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Control of coleader formations in the plane

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***Abstract*—This paper addresses the n-agent formation shape maintenance problem in the plane. We consider a class of directed information architectures associated with so called *minimally persistent coleader* formations. The formation shape is specified by certain interagent distances. Only one agent is responsible for maintaining each distance. We propose a control law where each agent executes its control using only the relative position measurements of agents it must maintain its distance to. The resulting nonlinear closed-loop system has a manifold of equilibria; thus the linearized system is nonhyperbolic. We apply center manifold theory to show local exponential stability of the desired formation shape that circumvents the non-compactness of the equilibrium manifold. Choosing stabilizing gains is possible if a certain submatrix of the rigidity matrix has all leading principal minors nonzero, and we show that this condition holds for all minimally persistent coleader formations with generic agent positions.**

I. INTRODUCTION

Much attention has been given recently to control of autonomous vehicle formations and mobile sensor networks due to many promising scientific and engineering applications. Applications include teams of UAVs performing military reconnaissance and surveillance missions in hostile environments, satellite formations for high-resolution Earth and deep space imaging, and submarine swarms for oceanic exploration and mapping. For large formations, an overarching requirement is decentralized implementation, where each agent operates using only local information.

Precisely controlled formations can maintain a network of mobile sensing agents in an optimal sensing configuration. In this paper, we consider the n-agent formation shape maintenance problem. The objective is to design decentralized motion control laws for each agent so that the agents cooperatively and autonomously restore the desired formation shape in the presence of small perturbations from the desired shape. Formation shape is restored by actively controlling a certain set of interagent distances.

We utilize information architectures as a basis for designing control laws that allow decentralized implementation of the formation shape control task. The state information exchange architecture is directed and decentralized in that: (a) only one agent is responsible for maintaining a given interagent distance, (b) each agent executes its control law using only its position information relative to the agents with which it is responsible for maintaining its distance. In [1], Yu et al present decentralized nonlinear control laws for a *minimally persistent leader-first-follower (LFF)* formation with cycles in the associated directed graph to restore formation shape in the presence of small distortions from the desired shape. They show that choosing stabilizing control gains is possible if a certain submatrix of the *rigidity matrix* has all leading principal minors nonzero and further prove that all minimally persistent LFF formations generically obey this principal minor condition. In [2], Krick et al present decentralized gradient-based control laws for a minimally rigid formation (with undirected information architecture, i.e. a setting where each distance must be maintained by both the agents associated with it) to restore formation shape in the presence of small distortions from the desired shape. Since the linearized system is nonhyperbolic, they utilize center manifold theory to prove local exponential stability.

In this paper, we consider formations in the plane with *minimally persistent coleader* structure. We present decentralized nonlinear control laws in the sense described above and analogous to [1]. The nonlinear closed-loop system has a manifold of equilibria, which implies that the linearized system is nonhyperbolic. We apply center manifold theory to show local exponential stability of the desired formation shape; a key challenge of the argument, in contrast to that of [2], is to circumvent the non-compactness of the equilibrium manifold. Again, it becomes possible to choose stabilizing control gains whenever a certain submatrix of the rigidity matrix has all leading principal minors nonzero, and we show that this condition holds for all coleader formations.

The paper is organized as follows. Section II presents background on the structure of minimally persistent formations and center manifold theory. In Section III, we describe the nonlinear equations of motion and show how center manifold theory can be applied to show local exponential stability of the desired formation shape. In Section IV we show that the principal minor condition holds for coleader formations. Section V gives concluding remarks and future research directions.

II. BACKGROUND

In this section, we review (1) the structure of information architectures for minimally persistent formations, and (2) center manifold theory, which offers tools for analyzing stability of dynamical systems near nonhyperbolic equilibrium points.

