Stochastic Stability via Robustness of Linear Systems

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***Abstract*— Robust stability and stochastic stability have separately seen intense study in control theory since its inception. In this work we establish relations between these properties for discrete-time systems. Specifically, we examine a robustness framework which models the inherent uncertainty and variation in the system dynamics which arise in model-based learning control methods such as adaptive control and reinforcement learning. We provide results which guarantee mean-square stability margins in terms of multiplicative noises which affect the nominal dynamics, as well as connections to prior work which together imply that robust stability and mean-square stability are, in a certain sense, equivalent.**

I. Introduction

The study of systems with multiplicative noise, also known as systems with stochastic parameters, has a long history in control theory [1], [2], [3]. In contrast with the well-known additive noise setting, multiplicative noise captures linear dependence of the noise on the state and control input, which occurs intrinsically in diverse modern control systems such as robotics [4], [5], networked systems with noisy communication channels [6], modern power networks with high penetration of intermittent renewables [7], turbulent fluid flow [8], biological movement systems [9] and finance [10]. For proper and safe operation of such systems, it is critical to ensure certain types of *stochastic stability* which ensure distributions of random states converge over time. Furthermore, multiplicative noise models can be used as a device for representing inherently stochastic uncertainty in dynamic model parameters that are estimated from noisy trajectory data. These models can then inform the design of controllers that are robust to structured parametric uncertainty [11], and they can complement alternative non-parametric models of uncertainty traditionally used in $\mathscr{H}_\infty$ control [12].

Although linear systems with multiplicative noise are simple enough to admit closed-form expressions for stability and optimal control via generalized Lyapunov and Riccati equations, it may be convenient or advantageous in certain contexts to certify or design controllers to achieve robust stability under static perturbations of the system from the nominal dynamics. Tools for design and certification of robustly stable linear systems have been developed and extensively tested, including $\mathscr{H}_\infty$ control design, which treats modeling error as a worst-case or adversarial disturbance [13], [14], and robust optimization over parametric state-space uncertainty sets using shared Lyapunov functions via convex semidefinite programming [15], [16], [17], low-order control structures [18], and System Level Synthesis [19].

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In this paper we consider a fundamental question:

What is a set of multiplicative noise distributions for which a stochastic system can be guaranteed mean-square stable, given knowledge only of the nominal system dynamics and robust stability within bounded static perturbations?

To the best of our knowledge, this question has not been examined explicitly heretofore in the literature. The answer to this question, as developed within the present work, represents results that are complementary and converse to our prior companion work [20], which like [21] showed that robustness could be achieved via stochastic stability in the discrete-time setting, similar to [22], [5] for the continuous-time setting.

In this paper we make the following contributions:

1) We demonstrate that robust stability under parametric perturbations implies mean-square stability under stochastic time-varying multiplicative noise (Theorem 1 and Corollary 4) with computationally tractable margins.
2) We show that Corollary 4 yields mean-square stability margins whose maximum size increases monotonically with the robust stability perturbations and collapse to zero when the allowable perturbation approaches zero.
3) Taken together with the results in [20], we demonstrate that mean-square stability and robust stability under parametric perturbations are "equivalent" in the sense that each implies the other with appropriate bounds on variations of the system matrices (§IV).

We introduce preliminary concepts in §II, develop the main result in §III, develop an equivalence relation in §IV, give a numerical example in §V, and conclude in §VI.

II. Notation and Preliminaries

$\mathbb{R}^{n\times m}$	Space of real-valued $n\times m$ matrices.
$\mathbb{S}^n$	Space of symmetric real-valued $n\times n$ matrices.
$\mathbb{S}^n_{++}$ ($\mathbb{S}^n_{+}$)	Space of symmetric real-valued positive (semi)definite $n\times n$ matrices.
$\rho(M)$	Spectral radius (greatest magnitude of an eigenvalue) of a square matrix M.
$\lVert M\rVert$	Spectral norm (greatest singular value) of a matrix M.
$M\otimes N$	Kronecker product of matrices M and N.
$\mathrm{vec}(M)$	Vectorization of matrix M by stacking its columns.
$M\succ(\succeq)\ 0$	Matrix M is positive (semi)definite.
$M\succ(\succeq)\ N$	Matrix M succeeds matrix N in a positive (semi)definite sense i.e. $M-N\succ(\succeq)0$.
I_n	Identity matrix of size $n\times n$.

