Anomaly Detection Under Multiplicative Noise Model Uncertainty

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***Abstract*— State estimators are crucial components of anomaly detectors that are used to monitor cyber-physical systems. Many frequently used state estimators are susceptible to model risk as they rely critically on the availability of an accurate state-space model. Modeling errors make it more difficult to distinguish whether deviations from expected behavior are due to anomalies or simply a lack of knowledge about the system dynamics. In this research, we account for model uncertainty through a multiplicative noise framework. Specifically, we propose two different state estimators in this setting to hedge against the model uncertainty risk namely, 1) multiplicative noise LQG, and 2) Wasserstein distributionally robust Kalman filter. The size of the residual from either estimator can then be compared against a threshold to detect anomalies. Finally, the proposed detectors are validated using numerical simulations. Extension of state-of-the-art anomaly detection in cyber-physical systems to handle model uncertainty represents the main novel contribution of the present work.**

I. Introduction

Cyber-Physical Systems (CPS) are physical processes that are tightly integrated with computation and communication systems for monitoring and control. Though advances in CPS design has equipped them with adaptability, resiliency, safety, and security features that exceed the simple embedded systems of the past, it often leaves open several points for attackers to strike. CPS security problems have attracted the attention of researchers worldwide recently; some state-of-the-art anomaly detection algorithms can be found in [1]–[3].

A common practice is to model a CPS as either a deterministic system or a stochastic system with additive Gaussian uncertainties. Motivated by the recent developments in distributionally robust optimization (DRO) techniques [4]–[6], authors in [7]–[9] have developed DRO anomaly detectors that remove assumptions on specific functional forms of the uncertainties in the stochastic CPS model. On the other hand, it is a common practice to assume that the true CPS dynamics are known exactly. Unfortunately, modeling and sampling errors are inherent and significant in working with real systems due to nonlinearities, learned (system identification, machine learning) models, adaptive models, or simply due to changing environmental conditions or aging. A multiplicative noise framework for capturing model uncertainty offers several compelling advantages over additive noise models. It provides a statistical description of the uncertainty that depends on the control input and state [10]–[12]. Using a multiplicative noise model, however, requires new tools to build and tune anomaly detectors that accommodate the more general functional form of the model.

State estimation is a crucial component in any model-based anomaly detector design, which depends on a state-space model for the system dynamics. This dependency causes limitations on the usage of the classical Kalman filter as it critically relies on the availability of an accurate state-space model, making it susceptible to model risk. Robust Kalman filtering with additive uncertainties was explored in [13], where the uncertain joint distribution of the states and outputs was accounted for. Another robust Kalman filter design was developed using a τ-divergence based family of distributions in [14]. In [15], a Wasserstein distributionally robust Kalman filter (W-DR-KF) was developed to account for distributional uncertainty. We propose to use a variant of W-DR-KF with control inputs in this paper.

Although stochastic modeling of CPS with additive uncertainty is well studied, there are no works to the best of our knowledge which have considered both multiplicative and additive noises together in the CPS security literature. The evolution of non-Gaussian state distributions under the effect of multiplicative noise invalidates use of the standard Kalman filter, as the separation principle advocated in linear quadratic Gaussian (LQG) setting in [16] no longer holds. Though [10] considered both multiplicative and additive noises in an optimal control setting, a restrictive Gaussian assumption was imposed on the uncertainties. The approach in this paper builds on the foundation established by [17], where the multiplicative noise-driven LQG (MLQG) problem was solved by posing a set of coupled algebraic Riccati equations, from which the optimal linear output feedback controller and estimator gains were jointly computed.

Contributions: This paper is part of our ongoing work [7], [8] to leverage results from distributionally robust optimization to design robust anomaly detectors. Specifically, the detector threshold corresponding to a desired false alarm rate in the setting considered in this paper was computed through the moment-based approaches explained [7]. In prior work we addressed detectors robust to non-Gaussian additive noise. In this work,

1) We design an anomaly detector for stochastic linear cyber-physical systems that is robust to modeling errors. To our knowledge, this is the first paper to consider tuning an anomaly detector for a system model that incorporates model uncertainty. We propose a multiplicative noise framework and integrate two alternative estimators to compute the residual: the MLQG or the W-DR-KF.

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2) We demonstrate our tuning methodology using a numerical simulation and show that multiplicative noises result in greater anomaly detector thresholds as long as mean square compensatability conditions are satisfied.

The rest of the paper is organized as follows. In §II, the problem of monitoring an uncertain CPS with model uncertainty is formulated. Then, the multiplicative noise driven LQG and the Wasserstein distributionally robust Kalman filter based state estimators are proposed in §III and §IV respectively. Subsequently, the anomaly detector design is presented in §V. The proposed idea is then demonstrated using a numerical simulation in §VI. Finally, the paper is closed in §VII along with directions for future research.

Notations & Preliminaries

The set of real numbers, integers are denoted by $\mathbb{R}, \mathbb{Z}$. The subset of real numbers greater than $a \in \mathbb{R}$ is denoted by $\mathbb{R}_{>a}$. The set of integers between two values $a, b \in \mathbb{Z}$ with $a < b$ is denoted by $[a : b]$. We denote by $\mathbb{S}^n$ the set of symmetric matrices in $\mathbb{R}^{n\times n}$ and the cone of positive definite (semi-definite) matrices on $\mathbb{S}^n$ as $\mathbb{S}^n_{++}(\mathbb{S}^n_{+})$. An identity matrix in dimension n is denoted by I_n. The Kronecker product of two matrices $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{p\times q}$ is denoted by $A\otimes B$ and the vectorization of a matrix $A \in \mathbb{R}^{m\times n}$ is denoted by $\mathrm{vec}(A) \in \mathbb{R}^{mn}$ and the matricization of vector $x \in \mathbb{R}^p$ is denoted by $\mathrm{mat}(x, n, m) \in \mathbb{R}^{n\times m}$ where $n \times m = p$. We denote by $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{P}(\mathbb{R}^d)$ the Borel $\sigma-$algebra on $\mathbb{R}^d$ and the space of probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ respectively. A probability distribution with mean μ and covariance Σ is denoted by $P(\mu, \Sigma)$, and $\mathcal{N}(\mu, \Sigma)$ if the distribution is normal. Given a constant $q \in \mathbb{R}_{\geq 1}$, the set of probability measures in $\mathcal{P}(\mathbb{R}^d)$ with finite $q-$th moment is denoted by $\mathcal{P}_q(\mathbb{R}^d) := \left\{\mu \in \mathcal{P}(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} \|x\|^q \, d\mu < \infty\right\}$. The type-$q$ Wasserstein distance $\forall q \geq 1$ between $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathcal{P}_q(\mathbb{R}^d)$ with $\Pi(\mathbb{Q}_1, \mathbb{Q}_2)$ being the set of all joint probability distributions on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals $\mathbb{Q}_1$ and $\mathbb{Q}_2$ is defined as

$$W_q(\mathbb{Q}_1, \mathbb{Q}_2) \triangleq \left(\inf_{\pi\in\Pi(\mathbb{Q}_1,\mathbb{Q}_2)} \int_{\mathbb{R}^d\times\mathbb{R}^d} \|z_1 - z_2\|^q \, \pi(dz_1, dz_2)\right)^{\frac{1}{q}}. \tag{1}$$