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A. Minimally Persistent Formations

We model the information architecture as a graph $G(V, E)$ where V is a set of vertices representing agents and E is a set of edges representing the set of interagent distances to be actively controlled to maintain formation shape. We assign the task of controlling a particular interagent distance to only one of the involved agents, which results in a directed information architecture (as opposed to assigning it to both agents which results in an undirected information architecture). In this case, G is a directed graph where a direction is assigned to every edge in E with an outward arrow from the agent responsible for controlling the interagent distance. In order to maintain formation shape, G is required to be *persistent*. Let $F(G, p)$ denote a formation of n agents in the plane where $G(V, E)$ is a directed graph and $p : V \rightarrow \Re^{2n}$ is a position function which maps each vertex to a position in the plane. A formation is called *minimally persistent* if the information architecture G is minimally persistent, and G is minimally persistent if it is *minimally rigid* and *constraint consistent*. A minimally rigid graph on n vertices has $2n - 3$ edges which are well-distributed according to Laman's Theorem [3]. A constraint consistent graph precludes certain directed information flow patterns that make it impossible to control formation shape. See [4] and [5] for a thorough treatment of persistence. For the purposes of this paper it suffices to note the following

Theorem 1. *A graph $G(V, E)$ is minimally persistent iff the following hold: $|E| = 2|V| - 3$, for any subgraph $G(V', E')$ there holds $|E'| \leq 2|V'| - 3$, and no vertex has more than two outgoing edges.*

This means in particular that one of the following happens for minimally persistent formations. Type (A) [*Coleader* formations]: As in the coleader case of this paper three agents (known as coleaders), have one outgoing edge each and all others (known as ordinary followers), have exactly two such edges each. Type (B) [*Leader-First-Follower* (LFF) formations]: One agent known as the leader has no outgoing edge, another known as the first follower has one out going edge to the leader, and the remaining, ordinary followers have two outgoing edges each. Type (C) [*Leader-Remote-Follower* (LRF) formations]: One agent known as the leader has no outgoing edge, another known as the remote follower has one out going edge to a agent other than the leader, and the remaining ordinary followers have two outgoing edges each. In this paper, we consider a minimally persistent formation of type (A) and call it a *coleader* formation. Figure 1 illustrates several examples of coleader formations with differing information flow patterns and coleader connectivity.

The distinction between LFF and coleader formations is important in the stability analysis for the formation shape maintenance control laws. In particular, for LFF formations it is possible to define a global coordinate basis to obtain a hyperbolic reduced-order system in which local stability can be ascertained via eigenvalue analysis of the linearized system (see [1]). The device used in [1] to obtain a global coordinate system that provides a hyperbolic reduced-order system does not apply for coleader formations. Consequently, one cannot draw conclusions about the local stability of the nonlinear system near the desired formation shape by analyzing the linearized system alone; more sophisticated techniques are needed. Center manifold theory provides tools for determining stability near nonhyperbolic equilibrium points.

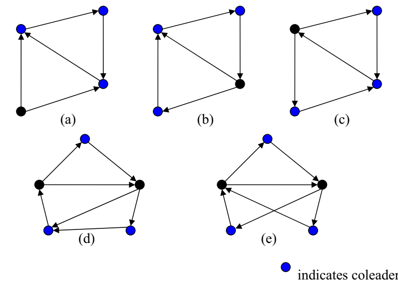

Fig. 1. Examples of coleader formations with connected coleaders: (a) cyclic coleaders, (b) inline coleaders, (c) v-coleaders. Examples of coleader formations with non-connected coleaders: (d) one-two coleaders, (e) distributed coleaders. Each coleader has only one interagent distance to maintain and so has only one outgoing arrow.

B. Center Manifold Theory

Standard treatments of center manifold theory can be found in e.g [6]–[8]. These concentrate on isolated equilibria. In the formation shape maintenance problem, the dynamic system has a manifold of non-isolated equilibrium points corresponding to the desired formation shape that for the coleader case is not even compact. In [9], Malkin proves a local stability result where trajectories converge to a point on an equilibrium manifold. More general results for equilibrium manifolds are presented by Aulbach in [10]. In [2], Krick emphasizes the importance of compactness for proving stability of the entire equilibrium manifold. Here, we state a result for stability of equilibrium manifolds that sidesteps the issue of compactness, and offer a concise proof using center manifold theory.