A. Generalized eigenvalues and semidefiniteness

Consider $A \in \mathbb{S}^n$ and $B \in \mathbb{S}^n_{++}$. If $\lambda_{\max}$ is the maximum *generalized eigenvalue* which solves the *generalized eigenvalue problem (GEVP)* $Av = \lambda Bv$, then $\lambda_{\max} B \succeq A$. This is a standard result which follows e.g. because the maximum generalized eigenvalue can be expressed as the generalized Rayleigh quotient [23], [24]

$$\lambda_{\max} = \max_{y \neq 0 \in \mathbb{R}^n} \frac{y^\intercal A y}{y^\intercal B y} \geq \frac{x^\intercal A x}{x^\intercal B x} \text{ for any } x \neq 0 \in \mathbb{R}^n,$$

so

$$\lambda_{\max} x^\intercal B x \geq x^\intercal A x \text{ for any } x \in \mathbb{R}^n,$$

which shows indeed $\lambda_{\max} B \succeq A$.

B. System concepts

We consider an autonomous *discrete-time linear system* or *linear difference inclusion* of the form

$$x_{t+1} = A_t x_t \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the *state* and $A_t \in \Omega_t \subseteq \mathbb{R}^{n \times n}$ is the *system matrix*. The subset Ω_t represents the possible A_t which may be realized at time t. If $\Omega_t = \Omega$ is a constant singleton, i.e. $A_t = A$ for all $t = 0, 1, \dots$, then the system is *linear time-invariant (LTI)*, and otherwise *linear time-varying (LTV)*. If Ω_t is a (possibly time-varying) singleton, then A_t follows a prescribed sequence of matrices and the system is *deterministic*, whereas if Ω_t is non-singleton and A_t are random matrices, then the system is *stochastic*.

The initial state is assumed to be a random vector distributed as $x_0 \sim \mathbb{X}_0$. A deterministic initial state is a special case where $\mathbb{X}_0$ is a degenerate distribution with all probability density concentrated at a single point.

C. Stability concepts

The system (1) is *(asymptotically) stable* if all realized state sequences $\{x_t\}_{t=0}^{\infty}$ converge to zero as $t \to \infty$ regardless of the realized initial state x_0. It is *almost-surely stable* if $\{x_t\}_{t=0}^{\infty}$ converges to zero as $t \to \infty$ almost surely. It is *quadratically stable* if there exists a matrix $P \in \mathbb{S}^n_{++}$ such that the *quadratic Lyapunov function* $V(x) = x^\intercal P x$ decreases along any trajectory of the system, i.e. for all $t = 0, 1, \dots$ and all $A_t \in \Omega_t$

$$\begin{aligned} V(x_{t+1}) - V(x_t) &= x_{t+1}^\intercal P x_{t+1} - x_t^\intercal P x_t \\ &= x_t^\intercal (A_t^\intercal P A_t - P) x_t < 0, \end{aligned}$$

equivalent to satisfaction of the *Lyapunov inequality* [16]

$$P \succ A_t^\intercal P A_t \text{ for all } A_t \in \Omega_t.$$

Quadratic stability is sufficient for stability [16]. Quadratic stability is closely related to the notions of the *joint spectral radius* and *simultaneous contractability* of Ω_t, whose study has a long history in mathematics [25], [26], [27], [28] and is important in the field of switched linear systems.

The system (1) is *k-moment stable* if the k^{th} statistical moment of the state converges to zero as $t \to \infty$ regardless of the initial state distribution $\mathbb{X}_0$. The special case of interest in this work is *mean-square-stability (ms-stability)* ($k = 2$) so $\lim_{t \to \infty} \mathbb{E}[x_t x_t^\intercal] = 0$. Ms-stability of (1) is sufficient for both mean stability ($k = 1$) and almost-sure stability (since there is no additive noise) [16]. A stochastic linear system with a time-invariant distribution of system matrices that are independent across time may be decomposed as

$$x_{t+1} = \left(A + \textstyle\sum_{i=1}^{p} \gamma_{t,i} \Delta_i\right) x_t$$

where $A = \mathbb{E}[A_t]$ and $\gamma_t = \begin{bmatrix} \gamma_{t,1} & \cdots & \gamma_{t,p} \end{bmatrix}^\intercal$ is a random vector independent across time with $\mathbb{E}[\gamma_t] = 0$ and $\mathbb{E}[\gamma_t \gamma_t^\intercal] = \operatorname{diag}(\sigma_1^2, \cdots, \sigma_p^2)$, where σ_i^2 and Δ_i are the $p \leq n$ nonzero eigenvalues and eigenvectors of the covariance of A_t, respectively. In this form, it is apparent that the $\gamma_{t,i} \Delta_i$ are *state-multiplicative noises*, the scalars σ_i represent the *noise levels*, and the pattern matrices Δ_i represent the *noise directions*. In this form, ms-stability is equivalent to the existence of a matrix $P \in \mathbb{S}^n_{++}$ that satisfies the generalized Lyapunov inequality [16]

$$P \succ A^\intercal P A + \textstyle\sum_{i=1}^{p} \sigma_i^2 \Delta_i^\intercal P \Delta_i \tag{2}$$

If the system is LTI, then all the mentioned notions of stability are equivalent, and further equivalent to $\rho(A) < 1$.