II. Problem Formulation

A. Uncertain CPS Model

We model an uncertain CPS for time $k \in \mathbb{N}$ using a stochastic discrete-time linear time varying (LTV) system:

$$x_{k+1} = A_k x_k + B_k u_k + w_k, \tag{2}$$
$$y_k = C_k x_k + v_k. \tag{3}$$

Here, $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, and $y_k \in \mathbb{R}^p$ are the system state, control input, and output at time k. The next-state $x_{k+1} \in \mathbb{R}^n$ is a random linear combination of the current state and process noise w_k, which is a zero-mean white noise process. Similarly, the output $y_k \in \mathbb{R}^p$ is a random linear combination of the states and the sensor noise v_k, which is a zero-mean white noise process. The initial state is a random variable $x_0 \sim P_{x_0}(0, \Sigma_{x_0})$. The system matrices are decomposed as

$$A_k = \left(\bar{A} + \hat{A}_k\right), \; B_k = \left(\bar{B} + \hat{B}_k\right), \; C_k = \left(\bar{C} + \hat{C}_k\right),$$
$$\hat{A}_k = \sum_{i=1}^{n_a} \gamma_{ki}\mathcal{A}_i, \quad \hat{B}_k = \sum_{j=1}^{n_b} \delta_{kj}\mathcal{B}_j, \quad \hat{C}_k = \sum_{l=1}^{n_c} \kappa_{kl}\mathcal{C}_l. \tag{4}$$

where $\bar{A}$, $\bar{B}$, $\bar{C}$ denote the nominal dynamics, control, and output matrices respectively. The multiplicative noise terms are modeled by the i.i.d. across time (white), zero-mean, mutually independent scalar random variables γ_{ki}, δ_{kj}, κ_{kl}, which have variances $\sigma^2_{a,i}$, $\sigma^2_{b,j}$, $\sigma^2_{c,l}$ for $i \in [1 : n_a]$, $j \in [1 : n_b]$, $l \in [1 : n_c]$ respectively with $n_a, n_b, n_c \in \mathbb{Z}_{>0}$. The pattern matrices $\mathcal{A}_i \in \mathbb{R}^{n\times n}$, $\mathcal{B}_j \in \mathbb{R}^{n\times m}$, and $\mathcal{C}_l \in \mathbb{R}^{p\times n}$ specify how each scalar noise term affects the system matrices. It is then evident from (2) and (3) that $\hat{A}_k, \hat{B}_k, \hat{C}_k$ quantify uncertainty about the nominal system matrices $\bar{A}, \bar{B}, \bar{C}$ respectively. The distributions of all the scalar multiplicative noise random variables are assumed to be known. On the other hand, the distributions of the process noise $\mathbb{P}_w$ and measurement noise $\mathbb{P}_v$ are not known exactly, but reside in the moment-restricted ambiguity sets

$$\mathcal{P}^w := \left\{\mathbb{P}_w \mid \mathbb{E}[w_k] = 0, \underline{\Sigma_w} \preceq \mathbb{E}[w_k w_k^\top] \preceq \overline{\Sigma_w}\right\}, \tag{5}$$
$$\mathcal{P}^v := \left\{\mathbb{P}_v \mid \mathbb{E}[v_k] = 0, \underline{\Sigma_v} \preceq \mathbb{E}[v_k v_k^\top] \preceq \overline{\Sigma_v}\right\}, \tag{6}$$

where the covariance bounds $(\underline{\Sigma_w}, \overline{\Sigma_w})$ and $(\underline{\Sigma_v}, \overline{\Sigma_v})$ are assumed known; they may be estimated from collected data via e.g. bootstrap sample averaging. Further, the ambiguity sets are assumed to contain the true covariances Σ_w and Σ_v respectively. For simplicity, we assume that x_0 and all the additive, multiplicative noises $w_k, v_k, \{\gamma_{ki}\}_{i=1}^{n_a}, \{\beta_{kj}\}_{j=1}^{n_b}, \{\kappa_{kl}\}_{l=1}^{n_c}$ are mutually independent of each other. We denote the first moment, second moment, and covariance of the state at time k as $\mu_{x_k} = \mathbb{E}[x_k]$, $V_k = \mathbb{E}\left[x_k x_k^\top\right]$, and $\Sigma_{x_k} = \mathbb{E}\left[(x_k - \mu_{x_k})(x_k - \mu_{x_k})^\top\right]$, respectively. Likewise, we denote the first moment, second moment, and covariance of the output at time k as $\mu_{y_k} = \mathbb{E}[y_k]$, $Y_k = \mathbb{E}\left[y_k y_k^\top\right]$, and $\Sigma_{y_k} = \mathbb{E}\left[(y_k - \mu_{y_k})(y_k - \mu_{y_k})^\top\right]$, respectively.

B. Review of Concepts

Here, we re-state some definitions from [17] on the mean squared versions of stabilizability, detectability and the resulting compensatability of systems given by (2) and (3).

Definition 1: The system in (2) is *mean-square stable* if $\forall x_0 \in \mathbb{R}^n, \exists V_\infty \in \mathbb{S}^n_+$ such that

$$\lim_{k\to\infty} V_k = \lim_{k\to\infty} \mathbb{E}\left[x_k x_k^\top\right] \to V_\infty.$$

Definition 2: The system in (2) is *mean-square stabilizable* if there exists a control gain matrix $K \in \mathbb{R}^{m\times n}$ such that using controls $u_k = Kx_k$ makes (2) mean-square stable.

Definition 3: The system in (2) and (3) is *mean-square compensatable* if there exists a control and filter gain matrices $K \in \mathbb{R}^{m\times n}$ and $L \in \mathbb{R}^{n\times p}$ such that the system

$$\begin{bmatrix} x_{k+1} \\ \hat{x}_{k+1} \end{bmatrix} = \begin{bmatrix} A_k & B_k K \\ LC_k & \bar{A} + \bar{B}K - L\bar{C} \end{bmatrix} \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix}$$

is mean-square stable.