Consider the nonlinear autonomous dynamic system

$$\dot{x} = f(x), \quad x \in \Re^n \tag{1}$$

where the function f is $\mathbf{C}^r$, $r \geq 2$ almost everywhere including a neighborhood of the origin.. Suppose the origin is an equilibrium point and that the Jacobian of f (we will use the notation $J_f(x)$) at the origin has m eigenvalues with zero real part and $n - m$ eigenvalues with negative real part. Then (1) can be transformed into the following form

$$\begin{aligned} \dot{\theta} &= A_c\theta + g_1(\theta, \rho) \\ \dot{\rho} &= A_s\rho + g_2(\theta, \rho) \quad (\theta, \rho) \in \Re^m \times \Re^{n-m} \end{aligned} \tag{2}$$

where A_c is a matrix having eigenvalues with zero real parts, A_s is a matrix having eigenvalues with negative real parts, and the functions g_1 and g_2 satisfy

$$\begin{aligned} g_1(0,0) = 0, \quad J_{g1}(0,0) = 0 \\ g_2(0,0) = 0, \quad J_{g2}(0,0) = 0. \end{aligned} \tag{3}$$

Definition 1. *An invariant manifold is called a* center manifold *for (2) if it can be locally represented as follows*

$$W^c(0) = \{(\theta, \rho) \in U \subset \Re^m \times \Re^{n-m} | \rho = h(\theta)\} \quad (4)$$

for some sufficiently small neighbourhood of the origin U where the function h satisfies $h(0) = 0$ and $J_h(0) = 0$.

We have the following standard result.

Theorem 2 ([8]). *Consider (2) where A_c has eigenvalues with zero real part, A_s has eigenvalues with negative real part, and g_1 and g_2 satisfy (3). There exists a $\mathbf{C}^r$ center manifold for (2) with local representation function $h : \Re^m \to \Re^{n-m}$. The dynamics of (2) restricted to any such center manifold is given by the following m-dimensional nonlinear system for ξ sufficiently small*

$$\dot{\xi} = A_c \xi + g_1(\xi, h(\xi)), \quad \xi \in \Re^m. \quad (5)$$

If the origin of (5) is stable (asymptotically stable) (unstable), then the origin of (2) is stable (asymptotically stable) (unstable). Suppose the origin of (5) is stable. Then if $(\theta(t), \rho(t))$ is a solution of (2) for sufficiently small $(\theta(0), \rho(0))$, there is a solution $\xi(t)$ of (5) such that as $t \to \infty$

$$\begin{aligned} \theta(t) &= \xi(t) + O(e^{-\gamma t}) \\ \rho(t) &= h(\xi(t)) + O(e^{-\gamma t}) \end{aligned} \quad (6)$$

where γ is a positive constant.

This result show that in order to determine stability near the nonhyperbolic equilibrium point of (1), one can analyze a reduced-order system, viz. (5). If the origin of the reduced-order system (5) is stable, then the solutions of the original system converge exponentially to a trajectory on the center manifold.

We have the following result when there is a manifold of equilibria. The proof is omitted due to space limitations and will be included in a forthcoming journal version [11]. Observe that although the theorem postulates and proves the existence of a center manifold, it makes no explicit compactness assumptions.

Theorem 3. *Consider (1) with $f \in \mathbf{C}^r$, $r \geq 2$ almost everywhere including a neighbourhood of the origin. Suppose there is a smooth m-dimensional ($m > 0$) manifold of equilibrium points $M_1 = \{x \in \Re^n | f(x) = 0\}$ for (1) that contains the origin. Suppose at the origin the Jacobian of f has m eigenvalues with zero real part and $n-m$ eigenvalues with negative real part. Then*

- *M_1 is a center manifold for (1).*
- *There are neighborhoods Ω_1 and Ω_2 of the origin such that $M_2 = \Omega_2 \cap M_1$ is locally exponentially stable*[1] *and for each $x(0) \in \Omega_1$ there is a point $q \in M_2$ such that $lim_{t \to \infty} x(t) = q$.*

[1] Saying that M_2 is locally exponentially stable means that there is a single exponent γ such that all trajectories converge to M_2 from the neighbourhood Ω_1 at least as fast as $e^{-\gamma t}$.