III. Mean-square stability via robust stability

We now give our most general result, which answers the fundamental question posed in the introduction §I.

Theorem 1: Suppose the LTI system

$$x_{t+1} = (A + \mu_i \Delta_i) x_t \tag{3}$$

is stable for any static bounded perturbation μ_i such that $|\mu_i| \leq \eta_i$ for $i = 1, \dots, p$ for some positive scalars $\eta_i > 0$. Then there exist computable scalars $\beta_i \in (0, \infty)$ such that the stochastic system

$$x_{t+1} = \left(A + \textstyle\sum_{i=1}^{p} \gamma_{t,i} \Delta_i\right) x_t \tag{4}$$

is ms-stable for any distribution of the independent across time random vector $\gamma_t = \begin{bmatrix} \gamma_{t,1} & \cdots & \gamma_{t,p} \end{bmatrix}^\intercal$ that satisfies $\mathbb{E}[\gamma_t] = 0$ and $\mathbb{E}[\gamma_t \gamma_t^\intercal] = \operatorname{diag}(\sigma_1^2, \cdots, \sigma_p^2)$ where $\sigma_i \leq \beta_i$ for $i = 1, \dots, p$.

Proof: Fix matrices $R_i \succ 0$ and $S_i \succ 0$ for $i = 1, \dots, p$ and define the matrix-valued functions $M_i(\theta_i)$ and $N_i(\theta_i)$ that solve the Lyapunov equations

$$M_i(\theta_i) = R_i + (A + \theta_i \Delta_i)^\intercal M_i(\theta_i)(A + \theta_i \Delta_i), \tag{5}$$

$$N_i(\theta_i) = S_i + (A - \theta_i \Delta_i)^\intercal N_i(\theta_i)(A - \theta_i \Delta_i), \tag{6}$$

which are well-defined and positive definite for any $|\theta_i| \leq \eta_i$ by the stability assumption on (3). Define the sum and difference matrices

$$P_i(\theta_i) := \frac{1}{2}\left(M_i(\theta_i) + N_i(\theta_i)\right), \tag{7}$$

$$D_i(\theta_i) := \frac{1}{2}\left(M_i(\theta_i) - N_i(\theta_i)\right), \tag{8}$$

$$Q_i := \frac{1}{2}\left(R_i + S_i\right), \tag{9}$$

and the weighted average matrix $P(\theta)$ as

$$P(\theta) := \sum_{i=1}^{p} w_i^2 P_i(\theta_i) \tag{10}$$

where $\theta = [\theta_1 \ \cdots \ \theta_p]^\mathsf{T}$. Let scalars $\phi_i \in (0, \eta_i]$, which are collected in the vector $\phi = \begin{bmatrix}\phi_1 & \cdots & \phi_p\end{bmatrix}^\mathsf{T}$, and $\psi_i \in (0, \infty)$ satisfy, for $i = 1, \ldots, p$,

$$Q_i + \phi_i \left[A^\mathsf{T} D_i(\phi_i)\Delta_i + \Delta_i^\mathsf{T} D_i(\phi_i) A\right] \succ 0, \tag{11}$$

$$\Delta_i^\mathsf{T} P_i(\phi_i)\Delta_i \succeq \psi_i^2 \Delta_i^\mathsf{T} P(\phi)\Delta_i. \tag{12}$$

We first demonstrate that such scalars exist. Following arguments similar to [29], by vectorization,

$$\begin{aligned}\operatorname{vec}(M_i(\theta_i)) &= [(A+\theta_i\Delta_i)^\mathsf{T} \otimes (A+\theta_i\Delta_i)^\mathsf{T}] \operatorname{vec}(M_i(\theta_i)) + \operatorname{vec}(R_i)\\ \operatorname{vec}(N_i(\theta_i)) &= [(A-\theta_i\Delta_i)^\mathsf{T} \otimes (A-\theta_i\Delta_i)^\mathsf{T}] \operatorname{vec}(N_i(\theta_i)) + \operatorname{vec}(S_i).\end{aligned}$$