Assumptions

1) The system given by (2) and (3) is *mean-square compensatable*.
2) The optimal state estimator at any time k given (2) and (3) is an affine function of the output y_k.

Problem 1: Under the above assumptions for a given stochastic CPS model specified by (2), (3), obtain residual data from an appropriate state estimator module that accounts for both multiplicative and additive noises, and subsequently design an anomaly detector threshold such that the worst case false alarm rate does not exceed a desired value.

III. Residuals via Multiplicative Noise LQG

Due to the multiplicative noises in (2) and (3), the state distribution will be non-Gaussian even when all primitive noise distributions are Gaussian. Further, the classical separation principle from the additive noise setting does not hold in presence of multiplicative noises [17]. This necessitates a framework where the optimal controller and the estimator gains are computed *jointly*. Here, we elaborate on obtaining the residual from CPS using the multiplicative noise-driven LQG and show that the residual covariance is a function of both additive and multiplicative noise covariance matrices.

A. Designing Multiplicative Noise-Driven LQG

Under both multiplicative and additive noises in the system, the optimal linear output feedback controller can be exactly computed through the combination of a multiplicative noise KF with a multiplicative noise LQR as described in [10], [17], [18]. We consider the multiplicative noise-driven linear-quadratic Gaussian (MLQG) optimal control problem, which requires finding an output feedback controller $u_k = \pi_k(y_{0:k})$ for a system given by (2) and (3):

$$\begin{aligned} \underset{\pi_k \in \Pi_k}{\text{minimize}} \quad & \lim_{T\to\infty} \frac{1}{T}\mathbb{E}_{\mathcal{E}_k}\left[\sum_{k=0}^{T-1} x_k^\top Q x_k + u_k^\top R u_k\right], \\ \text{subject to} \quad & (2), (3), \end{aligned} \tag{7}$$

where $\mathcal{E}_k = \left\{x_0, \{\hat{A}_k\}, \{\hat{B}_k\}, \{\hat{C}_k\}, \{w_k\}, \{v_k\}\right\}$, $Q \succeq 0$, $R \succ 0$. In this section, we assume that the covariance of both w_k and v_k take the maximum bound that is used in their respective ambiguity sets. That is, $\Sigma_w = \overline{\Sigma_w}, \Sigma_v = \overline{\Sigma_v}$. Then, the optimal linear compensator and filter gain matrices can be computed by solving coupled nonlinear matrix Riccati equations in symmetric matrix variables $P_1, P_2, P_3, P_4 \in \mathbb{S}^n_+$. We have stated the lengthy coupled Riccati equations in the appendix of [19]. The optimal linear compensator is

$$u_k = K\hat{x}_k, \quad \text{and} \tag{8}$$

$$\begin{aligned} \hat{x}_{k+1} &= (\bar{A} + \bar{B}K)\hat{x}_k + L(y_k - \bar{C}\hat{x}_k), \\ &= (\bar{A} + \bar{B}K + L\hat{C}_k)\hat{x}_k + LC_k e_k + Lv_k \end{aligned} \tag{9}$$

It is necessary to account for the multiplicative noise to achieve the minimum quadratic cost; furthermore, it is straightforward to find systems in (2) and (3) which are *mean-square unstable* when controlled by (multiplicative-noise-ignorant) LQG, meaning that it is necessary to account for multiplicative noise to achieve mean-square stability.

B. Residual from Multiplicative Noise LQG

We define the estimation error as $e_k = x_k - \hat{x}_k$. Then the estimation error evolves as follows

$$e_{k+1} = (\bar{A} - \hat{B}_k K - LC_k)e_k + (\hat{A}_k + \hat{B}_k K)x_k + w_k - Lv_k. \tag{10}$$

We now elaborate how to obtain the residual signal required for anomaly detection. Denote the residual $r_k \in \mathbb{R}^p$ as

$$r_k = y_k - \bar{C}\hat{x}_k. \tag{11}$$

Denote $E_k = \text{vec}\left(\mathbb{E}[e_k e_k^\top]\right), X_k = \text{vec}\left(\mathbb{E}[x_k x_k^\top]\right)$ and $R_k = \text{vec}\left(\mathbb{E}[r_k r_k^\top]\right)$. Then, r_k is not necessarily Gaussian due to the multiplicative noise and has zero mean with covariance matrix whose vectorized form is given by

$$R_k = \mathbb{E}\left[\hat{C}_k \otimes \hat{C}_k\right] X_k + \left(\bar{C} \otimes \bar{C}\right) E_k + \text{vec}\left(\Sigma_v\right). \tag{12}$$

Since, the optimal gain matrices K, L achieve mean-square compensation of the system (2) and (3), the covariance of the estimation error will have a steady state value and hence $E_\infty, X_\infty \in \mathbb{R}^{n^2}$ exists and thereby $R_\infty \in \mathbb{R}^{p^2}$ exists. Then, the covariance can be retrieved as

$$\Sigma_r = \text{mat}\left(R_\infty, p, p\right). \tag{13}$$

IV. Residuals via the Wasserstein Distributionally Robust Kalman Filter

To leverage a fault-detection approach, we require an estimator of some type to produce a prediction of the system behavior and hence a residual. Given the system described by (2) and (3), we aim to estimate the current state x_k at any time $k \in \mathbb{Z}_{>0}$ based on the output history $Y_k = (y_1, \ldots, y_k)$. In practice, the joint distribution of x_k and y_k is never directly observable and thus being uncertain, its distributional uncertainty should be taken into account in the estimation procedure, subject to both the multiplicative noise and the additive noises. The joint state-output process $z_k \in \mathbb{R}^d$, $d = n + p$ defined by $z_k = \begin{bmatrix} x_k^\top & y_k^\top \end{bmatrix}^\top, \forall k \in \mathbb{N}$ evolves as

$$z_k = \begin{bmatrix} A_{k-1} \\ C_k A_{k-1} \end{bmatrix} x_{k-1} + \begin{bmatrix} B_{k-1} \\ C_k B_{k-1} \end{bmatrix} u_{k-1} + \begin{bmatrix} I_n & 0 \\ C_k & I_p \end{bmatrix} \begin{bmatrix} w_{k-1} \\ v_k \end{bmatrix}, \tag{14}$$

and follows an unknown distribution $\mathbb{Q}_{z_k}$ in the neighborhood of a known nominal distribution $\mathbb{P}_{z_k}$ determined through the linear state-space model given in (2). We assume that the concatenated additive noise denoted by $\tilde{w}_{k-1} = \begin{bmatrix} w_{k-1}^\top & v_k^\top \end{bmatrix}^\top \in \mathbb{R}^d$ is a zero mean white noise. Further, $\tilde{w}_k \sim P_{\tilde{w}} \in \mathcal{P}^{\tilde{w}}$, where ambiguity set $\mathcal{P}^{\tilde{w}}$ is given by

$$\mathcal{P}^{\tilde{w}} := \left\{P_{\tilde{w}} \mid \mathbb{E}[\tilde{w}_k] = 0, \mathbb{E}[\tilde{w}_k \tilde{w}_k^\top] = \Sigma_{\tilde{w}} = \begin{bmatrix} \Sigma_w & 0 \\ 0 & \Sigma_v \end{bmatrix}\right\}. \tag{15}$$