In the formation shape maintenance problem, the manifold of equilibria will correspond to formation positions with the desired shape. In the plane, the manifold is three-dimensional due to the three possible Euclidean motions of the formation in the plane (two translational and one rotational). In the next section, we show how the results in this section can be applied to prove local exponential stability of the desired formation shape.

III. EQUATIONS OF MOTION

In this section, we present equations of motion for the formation shape maintenance problem and study the local stability properties of the desired formation shape. Suppose the formation is initially in the desired shape. Then the position of each agent is perturbed by a small amount and all agents move under distance control laws to meet their distance specifications in order to restore the desired formation shape. This shape is realized by every point on a three-dimensional equilibrium manifold. Even though the manifold is not compact, a direct application of Theorem 3 proves local exponential convergence to the invariant manifold.

A. Nonlinear Equations of Motion

Consider a minimally persistent formation $F(G, p)$ of n agents in the plane where the coleaders are agents $n-2$, $n-1$, and n. We define the rigidity function

$$r(p) = [..., ||p_j - p_k||^2, ...]^T \quad (7)$$

where the ith entry of r, viz. $||p_j - p_k||^2$, corresponds to an edge $e_i \in E$ connecting two vertices j and k. Let $d = [..., d_{jk}^{*2}, ...]$ represent a vector of the squares of the desired distances corresponding to each edge. We assume that there exist agent positions p such that $p = r^{-1}(d)$, i.e. the set of desired interagent distances corresponds to a realizable formation. Formation shape is controlled by controlling the interagent distance corresponding to each edge.

Following [1] and [2], we adopt a single integrator model for each agent:

$$\dot{p}_i = u_i. \quad (8)$$

Consider an ordinary follower agent denoted by i that is required to maintain constant distances d_{ij}^* and d_{ik}^* from agents j and k, respectively, and can measure the instantaneous relative positions of these agents. We use the same law as in [1] for the ordinary followers ($i = 1, ..., n-3$):

$$u_i = K_i(p_i^* - p_i) = K_i f_i(p_j - p_i, p_k - p_i, d_{ij}^*, d_{ik}^*) \quad (9)$$

where K_i is a gain matrix and p_i^* is the instantaneous target position for agent i in which the distances from agents j and k are correct. Since the perturbations from the desired shape are small, the instantaneous target positions are well-defined. For the coleaders ($i = n-2, n-1, n$), we have

$$\begin{aligned} u_i &= K_i(p_i^* - p_i) \\ &= K_i \frac{||p_j - p_i|| - d_{ij}^*}{||p_j - p_i||}(p_j - p_i) \end{aligned} \quad (10)$$

where K_i is a gain matrix and agent j is the agent from which coleader i is maintaining the constant distance d_{ij}^*.

Equations (9) and (10) represent the dynamics of the autonomous closed-loop system, which may be written in the form

$$\dot{p} = f(p) \tag{11}$$

where $f : \Re^{2n} \to \Re^{2n}$.

There is a *manifold* of equilibria for (11) given by

$$\Psi = \{p \in \Re^{2n} | p = r^{-1}(d)\} \tag{12}$$

corresponding to formations where all distance constraints are satisfied. The manifold Ψ is three-dimensional because a formation with correct distances has three degrees of freedom associated with the planar Euclidean motions (two for translation and one for rotation). Given these degrees of freedom, it is evident that Ψ is not compact.

B. Linearized Equations

We represent the position of the formation as $p(t) = \delta p(t) + \bar{p}$, where $\bar{p}$ is any equilibrium position with desired shape close to the perturbed formation, and the displacements $\delta p(t)$ are assumed to be small. In particular, for agent i we have $p_i(t) = \delta p_i(t) + \bar{p}_i$ where $\bar{p}_i$ corresponds to the position of agent i that meet its distance constraints. Let $p_i(t) = [x_i(t), y_i(t)]^T$, $\bar{p}_i = [\bar{x}_i, \bar{y}_i]^T$, and $\delta p_i(t) = [\delta x_i(t), \delta y_i(t)]^T$ in an arbitrary global coordinate system.