The eigenvalues of

$$\begin{aligned}G_i(\theta_i) &:= I_n \otimes I_n - (A+\theta_i\Delta_i)^\mathsf{T} \otimes (A+\theta_i\Delta_i)^\mathsf{T},\\ H_i(\theta_i) &:= I_n \otimes I_n - (A-\theta_i\Delta_i)^\mathsf{T} \otimes (A-\theta_i\Delta_i)^\mathsf{T},\end{aligned}$$

are, respectively,

$$\begin{aligned}&\{1 - \lambda_i(A+\theta_i\Delta_i)\lambda_j(A+\theta_i\Delta_i) \ : \ i, j = 1, \ldots, n\},\\ &\{1 - \lambda_i(A-\theta_i\Delta_i)\lambda_j(A-\theta_i\Delta_i) \ : \ i, j = 1, \ldots, n\},\end{aligned}$$

and all have magnitude less than 1 by the stability assumption on (3). Therefore $G_i(\theta_i)$ and $H_i(\theta_i)$ are invertible, so

$$\begin{aligned}\operatorname{vec}(M_i(\theta_i)) &= G_i(\theta_i)^{-1} \operatorname{vec}(R_i),\\ \operatorname{vec}(N_i(\theta_i)) &= H_i(\theta_i)^{-1} \operatorname{vec}(S_i).\end{aligned}$$

By Cramer's rule, $M_i(\theta_i)$ and $N_i(\theta_i)$, as well as the sum and difference matrices $P(\theta)$, $P_i(\theta_i)$, and $D_i(\theta_i)$, are rational functions of polynomials in θ_i and thus real analytic functions of θ, and thus continuous in θ. If $\phi_i = 0$ then (11) is satisfied since $Q_i \succ 0$ by construction. Likewise, if $\phi_i = 0$ then (12) is satisfied with $\psi_i^2 = 1$ since $P_i(0) = P(0)$. Therefore, by continuity of the singular values of the expressions of (11) and (12) in ϕ, there exist some $\phi_i > 0$ for $i = 1, \ldots, p$ for which (11) and (12) hold.

Fix a set of p weights $w_i \in (0, 1]$ such that $\sum_{i=1}^p w_i^2 = 1$. Now, by definition,

$$\begin{aligned}P(\phi) &= \textstyle\sum_{i=1}^p w_i^2 P_i(\phi_i) = \sum_{i=1}^p w_i^2 \cdot \frac{1}{2}\left[M_i(\phi_i) + N_i(\phi_i)\right]\\ &= \textstyle\sum_{i=1}^p w_i^2 \cdot \frac{1}{2}\left[R_i + (A+\phi_i\Delta_i)^\mathsf{T} M_i(\phi_i)(A+\phi_i\Delta_i)\right.\\ &\qquad\qquad \left. + S_i + (A-\phi_i\Delta_i)^\mathsf{T} N_i(\phi_i)(A-\phi_i\Delta_i)\right]\\ &= \textstyle\sum_{i=1}^p w_i^2 \left[Q_i + A^\mathsf{T} P_i(\phi_i) A + \phi_i^2 \Delta_i^\mathsf{T} P_i(\phi_i)\Delta_i\right.\\ &\qquad\qquad \left. + \phi_i\left(A^\mathsf{T} D_i(\phi_i)\Delta_i + \Delta_i^\mathsf{T} D_i(\phi_i) A\right)\right]\\ &= A^\mathsf{T} P(\phi) A + \textstyle\sum_{i=1}^p w_i^2 \left[\phi_i^2 \Delta_i^\mathsf{T} P_i(\phi_i)\Delta_i + Q_i\right.\\ &\qquad\qquad \left. + \phi_i\left(A^\mathsf{T} D_i(\phi_i)\Delta_i + \Delta_i^\mathsf{T} D_i(\phi_i) A\right)\right]\\ &\succ A^\mathsf{T} P(\phi) A + \textstyle\sum_{i=1}^p w_i^2 \phi_i^2 \Delta_i^\mathsf{T} P_i(\phi_i)\Delta_i \qquad \text{(by (11))}\\ &\succeq A^\mathsf{T} P(\phi) A + \textstyle\sum_{i=1}^p w_i^2 \psi_i^2 \phi_i^2 \Delta_i^\mathsf{T} P(\phi)\Delta_i. \qquad \text{(by (12))}\end{aligned}$$

Since $P(\phi)$ is the sum of positive definite matrices, $P(\phi) \succ 0$, and the claim follows with $\beta_i = w_i \psi_i \phi_i$ by (2). ∎