Covariance values Σ_w and Σ_v that are within their respective bounds given in their ambiguity sets (5), (6) respectively can be used in this setting. We model this distributional and covariance uncertainty of z_k at time k through an ambiguity set $\mathcal{P}^{z_k}$, that is, a family of distributions on $\mathbb{R}^d$, that govern the concatenated dynamics z_k in view of the available data or that are sufficiently close to a prescribed nominal distribution, $\mathbb{P}_{z_k}$. The ambiguity set

$$\mathcal{P}^{z_k} = \left\{ \mathbb{Q}_{z_k} \in \mathcal{P}_2(\mathbb{R}^d) \mid W_2(\mathbb{Q}_{z_k}, \mathbb{P}_{z_k}) \leq \rho_k \right\}, \qquad (16)$$

can be interpreted as a ball of radius $\rho_k \geq 0$, in the space of distributions. The Wasserstein radius, ρ_k quantifies the amount of distrust we have over the nominal distribution $\mathbb{P}_{z_k}$ at time k. Further, $\mathcal{P}^{z_k}$ is centered at the nominal distribution $\mathbb{P}_{z_k}$ which is assumed to be a normal distribution $\mathbb{P}_{z_k} = \mathcal{N}_d(\mu_{z_k}, \Sigma_{z_k})$ with covariance matrix $\Sigma_{z_k} \succ 0$. However, due to the multiplicative noise in (2), the true variance of the distribution that governs z_k might be bigger than Σ_{z_k} due to the nominal model alone. The true unknown *possibly non-Gaussian* distribution of z_k referred to as $\mathbb{Q}^{\dagger}_{z_k}$ may lie in $\mathcal{P}^{z_k}$. We seek an estimator that minimizes the worst-case mean square error across all distributions in the ambiguity set given by (16).

A. Computing the parameters of nominal distribution $\mathbb{P}_{z_k}$

We assume that a controller is used which is linear in the estimated state. This could come from optimal and/or robust control design procedures. Ideally, the controller would be designed concurrently with the state estimator, as in MLQG. However, incorporating such coupling is non-trivial in the Wasserstein distributionally robust framework. As an approximation, we assume that the controller is characterized by the same gain matrix K which results from the coupled Riccati equations of MLQG, i.e. $u_{k-1} = K\hat{x}_{k-1}$. Then the stochastic dynamics given in (14) can be re-written as

$$\begin{aligned} z_k = &\underbrace{\begin{bmatrix} \bar{A}_{cl} \\ \bar{C}\bar{A}_{cl} \end{bmatrix}}_{\tilde{A}} x_{k-1} - \underbrace{\begin{bmatrix} \bar{B}K \\ \bar{C}\bar{B}K \end{bmatrix}}_{\tilde{E}} e_{k-1} + \underbrace{\begin{bmatrix} I_n & 0 \\ \bar{C} & I_p \end{bmatrix}}_{\tilde{W}} \tilde{w}_{k-1} \\ &+ \underbrace{\begin{bmatrix} \hat{A}_{cl,k-1} \\ (\bar{C}+\hat{C}_k)\hat{A}_{cl,k-1} + \hat{C}_k\bar{A}_{cl} \end{bmatrix}}_{\tilde{\tilde{A}}} x_{k-1} + \underbrace{\begin{bmatrix} 0 & 0 \\ \hat{C}_k & 0 \end{bmatrix}}_{\tilde{\tilde{W}}} \tilde{w}_{k-1} \\ &- \underbrace{\begin{bmatrix} \hat{B}_{k-1}K \\ (\bar{C}+\hat{C}_k)\hat{B}_{k-1}K + \hat{C}_k\bar{B}K \end{bmatrix}}_{\tilde{\tilde{E}}} e_{k-1}, \end{aligned} \qquad (17)$$

where $\bar{A}_{cl} = \bar{A} + \bar{B}K$ and $\hat{A}_{cl,k-1} = \hat{A}_{k-1} + \hat{B}_{k-1}K$. The deterministic mean dynamics and covariance of z_k obtained after taking expectation of (17) with respect to all the multiplicative and the additive noises is then given by

$$\mu_{z_k} = \tilde{A}\hat{x}_{k-1}, \qquad (18)$$

$$\begin{aligned} \Sigma_{z_k} = &\left(\tilde{A} + \tilde{\tilde{A}}\right) \Sigma_{x_{k-1}} \left(\tilde{A} + \tilde{\tilde{A}}\right)^\top + \left(\tilde{W} + \tilde{\tilde{W}}\right) \Sigma_{\tilde{w}} \left(\tilde{W} + \tilde{\tilde{W}}\right)^\top \\ &+ \left(\tilde{E} + \tilde{\tilde{E}}\right) \mathbb{E}[e_{k-1}e_{k-1}^\top] \left(\tilde{E} + \tilde{\tilde{E}}\right)^\top + \Upsilon_{k-1}, \end{aligned} \qquad (19)$$

where Υ_{k-1} accumulates all the cross terms involving $\mathbb{E}[x_{k-1}e_{k-1}^\top], \mathbb{E}[e_{k-1}x_{k-1}^\top]$ and it is evident that Σ_{z_k} has contributions from both additive and all the multiplicative noise covariances $\{\sigma_{a,i}^2\}_{i=1}^{n_a}, \{\sigma_{b,j}^2\}_{j=1}^{n_b}$, and $\{\sigma_{c,l}^2\}_{l=1}^{n_c}$.