From [1], the linearized equation for the ordinary followers ($i = 1, ..., n-3$) is given by

$$\begin{bmatrix} \dot{\delta x}_i \\ \dot{\delta y}_i \end{bmatrix} = K_i R_{ei}^{-1} R_{ij,ik} \begin{bmatrix} \delta x_i \\ \delta y_i \\ \delta x_j \\ \delta y_j \\ \delta x_k \\ \delta y_k \end{bmatrix} \tag{13}$$

where K_i is a 2×2 gain matrix,

$$R_{ei} = \begin{bmatrix} (\bar{p}_j - \bar{p}_i)^T \\ (\bar{p}_k - \bar{p}_i)^T \end{bmatrix}$$

and

$$R_{ij,ik} = \begin{bmatrix} (\bar{p}_i - \bar{p}_j)^T & (\bar{p}_j - \bar{p}_i)^T & 0 \\ (\bar{p}_i - \bar{p}_k)^T & 0 & (\bar{p}_k - \bar{p}_i)^T \end{bmatrix}.$$

Similarly, the linearized equation for the coleaders ($i = n-2, n-1, n$) is given by

$$\begin{bmatrix} \dot{\delta x}_i \\ \dot{\delta y}_i \end{bmatrix} = K_i R_{ei}^{-1} R_{ij,00} \begin{bmatrix} \delta x_i \\ \delta y_i \\ \delta x_j \\ \delta y_j \end{bmatrix} \tag{14}$$

where K_i is a 2×2 gain matrix,

$$R_{ei} = \begin{bmatrix} \bar{x}_j - \bar{x}_i & \bar{y}_j - \bar{y}_i \\ \bar{y}_i - \bar{y}_j & \bar{x}_j - \bar{x}_i \end{bmatrix}$$

and

$$R_{ij,00} = \begin{bmatrix} (\bar{p}_i - \bar{p}_j)^T & (\bar{p}_j - \bar{p}_i)^T \\ 0 & 0 \end{bmatrix}.$$

Putting the equations together, we have

$$\dot{\delta p} = K R_e^{-1} \begin{bmatrix} R_{1,n-3} \\ r_{n-2} \\ 0 \\ r_{n-1} \\ 0 \\ r_n \\ 0 \end{bmatrix} \delta p \tag{15}$$

where $K = diag[K_1, ..., K_n]$ is a block diagonal gain matrix with each block of size 2×2 to be specified, $R_e = diag[R_{e1}, ..., R_{en}]$ is a block diagonal matrix with each block being a 2×2 submatrix of the rigidity matrix, and $[R_{1,n-3}^T, r_{n-2}^T, r_{n-1}^T, r_n^T]^T \in \Re^{2n-3 \times 2n}$ is the rigidity matrix. For a graph theoretic introduction to the rigidity matrix, see e.g. [12]; for examples of its application to formation control, see e.g. [1] and [2].

Expanding in a Taylor series about an equilibrium position, we can express (11) in the form

$$\dot{\delta p} = J_f(\bar{p})\delta p + g(\delta p) \tag{16}$$

where the first term represents the linearized system given by (15) and the second term represents the nonlinear part of order two or higher. The Jacobian $J_f(\bar{p})$ is rank deficient by three because of the three rows of zeros; consequently, three of its eigenvalues are zero. Thus the equilibrium position is non-hyperbolic, and we can seek to apply center manifold theory as developed in Section II to determine local stability of the equilibrium position. Since $J_f(\bar{p})$ has three zero eigenvalues, there exists an invertible matrix Q such that

$$Q J_f(\bar{p}) Q^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & A_s \end{bmatrix}. \tag{17}$$

where $A_s \in \Re^{2n-3 \times 2n-3}$ is a nonsingular matrix. Let $[\theta, \rho]^T = Q\delta p$ where $\theta \in \Re^3$ and $\rho \in \Re^{2n-3}$. Then (16) can be expressed in the form

$$\begin{aligned} \dot{\theta} &= g_1(\theta, \rho) \\ \dot{\rho} &= A_s \rho + g_2(\theta, \rho) \end{aligned} \tag{18}$$

where g_1 comprises the first three entries of $Qg(Q^{-1}[\theta, \rho]^T)$ and satisfies $g_1(0,0) = 0$ and $J_{g1}(0,0) = 0$, and g_2 comprises the last $2n-3$ entries of $Qg(Q^{-1}[\theta, \rho]^T)$ and satisfies $g_2(0,0) = 0$ and $J_{g2}(0,0) = 0$. This is in the normal form for center manifold theory.