Regarding interpretation of the assumptions and claim of Theorem 1, we have the following facts. The system matrix $A + \mu_i \Delta_i$ of system (3) is a member of a cross-shaped subset of $\mathbb{R}^{n \times n}$ defined by the directions Δ_i and sized by the bounds η_i. By contrast, the random system matrices of system (4) are generally *not restricted* to any subset of $\mathbb{R}^{n \times n}$ i.e. the distribution of γ_t may have unbounded support and $\{\Delta_i\}_{i=1}^p$ may span all of $\mathbb{R}^{n \times n}$ if $p \geq n$. It is permissible for the system (3) to be time-varying and stochastic, as static perturbations μ_i are simply a special case. The diagonal (auto)covariance $\mathbb{E}[\gamma_t \gamma_t^\mathsf{T}] = \operatorname{diag}(\sigma_1^2, \cdots, \sigma_p^2)$ means that the $\gamma_{t,i}$ are uncorrelated across index i. If the system (3) cannot be certified as robustly stable, i.e. $\eta_i = 0$ for some $i = 1, \ldots, p$, then the assumption of Theorem 1 fails and stochastic system (4) cannot be certified as ms-stable.

Regarding selection of the scalar parameters, we have the practical Algorithm 1. Note that, for sufficiently small tolerance ε, the bisection in line 3 will terminate with $\phi_i > 0$ after a finite number of computations due to the existence of $\phi_i > 0$ shown in the proof of Theorem 1. Likewise the GEVP in line 9 is well-posed because $P_i(\phi_i) \succ 0$. The GEVP in line 9 ensures $P_i(\phi_i) \succeq \psi_i^2 P(\phi)$, which implies (12) holds as required. We make no claim that this algorithm produces the *largest* margins $\beta_i = w_i \psi_i \phi_i$, which would require a joint search over ψ and ϕ which appear together in (12).

Algorithm 1 Margin computation for Theorem 1

Input: System matrix A, perturbation directions Δ_i and robustness bounds η_i for $i = 1, \ldots p$, bisection tolerance $\varepsilon > 0$, penalty skew scale $\zeta > 0$.

1: **for** $i = 1, \ldots p$ **do**
2: Fix $R_i = \zeta I_n$, $S_i = (1/\zeta) I_n$, $Q_i = (R_i + S_i)/2$.
3: Find $\phi_i > 0$ that satisfies (11) via bisection [30] starting from the interval $[t^-, t^+] = [0, 1]$ with tolerance ε:
4: **while** $t^+ - t^- > \varepsilon$ **do**
5: $t^m = (t^+ + t^-)$
6: **if** (11) holds with $\phi_i = t^m \eta_i$ **then**
7: $t^- \leftarrow t^m$
8: **else**
9: $t^+ \leftarrow t^m$
10: Set $\phi_i = t^-$.
11: Compute matrix $P_i(\phi_i)$ from (7).
12: Compute matrix $P(\phi)$ from (10).
13: **for** $i = 1, \ldots p$ **do**
14: Solve the GEVP of $P(\phi) v = \lambda P_i(\phi_i) v$ for the maximum generalized eigenvalue $\lambda_{\max}$.
15: Set $\psi_i = 1/\sqrt{\lambda_{\max}}$.
16: Set $w_i = 1/\sqrt{p}$.

Output: Scalars $\beta_i = w_i \psi_i \phi_i$ for $i = 1, \ldots p$.

There is flexibility in choosing the weights w_i; they may be chosen equally as $w_i = 1/\sqrt{p}$, or may be assigned relatively greater weight in directions Δ_i for which ms-stability under greater noise variance is desired. As a special case when $p = 1$, the greatest possible weight is $w_i = 1$, in which case ms-stability in only a single direction Δ_i is guaranteed. Likewise, there is flexibility in choosing the penalty matrices R_i and S_i; optimizing with respect to these matrices leads to semidefinite programs discussed later after Corollaries 3 and 4. In Algorithm 1 we set R_i and S_i to scaled identity matrices with a single tunable skew scale parameter ζ to investigate the effect of differing R_i and S_i in the numerical experiments.

We now give specializations of Theorem 1 which offer practical utility and theoretical insight.

Corollary 2: Suppose the LTI system

$$x_{t+1} = \left(A + \sum_{i=1}^{p} \mu_i \Delta_i\right) x_t \tag{13}$$

is stable for any static bounded perturbations μ_i such that $|\mu_i| \leq \eta_i$ for $i = 1, \ldots, p$. Then the claim of Theorem 1 holds.

