in the minimum value of the penalty parameter θ with the number of significantly weighted states being estimated $|S|$. The bottom graph shows the cost variation of the easiest and hardest set of states to be estimated. The network is a 21 node path graph with $|\mathcal{K}| = 3$ randomly placed sensors.

V. Conclusion

We have derived a performance bound for robust estimation in networks. A future direction for this research would be to further understand the combined performance and cost of actuator and sensor set optimization for different network structures.

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