To apply Theorem 3 we simply need the matrix A_s to be Hurwitz. Here, A_s must be *made* Hurwitz by a suitable choice of the gain matrices $K_1, ..., K_n$. Showing that such a choice of gains is indeed possible is the topic of the next section. Further, for each $\bar{p} \in \Psi$ there is a neighborhood of $\bar{p}$ throughout which the Jacobian is in C^2. Thus, if for a given $\bar{p}$, $J_f(\bar{p})$ has three zero eigenvalues and $2n-3$ eigenvalues in the open left half plane, then Theorem 3 proves that there is a neighborhood of $\bar{p}$ such that all trajectories commencing in this neighborhood converge to a point on Ψ. This point *may not be* $\bar{p}$.

IV. CHOOSING GAINS AND THE PRINCIPAL MINOR CONDITION

In this section we show that it is possible to choose the gain matrices for each agent such that all nonzero eigenvalues of the linearized system have negative real parts. This is the case if a certain submatrix of the rigidity matrix has all leading principal minors nonzero. That this condition is satisfied by all coleader formations is shown in the following. The arguments are similar but not identical to those of [1].

Let the gain matrices $K_1, ..., K_n$ be chosen as follows:

$$K_i = \Lambda_i R_{e,i}, \quad i = 1, ..., n \tag{19}$$

where the Λ_i are diagonal matrices. Reorder the coleader coordinates as $q = [x_1, y_1, ..., x_{n-3}, y_{n-3}, x_{n-2}, x_{n-1}, y_n, y_{n-2}, y_{n-1}, x_n]^T$. An equilibrium position $\bar{q}$ is defined from $\bar{p}$ in the same manner as q is defined from p. Then the linearized system has the form

$$\dot{\delta q} = \Lambda \begin{bmatrix} \hat{R} & R_{12} \\ 0 & 0 \end{bmatrix} \delta q = J_q(\bar{q}) \delta q \tag{20}$$

where $\Lambda \in \Re^{2n \times 2n}$ is a diagonal matrix whose diagonal entries can be chosen independently and $\hat{R} \in \Re^{2n-3 \times 2n-3}$ is a submatrix of the rigidity matrix R that we will now define. Recall that R has two columns associated with each agent: one comprised of x-coordinates and one of y-coordinates. The matrix $\hat{R}$ is obtained by removing the three columns from the rigidity matrix R as follows: one associated with each coleader and not all of x or y-type (i.e. one must remove two x-type and one y-type or vice versa). We have the following result from [1].

Theorem 4. *Suppose $\hat{R}$ defined above is a nonsingular matrix with every leading principal minor nonzero and let $\Lambda = diag(\Lambda_1, \Lambda_2)$ with $\Lambda_1 \in \Re^{2n-3 \times 2n-3}$ diagonal and $\Lambda_2 \in \Re^3$. Then there exists a diagonal matrix Λ_1 such that the real parts of all nonzero eigenvalues of the linearized system are negative.*

Thus, $2n-3$ eigenvalues of $J_q(\bar{q})$ have negative real part and clearly the remaining three eigenvalues are zero due to the rank deficiency of $J_q(\bar{q})$. To make use of Theorem 4, we now need to show that $\hat{R}$ satisfies the principal minor condition for all coleader formations. Let $V' = \{v_1, ..., v_{n-3}\}$ represent the set of ordinary followers, and let agents v_{n-2}, v_{n-1}, and v_n correspond to the coleaders. We have the following result:

Theorem 5. *Consider any minimally persistent coleader formation $F(G, p)$ of n agents at generic positions in the plane. Then there exists an ordering of the vertices of F and an ordering of the pair of outgoing edges for each vertex such that all leading principal minors of the associated $\hat{R}$ are generically nonzero.*

We note the following structure of $\hat{R} \in \Re^{2n-3 \times 2n-3}$

$$\hat{R} = \begin{bmatrix} R(V') & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} \end{bmatrix}. \tag{21}$$

$R(V') \in \Re^{2n-6 \times 2n-6}$ is the principal submatrix of $\hat{R}$ obtained by retaining the rows and columns corresponding to the elements of V'. Additionally, consider a subset of ordinary follower vertices $V_1 \subseteq V'$ and define $R(V_1)$ as the principal submatrix of $\hat{R}$ obtained by retaining columns corresponding to the elements of V_1. We have the following results from [1] for LFF formations, which extend to coleader formations with identical proof.