Proof: The claim follows immediately from Theorem 1 since $A + \mu_j \Delta_j = A + \sum_{i=1}^{p} \mu_i \Delta_i$ when $\mu_i = 0$ for all $i \neq j$. ■

In Corollary 2, the system (13) is a *polytopic linear difference inclusion (PLDI)* in the language of e.g. [16], meaning that the system matrix $A + \sum_{i=1}^{p} \mu_i \Delta_i$ is always a member of a convex polytope. We assume the special form of a *symmetric box polytope* for convenience when obtaining the ms-stability result by means of Theorem 1.

Corollary 3: Consider the notation and assumption (3) of Theorem 1. Suppose additionally that there exist matrices R_i and S_i such that the solutions of (5), (6), (7) are equal at η_i for each index $i = 1, \ldots, p$ individually:

$$M_i(\eta_i) = N_i(\eta_i) = P_i(\eta_i).$$

Then the stochastic system (4) is ms-stable for any noise levels $\sigma_i \leq \beta_i$ with $\beta_i = w_i \psi_i \eta_i$ for $i = 1, \ldots, p$.

Proof: By assumption, from the definition (8) we have $D_i(\eta_i) = 0$ for $i = 1, \ldots, p$ so $\phi_i = \eta_i$ is valid in (11) as $Q_i \succ 0$, and the claim follows. ■

Corollary 4: Consider the notation and assumption (3) of Corollary 3. Suppose additionally that there exist matrices R_i and S_i such that, such that the solutions of (5), (6), (7), (10) are equal at η_i, η_j for any pair of indices $i, j = 1, \ldots, p$ simultaneously:

$$M_i(\eta_i) = N_i(\eta_i) = M_j(\eta_j) = N_j(\eta_j) = P(\eta). \tag{14}$$

Then the stochastic system (4) is ms-stable for any noise levels $\sigma_i \leq \beta_i$ with $\beta_i = w_i \eta_i$ for $i = 1, \ldots, p$.

Proof: By assumption, $\psi_i = 1$ suffices and makes (12) hold with equality, and the claim follows. ■

The conditions of Corollaries 3 and 4 may be cast as linear matrix inequality (LMI) constraints in a feasibility semidefinite program (SDP). For example, the assumption of Corollary 4 is equivalent to quadratic stability of (13), which may be equivalently posed as the SDP

$$\begin{aligned}
&\text{find} && P \in \mathbb{S}^n_{++} && (15)\\
&\text{such that} && P \succ \left(A + \sum_{i=1}^{p} s_i \eta_i \Delta_i\right)^\intercal P \left(A + \sum_{i=1}^{p} s_i \eta_i \Delta_i\right) && (16)\\
& && \text{for } i = 1, \ldots, p, \quad \text{for any } s_i \in \{1, -1\}. && (17)
\end{aligned}$$

This is a standard method for certifying robust stability of linear systems in a convex polytope of state-space matrices [16]. If the SDP (16) solves successfully, then the ms-stability claim of Corollary 4 follows immediately, and the matrix P is called a *shared quadratic Lyapunov matrix*. Thus the SDP (16) may be considered an alternative to Algorithm 1.

Corollary 5: Assume only that the the spectral norm robust stability condition

$$\left\| A + \sum_{i=1}^{p} \mu_i \Delta_i \right\| < 1$$

holds for any bounded perturbations μ_i such that $|\mu_i| \leq \eta_i$ for $i = 1, \ldots, p$. Then the claim of Corollary 4 holds.

Proof: Fix a matrix $P \succ 0$ arbitrarily. Fix the sign scalars $s_i \in \{1, -1\}$ for $i = 1, \ldots, p$ arbitrarily, and define the associated system matrix $A' := A + \sum_{i=1}^{p} s_i \eta_i \Delta_i$. Then

$$P - A'^\intercal P A' \succeq P - \|A'\|^2 P = \left(1 - \|A'\|^2\right) P \succ 0$$

where the final inequality follows by assumption. This shows that $P = P(\eta)$ satisfies the assumption (14) in Corollary 4, and the claim follows. ■

IV. Equivalence of mean-square and robust stability

We first state the following result for completeness, which was proved in the authors' prior work [20]:

Theorem 7 of [20] (paraphrase): Suppose

$$x_{t+1} = \left(A + \sum_{i=1}^{p} \gamma_{t,i} \Delta_i\right) x_t \tag{18}$$

is ms-stable with $\mathbb{E}[\gamma_t] = 0$, $\mathbb{E}[\gamma_t \gamma_t^\intercal] = \operatorname{diag}(\sigma_1^2, \cdots, \sigma_p^2)$ where $\sigma_i > 0$ for $i = 1, \ldots, p$, and $\mathbb{E}[\gamma_t \gamma_\tau^\intercal] = 0$ for any $t \neq \tau$. Then there exist scalars η_i in the interval $0 < \eta_i < \sqrt{\zeta_i^2 + \sigma_i^2} - \zeta_i \leq \sigma_i$, where ζ_i are positive scalars, such that the LTI system

$$x_{t+1} = \left(A + \sum_{i=1}^{p} \mu_i A_i\right) x_t \tag{19}$$

is stable for any $|\mu_i| \leq \eta_i$ by virtue of a shared quadratic Lyapunov matrix.