B. Wasserstein Robust Estimator

The nominal distribution $\mathbb{P}_{z_k}$ is uniquely determined by the marginal distribution $\mathbb{P}_{x_0} = \mathcal{N}_n(\hat{x}_0, \Sigma_{x_0})$ of the initial state x_0 and the conditional distributions $\mathbb{P}_{z_k|x_{k-1}} = \mathcal{N}_d(\mu_{z_k}, \Sigma_{z_k})$ of z_k given x_{k-1} for all $k \in \mathbb{N}$. Here, the construction of $\mathbb{P}_{z_k|Y_{k-1}}$ resembles the prediction step of the classical Kalman filter but uses the least favorable distribution $\mathbb{Q}^\star_{x_{k-1}|Y_{k-1}}$ instead of the nominal distribution $\mathbb{P}_{x_{k-1}|Y_{k-1}}$. By assumption 2, the desired state estimator at all time steps $k \in \mathbb{N}$ will be

$$\psi_k(y_k) = Gy_k + g, \qquad (20)$$

where G, g are of appropriate dimensions and $\mathcal{L}$ defines the family of all measurable functions representing the class of such affine estimators from $\mathbb{R}^p$ to $\mathbb{R}^n$. In the update step, the pseudo-nominal a priori estimate $\mathbb{P}_{z_k|Y_{k-1}}$ is updated by the measurement y_k and robustified against model uncertainty to yield a refined a posteriori estimate $\mathbb{Q}_{x_k|Y_k}$ which is found by solving the minimax problem

$$\inf_{\psi_k \in \mathcal{L}} \sup_{\mathbb{Q}_{z_k} \in \mathcal{P}^{z_k|Y_{k-1}}} \mathbb{E}^{\mathbb{Q}_{z_k}}[e_\psi] \qquad (21)$$

equipped with the Wasserstein ambiguity set

$$\mathcal{P}^{z_k|Y_{k-1}} = \left\{ \mathbb{Q}_{z_k} \in \mathcal{P}_2(\mathbb{R}^d) \mid W_2(\mathbb{Q}_{z_k}, \mathbb{P}_{z_k|Y_{k-1}}) \leq \rho_k \right\}, \qquad (22)$$

where $e_\psi = \|x_k - \psi_k(y_k)\|^2$. Here, the Wasserstein radius ρ_k quantifies our distrust in the pseudo-nominal apriori estimate and can therefore be interpreted as a measure of model uncertainty at time k. Invoking the minimax theorem in [15], with $\mathcal{P}^{z_k|Y_{k-1}}$ given by (22), we see that

$$\inf_{\psi_k \in \mathcal{L}} \sup_{\mathbb{Q}_{z_k} \in \mathcal{P}^{z_k|Y_{k-1}}} \mathbb{E}^{\mathbb{Q}_{z_k}}[e_\psi] = \sup_{\mathbb{Q}_{z_k} \in \mathcal{P}^{z_k|Y_{k-1}}} \inf_{\psi_k \in \mathcal{L}} \mathbb{E}^{\mathbb{Q}_{z_k}}[e_\psi], \qquad (23)$$

where the optimal solutions $\psi_k^\star$ and $\mathbb{Q}^\star_{z_k}$ of the two dual problems in (23) represent the respective minimax strategies implying that $(\psi_k^\star, \mathbb{Q}^\star_{z_k})$ forms a saddle point of the underlying zero-sum game. Subsequently, by invoking Theorem 2.5 from [15], (23) can be efficiently solved using the Frank-Wolfe algorithm to obtain the optimal solution $S^\star_{k,z_k}$ and the least favorable conditional distribution $\mathbb{Q}^\star_{z_k|Y_{k-1}} = \mathcal{N}_d(\mu_{z_k}, S^\star_{k,z_k})$ of z_k given Y_{k-1}. Then we obtain the parameters that define the least favorable conditional distribution of x_k given Y_k, $\mathbb{Q}^\star_{x_k|Y_k} = \mathcal{N}_n(\hat{x}_k, V_k)$ as

$$\psi_k^\star(y_k) = \hat{x}_k = \mu_{z_k,x} + \Theta_k(y_k - \mu_{z_k,y}), \qquad (24)$$

$$\Sigma_{x_k} = S^\star_{k,xx} - \Theta_k S^\star_{k,yx}, \qquad (25)$$

where $\Theta_k = S^\star_{k,xy}\left(S^\star_{k,yy}\right)^{-1}$ is the Wasserstein Kalman gain. Note that the Kalman gain Θ_k is a function of the Wasserstein radius ρ_k at time k.

C. Residual from W-DR-KF

The residual $r_k \in \mathbb{R}^p$ can be calculated as

$$r_k = C_k A_{k-1} e_{k-1} + C_k w_k + v_k. \qquad (26)$$

Again r_k has zero mean as it is a linear combination of zero-mean random vectors. Further, its covariance matrix Σ_{r_k} at time k given by

$$\Sigma_{r_k} = C_k A_{k-1} \mathbb{E}[e_{k-1} e_{k-1}^\top] A_{k-1}^\top C_k^\top + C_k \Sigma_w C_k^\top + \Sigma_v. \tag{27}$$

Since the given CPS model is assumed to be mean-square compensatable, the pair of matrices (K, Θ_k) will result in mean-square compensatability. Then it is guaranteed for a steady state estimation error covariance, $\hat{E}_\infty \in \mathbb{S}_+^n$ to exist and in which case the residual covariance would be

$$\Sigma_r = C_k A_{k-1} \hat{E}_\infty A_{k-1}^\top C_k^\top + C_k \Sigma_w C_k^\top + \Sigma_v. \tag{28}$$

Remark: Obtaining the pair (K, Θ_k) that will result in mean-square compensatability is still unexplored in this setting. Such a design procedure is left for future work.

V. Anomaly Detector Design with Residuals from Multiplicative Noise and Robust Filtering

We now present how to analyze the residual obtained from either of the two above presented state estimators and elaborate the procedure to construct the corresponding anomaly detector threshold in this section. Note that the covariance of the residual computed from either of the approach (12), (27) is a function of covariance matrices of both the additive and multiplicative noises. This is in sharp contrast to the case in [1], [7], [8] where the residual covariance was just a function of the additive noise covariance. Further, to account for the changes in the covariance of the residual, we form a quadratic distance measure as

$$q_k = \begin{cases} r_k^\top \Sigma_r^{-1} r_k, & \text{if } r_k \text{ is from (11)}, \\ r_k^\top \Sigma_{r_k}^{-1} r_k, & \text{if } r_k \text{ is from (26)}. \end{cases} \tag{29}$$

Then, for a given q_k from (29) and a threshold $\alpha \in \mathbb{R}_{>0}$ corresponding to a desired false alarm rate $\mathcal{F}$, the anomaly detector can be designed such that alarm time(s) $k^\star \in \mathbb{N}$ are produced according to the following rules

$$\begin{cases} q_k \leq \alpha, & \text{no alarm}, \\ q_k > \alpha, & \text{alarm: } k^\star = k. \end{cases} \tag{30}$$