Lemma 1. *For any minimally persistent coleader formation, $\hat{R}$ is generically nonsingular, and $R(V_1)$ is generically nonsingular for every $V_1 \subseteq V'$.*

Lemma 1 establishes that the largest leading principal minor is generically nonzero and that all even order leading principal minors up to size $2n-6$ are generically nonzero. The proof that all odd order leading principal minors up to size $2n-7$ are also generically nonzero relies on an appropriate ordering of edges and is identical to the proof in [1]. It now remains to show that the second and third largest leading principal minors (of size $2n-5$ an $2n-4$) are generically nonzero. We have the following two results that treat separately the case where the coleaders are connected and the case where the coleaders are not connected.

Lemma 2. *Suppose at most one coleader has its outgoing edge to V', the set of ordinary followers. Then the second and third largest leading principal minors of $\hat{R}$ are generically nonzero.*

Proof: If at most one coleader has its outgoing edge to V', then there exists an ordering of the coleaders such that the second and third largest leading principal submatrices of $\hat{R}$ have the structure

$$M_{2n-5} = \begin{bmatrix} R(V') & \times \\ 0 & x_{n-2} - \{x_{n-1}, x_n\} \end{bmatrix}$$

$$M_{2n-4} = \begin{bmatrix} M_{2n-5} & \times \\ 0 & x_{n-1} - x_n \end{bmatrix}$$

where $\times$ is a "don't care" vector (only v_n may have an edge to V'). Since $R(V')$ is generically nonsingular, then M_{2n-5} is generically nonsingular, which then implies that M_{2n-4} is generically nonsingular. ■

Lemma 3. *Suppose at least two coleaders have their outgoing edges to V', the set of ordinary followers. Then the second and third largest leading principal minors of $\hat{R}$ are generically nonzero.*

Proof: Assume that coleaders labeled $n-2$ and $n-1$ have outgoing edges to vertices i and j, respectively, both in V'. Observe that the hypothesis permits $i = j$. The argument below applies regardless of whether $i = j$ and regardless of whether the sole outgoing edge of n is to an element of V'. Introduce two artificial agents labeled $n+1$ and $n+2$. Assume that $n+1$ has only one outgoing edge and that to $n+2$; $n+2$ has no outgoing edge; $n-2$ (which was a coleader) has an additional outgoing edge to $n+1$, $n-1$ an additional edge to $n+2$ and n an additional edge to either $n+1$ or $n+2$.

Using the fact that G is minimally persistent and Theorem 1, it can be shown that resulting graph is minimally persistent with a LFF structure, with $n+2$ the leader and $n+1$ the first follower. Call this new graph $G_{LFF}=(V_2,E_2)$. Consider the matrices below where B_l is the l-th order principal submatrix of the matrix obtained by removing the three last columns of the rigidity matrix of the artificial graph of G_{LFF}.

$$B_{2n-5}=\begin{bmatrix} R(V') & a_1 \\ b_1 & x_{n-2}-x_i \end{bmatrix}$$

$$B_{2n-4}=\begin{bmatrix} R(V') & a_1 & a_2 \\ b_1 & x_{n-2}-x_i & y_{n-2}-y_i \\ 0 & x_{n-2}-x_{n+1} & x_{n-2}-x_{n+1} \end{bmatrix}$$

$$B_{2n-3}=\begin{bmatrix} R(V') & a_1 & a_2 & a_3 \\ b_1 & x_{n-2}-x_i & y_{n-2}-y_i & 0 \\ 0 & x_{n-2}-x_{n+1} & x_{n-2}-x_{n+1} & 0 \\ b_2 & 0 & 0 & x_{n-1}-x_j \end{bmatrix}$$