With Corollary 4 of the present work and Theorem 7 of [20] in hand, we have the following equivalence result.

Corollary 6: The following two statements are equivalent:

1) There exist scalars $\sigma_i > 0$ such that

$$x_{t+1} = \left(A + \sum_{i=1}^{p} \gamma_{t,i} \Delta_i\right) x_t$$

is ms-stable for any distribution of the independent across time random vector $\gamma_t = \begin{bmatrix} \gamma_{t,1} & \cdots & \gamma_{t,p} \end{bmatrix}^\intercal$ that satisfies $\mathbb{E}[\gamma_t] = 0$ and $\mathbb{E}[\gamma_t \gamma_t^\intercal] = \operatorname{diag}(\sigma_1^2, \cdots, \sigma_p^2)$.

2) There exist scalars $\eta_i > 0$ such that

$$x_{t+1} = \left(A + \sum_{i=1}^{p} \mu_i A_i\right) x_t$$

is stable for any $|\mu_i| \leq \eta_i$ by virtue of a shared quadratic Lyapunov matrix.

Moreover, the scalars $\sigma_i > 0$ and $\eta_i > 0$ may be computed by solving semidefinite programs.

Proof: The claim follows immediately because
1) The assumptions of Corollary 4 of the present work and the guarantee of Theorem 7 of [20] coincide.
2) The assumptions of Theorem 7 of [20] and the guarantee of Corollary 4 of the present work coincide. ■

Corollary 6 shows that robust stability by virtue of a shared quadratic Lyapunov matrix and ms-stability in the same perturbation directions Δ_i are "equivalent" in the precise mathematical sense established here, analogous in spirit to the "equivalence" of norms on finite-dimensional vector spaces.

However, in general some amount of the ms-stability and robust stability margins captured by the scalars σ_i and η_i respectively, are lost with each application of these results. From Corollary 4 the σ_i are upper bounded by $w_i^2\eta_i \leq \eta_i$, and likewise Theorem 7 of [20] the η_i are upper bounded by $\sqrt{\zeta_i^2+\sigma_i^2}-\zeta_i \leq \sigma_i$. Although in certain special cases the inequalities in the two bounds may hold with equality e.g. the scalar case, a single uncertainty ($p=1$), or special alignments of A and Δ_i, in general they are strict so that a loss of margin occurs. Quantifying the general amount of margin loss in a constructive mathematical way is difficult, but becomes evident when computing the values required to make each result hold for a particular problem instance. Accordingly, care may be required in settings where these results are applied multiple times or recursively.

Equivalence between stochastic stability and deterministic stability was studied previously in the context of switched linear systems with arbitrary switching signals by [28] where it was found that, roughly speaking, k-moment stability "converges" to deterministic stability for arbitrary switching signals as k tends to infinity. Although related, this result must be distinguished from Corollary 6 for two key reasons: 1) like all results based on joint spectral radius, [28] generally requires boundedness of the switched system matrices, in contrast to the possibly unbounded support of the multiplicative noise distributions considered in this paper, and 2) Corollary 6 involves only ms-stability, not the limit of k-moment stability as k tends to infinity.

V. Numerical Example

In this section we demonstrate how this result may be applied in a practical setting. Consider the system with nominal system matrix

$$A = \begin{bmatrix} 0.976 & 0.097 \\ -0.477 & 0.937 \end{bmatrix}$$

and perturbation directions

$$\Delta_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

which arises as a discrete-time model of a non-inverted rigid pendulum with uncertain restoring constant and friction constant. The perturbation bounds

$$\eta_1^{\max} = 0.3924, \quad \eta_2^{\max} = 0.0400,$$

can be verified as valid to ensure stability of (3). Choosing $w_1^2 = w_2^2 = 0.5$, we use Algorithm 1 and SDP (16) to compute ms-stability margins according to Theorem 1 and Corollary 4. We examine the relationship between the assumed robust stability margins η_i and the ms-stability margins β_i by scaling the robust stability margins as $\eta_i = a\eta_i^{\max}$ for various $a \in (0,1]$; each setting of a may be considered a new problem instance. We also vary the skew scale parameter ζ in Algorithm 1 to observe its effect. The results are plotted in Figure 1. Since the assumption of Corollary 4 is stronger than that of Theorem 1, the SDP (16) may be infeasible; in such events we omit the data point from the plot, explaining why the SDP path terminates much earlier than those of Algorithm 1. Theorem 1 claims that the system (4) is ms-stable for $\sigma_i \leq \beta_i$, which is confirmed by checking the generalized Lyapunov inequality (2) with the matrix P and noise levels β_i found by each method.