If $\mathbb{Q}_{z_k}^\dagger$ was Gaussian, then we can define $\mathcal{P}^{z_k|Y_{k-1}}$ in the space of normal distributions. Subsequently, r_k would be Gaussian and thereby q_k would follow the chi-squared distribution, meaning that for a given $\mathcal{F}$, the chi-squared detector described as in [2] can be used to obtain the required detector threshold. However, in reality due to the multiplicative noise, $\mathbb{Q}_{z_k}^\dagger$ is *non-Gaussian* and thereby the chi-squared detector is not appropriate. We instead utilize a moment-based approach for constructing the threshold. We propose to use the higher order moment based anomaly detector design proposed in [7] to design the detector threshold in this setting. The residual q_k from either of the two approaches is collected for a sufficiently long period of time to form the s-moments based ambiguity set $\mathcal{P}_q^s := \left\{ \mathbb{P}_q \mid \mathbb{E}[q_k^s] = M_q^s \right\}$. The optimal threshold $\alpha_{q,s}^\star$ [1] satisfying

$$\sup_{\mathbb{P}_q \in \mathcal{P}_q^s} \mathbb{P}_q \left[q_k > \alpha_{q,s}^\star \right] \leq \mathcal{F}, \tag{31}$$

can then be obtained by directly invoking Theorem 4 in [7] corresponding to a given desired false alarm rate $\mathcal{F}$.

VI. Numerical Results

We consider an inverted pendulum with a torque-producing actuator whose dynamics have been linearized about the vertical equilibrium. That is, the pendulum of mass m is suspended by a massless rod of length l and the angle θ is measured from the downward vertical with positive counter clockwise direction. The corresponding nonlinear differential equation of the pendulum mass is

$$\ddot{\theta} = m_c \sin(\theta) + \tau, \tag{32}$$

where $m_c = -\frac{g}{l}$ denotes the uncertain mass constant. Let us denote the state vector by $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} \theta & \dot{\theta} \end{bmatrix}$ and the torque input by $u = \tau$. Then, the corresponding discrete time dynamics obtained through the forward Euler discretization of the linearized dynamics of (32) around the equilibrium point $\tilde{x} = (\pi, 0)$ with step size Δt is

$$x_{k+1} = \begin{bmatrix} 1 & \Delta t \\ m_c \Delta t & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ \Delta t \end{bmatrix} u_k + w_k. \tag{33}$$

Uncertainty on the mass constant m_c corresponds to uncertainty on the matrix A. We consider an example where the true mass constant is $m_c = 10$, but the nominal model underestimates it as $m_c = 5$. We take a step size $\Delta t = 0.1$. At discrete time instances, the sensor returns a noisy measurement of the angular position of pendulum. Hence the corresponding linearized noisy output model is,

$$y = \theta + v_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v_k. \tag{34}$$

Both w_k and v_k are sampled from the multivariate Laplacian (which has heavier tails than Gaussian with same mean and covariance) with zero-mean and covariance $\Sigma_w = 2I_n$, $\Sigma_v = 2I_p$ respectively. The state and control penalty matrices are $Q = I_n, R = I_p$ respectively. The residual data was computed using the multiplicative noise-driven LQG approach explained in III and the anomaly detector was tuned for a desired false alarm rate of $\mathcal{F} = 5\%$. The multiplicative noise was considered to exist only in the system dynamics matrix A, with the direction matrix being $\mathcal{A}_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $\gamma_{k,1} \sim \mathcal{N}(0, 0.01)$. We used Theorem 4 in [7] to compute the anomaly detector thresholds while using a bisection tolerance of $\epsilon = 10^{-4}$. Through simulation, we collected the residual data for $T = 10^4$ time steps under two different settings namely, 1) without any multiplicative noise $(\sigma_{a,1}^2 = 0)$ using the standard LQG, and 2) with multiplicative noise $(\sigma_{a,1}^2 = 0.01)$ using multiplicative noise-driven LQG.

[1]The two subscripts q, s in $\alpha_{q,s}^\star$ denote the random variable and the number of moments considered respectively.

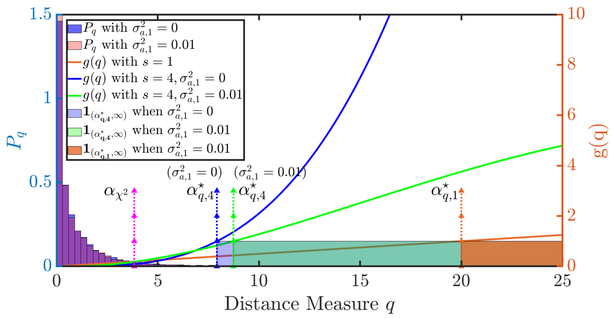

Fig. 1. **Detector Threshold With and Without Multiplicative Noise:** The moment based polynomials $g(q)$ in orange, green and blue bounding their respective indicator functions in shaded orange, green and blue colors are shown. It is evident that the threshold $\alpha^\star_{q,4}$ with the multiplicative noise is greater than the one without it.

A. Discussion of Results

The results of simulating the system given by (33) and (34) are shown in Figure 1. When the standard LQG was employed on the nominal system without any multiplicative noise, it resulted in detector thresholds $\alpha^\star_{q,1} = 20.0, \alpha^\star_{q,4} = 7.92$ with false alarm rates being 0% and 0.9% respectively. Under the multiplicative noise setting, the controller and the estimation gain matrices computed using the multiplicative noise-driven LQG resulted in a mean-square compensatability (verified via the convergence of the coupled Riccati equations) and subsequently resulted in detector thresholds $\alpha^\star_{q,1} = 20.0, \alpha^\star_{q,4} = 8.74$ with false alarm rates being 0% and 0.8% respectively. However, standard LQG was not able to mean square stabilize the system under non-zero multiplicative noise. Further, when the residual data from the system with multiplicative noise was evaluated against the thresholds $\alpha^\star_{q,4}$ obtained using standard LQG and multiplicative LQG, it resulted in 1.2% and 0.9% false alarms respectively. This trend of increasing false alarm rate with uncertainty in just A matrix would only worsen when all other multiplicative noises on all matrices are present with a stronger covariance. This clearly motivates that multiplicative noise LQG is needed to guarantee reduced false alarm rates. While it is true that multiplicative noise LQG is capable of handling CPSs with modeling uncertainty, it is sensitive to the covariance of the multiplicative noise. For instance in the considered example, mean square compensation was possible for values of $\sigma^2_{a,1} \leq 0.07$. In general, achieving mean square compensation becomes increasingly difficult when other multiplicative noise covariance $\sigma^2_{b,j}, \sigma^2_{c,l}$ for $j = 1, \ldots, n_b$ and $l = 1, \ldots, n_c$ are also present. Hence, multiplicative noise-driven LQG framework is valid as long as the gain matrices pair (K, L) achieve mean square compensation.