$$B_{2n-2}=$$

$$\begin{bmatrix} R(V') & a_1 & a_2 & a_3 & a_4 \\ b_1 & x_{n-2}-x_i & y_{n-2}-y_i & 0 & 0 \\ 0 & x_{n-2}-x_{n+1} & x_{n-2}-x_{n+1} & 0 & 0 \\ b_2 & 0 & 0 & x_{n-1}-x_j & y_{n-1}-y_j \\ 0 & 0 & 0 & x_{n-1}-x_{n+2} & y_{n-1}-y_{n+2} \end{bmatrix}$$

We note that all the even-dimensioned submatrices above are generically nonsingular, since they are the even-dimensioned leading principal submatrices of a LFF structure (see [1]).

The third largest leading principal submatrix of $\hat{R}$ is given by $M_{2n-5}=B_{2n-5}$. Suppose B_{2n-5} is not generically nonsingular. Then because of the underlying symmetry of the x and y columns, neither is the matrix

$$\begin{bmatrix} R(V') & a_2 \\ b_1 & y_{n-2}-y_i \end{bmatrix}.$$

But this implies that B_{2n-4} is generically nonsingular, which establishes a contradiction. Therefore, M_{2n-5} is generically nonsingular.

The second largest leading principal submatrix of $\hat{R}$ is

$$M_{2n-4}=\begin{bmatrix} R(V') & a_1 & a_2 \\ b_1 & x_{n-2}-x_i & 0 \\ b_2 & 0 & x_{n-1}-x_j \end{bmatrix}.$$

We now argue that this matrix is nonsingular. Since B_{2n-2} is generically nonsingular, an argument similar to above establishes the generic nonsingularity of B_{2n-3}, which implies the generic nonsingularity of

$$\begin{bmatrix} R(V') & a_1 & a_2 & a_3 \\ b_1 & x_{n-2}-x_i & y_{n-2}-y_i & 0 \\ b_2 & 0 & 0 & x_{n-1}-x_j \\ 0 & x_{n-2}-x_{n+1} & x_{n-2}-x_{n+1} & 0 \end{bmatrix}.$$

Finally, another argument similar to above then establishes the generic nonsingularity of M_{2n-4}. ■

Thus, all leading principal minors of $\hat{R}$ are generically nonzero for all coleader formations. Therefore, one can choose the diagonal matrix Λ such that the real parts of all nonzero eigenvalues of the linearized system (20) are negative (and accordingly the matrix A_s in (18) is Hurwitz). The stabilizing gains are designed for a particular equilibrium point in Ψ. It is important to note here that the control gains proposed in (19) may not be stabilizing for all other points in Ψ. Theorem 3 can be directly applied to show that for each $\bar{p}\in\Psi$, there is a neighbourhood $\Omega(\bar{p})$ of $\bar{p}$ such that for any initial formation position $p(0)\in\Omega(\bar{p})$ there is a point $p^*\in\Psi$ such that $lim_{t\to\infty}p(t)=p^*$ at an exponential rate, i.e. the formation converges locally exponentially to the desired shape.

V. CONCLUDING REMARKS

In this paper, we have addressed the n-agent formation shape maintenance problem for coleader formations. We presented decentralized nonlinear control laws that restore desired formation shape in the presence of small perturbations from the nominal shape. The nonlinear system has a manifold of equilibria, which implies that the linearized system is nonhyperbolic. We applied center manifold theory to show local exponential stability of the equilibrium formation with desired shape. We have also shown that a principal minor condition holds for all coleader formations, which allows a choice of stabilizing gain matrices.

There are several possible directions for future research. First, the stability results here are local, and an immediate task would be to determine the size of the region of attraction. Second, one can show that the stability properties of our control law are translationally, but not rotationally invariant. One could investigate whether or not it is possible to constrain the gain matrices in order to obtain rotational invariance. Preliminary calculations suggest that this will not always be possible. Finally, non-minimally persistent formations will eventually be of interest because it may be desirable to control more than the minimum number of distances for formation shape maintenance in order to obtain a level of robustness.

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