Several interesting phenomena can be observed. First, it is clear that indeed useful nonzero ms-stability margins β_i are obtained, and are of the same order of magnitude as (but lesser than) η_i. For example, for $a = 1.0$, $\zeta = 1$ we have $\eta_1 = 0.3924$ and $\beta_1 = 0.05195$, and $\eta_2 = 0.0400$ and $\beta_2 = 0.0370$. Second, it is apparent that the margins from Algorithm 1 do not monotonically increase in each component with a; this is due to the bisection search over each component ϕ_i individually and their interaction in (12). By contrast, the ms-stability margins β_i of SDP (16) are proportional to the robustness margins η_i as $\beta_i = w_i\eta_i$ by Corollary 4, and thus do increase linearly with a. Third, we observe that ζ has a significant impact on the ms-stability margins found by Algorithm 1; on this problem $\zeta = 1$ happens to give larger margins than $\zeta = 5$ and $\zeta = 0.2$. This suggests opportunities for optimizing margins with respect to R_i and S_i, which is left to future work. Lastly, the inset plot reveals that all methods yield similar ms-stability margins near the origin.

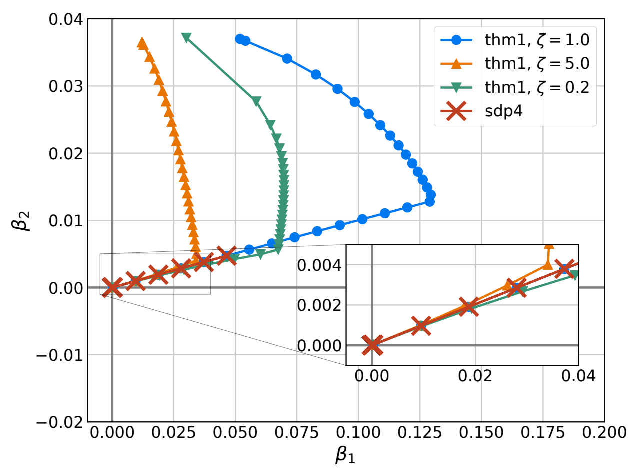

Fig. 1: Certified ms-stability levels β_i found by Algorithm 1 (thm1) and the SDP (16) (sdp4). As a varies linearly from $a = 0.001$ to $a = 1$ paths are swept out from near the origin to larger positive values.

VI. Conclusions and Future Work

We developed guarantees of mean-square stability of a stochastic system based solely on knowledge of nominal system dynamics and robust stability under static parametric perturbations. Further, we connected these results with existing converse results which together show that mean-square stability and robust stability are equivalent, up to a diminishing scaling of the mean-square and robust stability margins.

In principle, these results could be adapted for use in designing ms-stabilizing controllers by designing robustly stabilizing controllers that make the closed-loop system satisfy the assumptions of e.g. Theorem 1, analogous to the robust control design methods proposed in [20]. Indeed, many design tools for robust control exist already e.g. via $\mathscr{H}_\infty$ methods and dynamic game theory [31], [32] or system-level synthesis [19]. However, such an approach would be somewhat unsatisfying, as synthesis of optimal controllers in the presence of multiplicative noise is already relatively straightforward and tractable; see e.g. [2], [16], [33], [34] in the state-feedback setting and [35] in the output-feedback setting. However, in the setting where the nominal system dynamics and multiplicative noise statistics of the true system are not known *a priori* and must be estimated from data, e.g. as in model-based methods in adaptive control and reinforcement learning, such an approach may be effective. To elaborate, one could use existing design methods to find a controller which robustly stabilizes a set of presumptive LTI systems contained in an empirical uncertainty set e.g. defined by statistical bootstrap samples, then leverage Theorem 1 or its corollaries to certify ms-stability of the true system. Such applications will be examined in future work.

Future work will also explore ways to extend these results and methods to continuous-time systems by using (mean-square) stability-preserving transformations between continuous and discrete time [36], [37] and Lyapunov equation relations [38].

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