VII. Conclusion

An extension of the state-of-the-art anomaly detection algorithms for CPS with modeling errors via the multiplicative noise framework was discussed in this paper. Two robust state estimators namely multiplicative noise-driven LQG and a Wasserstein distributionally robust Kalman filter were used to hedge against the model risk to construct the state estimate. The proposed method was demonstrated using a numerical simulation. Future work seeks to investigate the setting where the multiplicative noise distributions are unknown and to obtain online estimates of the system dynamics through system identification technique combined with the above filtering procedure for implementing data-driven distributionally robust anomaly detection for vulnerable CPS.

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APPENDIX I
MOMENT DYNAMICS

Recall the closed-loop system equations:

$$\begin{aligned}
x_{k+1} &= A_k x_k + B_k u_k + w_k,\\
\hat{x}_{k+1} &= \bar{A}\hat{x}_k + \bar{B}u_k + L(y_k - \hat{y}_k),\\
u_k &= K\hat{x}_k,\\
y_k &= C_k x_k + v_k,\\
\hat{y}_k &= \bar{C}\hat{x}_k,
\end{aligned}$$

and the state- and output-residuals

$$\begin{aligned}
e_k &= x_k - \hat{x}_k,\\
r_k &= y_k - \hat{y}_k.
\end{aligned}$$

Denote

$$\begin{aligned}
\Sigma'_A &= \mathbb{E}\left[\hat{A}_k \otimes \hat{A}_k\right] = \sum_{i=1}^{n_a} \sigma_{a,i}^2 (\mathcal{A}_i \otimes \mathcal{A}_i),\\
\Sigma'_B &= \mathbb{E}\left[\hat{B}_k \otimes \hat{B}_k\right] = \sum_{j=1}^{n_b} \sigma_{b,j}^2 (\mathcal{B}_j \otimes \mathcal{B}_j),\\
\Sigma'_C &= \mathbb{E}\left[\hat{C}_k \otimes \hat{C}_k\right] = \sum_{l=1}^{n_c} \sigma_{c,l}^2 (\mathcal{C}_l \otimes \mathcal{C}_l).
\end{aligned}$$

Hence, we have the identities

$$\begin{aligned}
\mathbb{E}\left[A_k \otimes A_k\right] &= \bar{A} \otimes \bar{A} + \Sigma'_A,\\
\mathbb{E}\left[B_k \otimes B_k\right] &= \bar{B} \otimes \bar{B} + \Sigma'_B,\\
\mathbb{E}\left[C_k \otimes C_k\right] &= \bar{C} \otimes \bar{C} + \Sigma'_C.
\end{aligned}$$

While studying the moment dynamics, we shall readily employ the zero-mean and zero-correlation assumptions of $\hat{A}_k$, $\hat{B}_k$, $\hat{C}_k$, w_k, and v_k in the following derivations.

A. *First moment dynamics*

The expected output-residual is

$$\begin{aligned}
\mathbb{E}[r_k] &= \mathbb{E}[y_k - \hat{y}_k]\\
&= \mathbb{E}[C_k x_k + v_k - \bar{C}\hat{x}_k]\\
&= \mathbb{E}[C_k x_k] - \bar{C}\mathbb{E}[\hat{x}_k]\\
&= \bar{C}\mathbb{E}[x_k - \hat{x}_k]\\
&= \bar{C}e_k. \qquad (35)
\end{aligned}$$

The expected state-residual evolves as

$$\begin{aligned}
\mathbb{E}[e_{k+1}] &= \mathbb{E}[x_{k+1} - \hat{x}_{k+1}]\\
&= \mathbb{E}[x_{k+1}] - \mathbb{E}[\hat{x}_{k+1}]\\
&= \mathbb{E}\left[A_k x_k + B_k K\hat{x}_k + w_k\right] - \mathbb{E}\left[\bar{A}\hat{x}_k + \bar{B}K\hat{x}_k + L(y_k - \hat{y}_k)\right] \qquad (36)\\
&= \bar{A}\mathbb{E}[x_k] + \bar{B}K\mathbb{E}[\hat{x}_k] - (\bar{A} + \bar{B}K)\mathbb{E}[\hat{x}_k] - L\mathbb{E}[y_k - \hat{y}_k]\\
&= \bar{A}\mathbb{E}[x_k] + \bar{B}K\mathbb{E}[\hat{x}_k] - (\bar{A} + \bar{B}K)\mathbb{E}[\hat{x}_k] - L\bar{C}e_k\\
&= \bar{A}\mathbb{E}[x_k] - \bar{A}\mathbb{E}[\hat{x}_k] + \bar{B}K\mathbb{E}[\hat{x}_k] - \bar{B}K\mathbb{E}[\hat{x}_k] - L\bar{C}e_k\\
&= (\bar{A} - L\bar{C})e_k. \qquad (37)
\end{aligned}$$

B. Second moment dynamics

For the state and state-estimate second moment dynamics, denote

$$X_k = \operatorname{vec}\mathbb{E}\left[x_k x_k^\top\right], \quad \tilde{X}_k = \operatorname{vec}\mathbb{E}\left[x_k \hat{x}_k^\top\right],$$
$$\breve{X}_k = \operatorname{vec}\mathbb{E}\left[\hat{x}_k x_k^\top\right], \quad \hat{X}_k = \operatorname{vec}\mathbb{E}\left[\hat{x}_k \hat{x}_k^\top\right].$$

We have

$$\begin{aligned} X_{k+1} &= \operatorname{vec}\mathbb{E}\left[x_{k+1} x_{k+1}^\top\right] \\ &= \operatorname{vec}\mathbb{E}\left[(A_k x_k + B_k K \hat{x}_k + w_k)(A_k x_k + B_k K \hat{x}_k + w_k)^\top\right] \\ &= \left(\bar{A}\otimes\bar{A} + \Sigma'_A\right) X_k + ((\bar{B}K)\otimes\bar{A})\tilde{X}_k + (\bar{A}\otimes(\bar{B}K))\breve{X}_k + \left(\bar{B}\otimes\bar{B} + \Sigma'_B\right)(K\otimes K)\hat{X}_k + \operatorname{vec}(\Sigma_w), \end{aligned} \tag{38}$$

and

$$\begin{aligned} \tilde{X}_{k+1} &= \operatorname{vec}\mathbb{E}\left[x_{k+1} \hat{x}_{k+1}^\top\right] \\ &= \operatorname{vec}\mathbb{E}\left[(A_k x_k + B_k K \hat{x}_k + w_k)\left(LC_k x_k + (\bar{A} + \bar{B}K - L\bar{C})\hat{x}_k + Lv_k\right)^\top\right] \\ &= ((L\bar{C})\otimes\bar{A})X_k + \left((\bar{A} + \bar{B}K - L\bar{C})\otimes\bar{A}\right)\tilde{X}_k + ((L\bar{C})\otimes(\bar{B}K))\breve{X}_k + \left((\bar{A} + \bar{B}K - L\bar{C})\otimes(\bar{B}K)\right)\hat{X}_k, \end{aligned} \tag{39}$$

and

$$\begin{aligned} \breve{X}_{k+1} &= \operatorname{vec}\mathbb{E}\left[\hat{x}_{k+1} x_{k+1}^\top\right] \\ &= \operatorname{vec}\mathbb{E}\left[\left(LC_k x_k + (\bar{A} + \bar{B}K - L\bar{C})\hat{x}_k + Lv_k\right)(A_k x_k + B_k K \hat{x}_k + w_k)^\top\right] \\ &= (\bar{A}\otimes(L\bar{C}))X_k + ((\bar{B}K)\otimes(L\bar{C}))\tilde{X}_k + \left(\bar{A}\otimes(\bar{A} + \bar{B}K - L\bar{C})\right)\breve{X}_k + \left((\bar{B}K)\otimes(\bar{A} + \bar{B}K - L\bar{C})\right)\hat{X}_k, \end{aligned} \tag{40}$$

and

$$\begin{aligned} \hat{X}_{k+1} &= \operatorname{vec}\mathbb{E}\left[\hat{x}_{k+1} \hat{x}_{k+1}^\top\right] \\ &= \operatorname{vec}\mathbb{E}\left[\left(LC_k x_k + (\bar{A} + \bar{B}K - L\bar{C})\hat{x}_k + Lv_k\right)\left(LC_k x_k + (\bar{A} + \bar{B}K - L\bar{C})\hat{x}_k + Lv_k\right)^\top\right] \\ &= (L\otimes L)(\bar{C}\otimes\bar{C} + \Sigma'_C)X_k + \left((\bar{A} + \bar{B}K - L\bar{C})\otimes(L\bar{C})\right)\tilde{X}_k \\ &\quad + \left((L\bar{C})\otimes(\bar{A} + \bar{B}K - L\bar{C})\right)\breve{X}_k + \left((\bar{A} + \bar{B}K - L\bar{C})\otimes(\bar{A} + \bar{B}K - L\bar{C})\right)\hat{X}_k + (L\otimes L)\operatorname{vec}(\Sigma_v). \end{aligned} \tag{41}$$

Define

$$\mathcal{X}_k := \begin{bmatrix} X_k \\ \tilde{X}_k \\ \breve{X}_k \\ \hat{X}_k \end{bmatrix}, \qquad \text{and} \qquad \mathcal{V} := \begin{bmatrix} \operatorname{vec}(\Sigma_w) \\ \operatorname{vec}(\Sigma_v) \end{bmatrix}.$$

By gathering the matrix coefficients in equations (38), (39), (40), (41) as

$$H := \begin{bmatrix} \bar{A}\otimes\bar{A} + \Sigma'_A & (\bar{B}K)\otimes\bar{A} & \bar{A}\otimes(\bar{B}K) & \left(\bar{B}\otimes\bar{B} + \Sigma'_B\right)(K\otimes K) \\ (L\bar{C})\otimes\bar{A} & (\bar{A} + \bar{B}K - L\bar{C})\otimes\bar{A} & (L\bar{C})\otimes(\bar{B}K) & (\bar{A} + \bar{B}K - L\bar{C})\otimes(\bar{B}K) \\ \bar{A}\otimes(L\bar{C}) & (\bar{B}K)\otimes(L\bar{C}) & \bar{A}\otimes(\bar{A} + \bar{B}K - L\bar{C}) & (\bar{B}K)\otimes(\bar{A} + \bar{B}K - L\bar{C}) \\ (L\otimes L)(\bar{C}\otimes\bar{C} + \Sigma'_C) & (\bar{A} + \bar{B}K - L\bar{C})\otimes(L\bar{C}) & (L\bar{C})\otimes(\bar{A} + \bar{B}K - L\bar{C}) & (\bar{A} + \bar{B}K - L\bar{C})\otimes(\bar{A} + \bar{B}K - L\bar{C}) \end{bmatrix}$$

and

$$\Phi := \begin{bmatrix} I_n\otimes I_n & 0_{n^2\times 1} \\ 0_{n^2\times n^2} & 0_{n^2\times 1} \\ 0_{n^2\times n^2} & 0_{n^2\times 1} \\ 0_{n^2\times n^2} & L\otimes L \end{bmatrix}$$

we have the compact representation of (38), (39), (40), (41) as

$$\mathcal{X}_{k+1} = H\mathcal{X}_k + \Phi\mathcal{V} \tag{42}$$

For the state- and output-residual second moments, denote

$$E_k = \operatorname{vec}\mathbb{E}\left[e_k e_k^\top\right]$$
$$R_k = \operatorname{vec}\mathbb{E}\left[r_k r_k^\top\right]$$

We have

$$E_k = \operatorname{vec}\mathbb{E}\left[e_k e_k^\top\right] = \operatorname{vec}\mathbb{E}\left[(x_k-\hat{x}_k)(x_k-\hat{x}_k)^\top\right] = X_k - \tilde{X}_k - \breve{X}_k + \hat{X}_k \tag{43}$$

and

$$\begin{aligned}R_k &= \operatorname{vec}\mathbb{E}\left[r_k r_k^\top\right]\\ &= \operatorname{vec}\mathbb{E}\left[(y_k-\hat{y}_k)(y_k-\hat{y}_k)^\top\right]\\ &= \operatorname{vec}\mathbb{E}\left[(C_k x_k + v_k - \bar{C}\hat{x}_k)(C_k x_k + v_k - \bar{C}\hat{x}_k)^\top\right]\\ &= \operatorname{vec}\mathbb{E}\left[(\bar{C}(x_k-\hat{x}_k)+(C_k-\bar{C})x_k+v_k)(\bar{C}(x_k-\hat{x}_k)+(C_k-\bar{C})x_k+v_k)^\top\right]\\ &= (\bar{C}\otimes\bar{C})E_k + \Sigma_C' X_k + \operatorname{vec}(\Sigma_v)\end{aligned} \tag{44}$$

C. Steady-state moments

By assumption, $\bar{A}-L\bar{C}$ is Schur stable. Therefore, by (37), $\mathbb{E}[e_k]\to 0$ as $k\to\infty$ regardless of the initial state-residual e_0. Consequently, by (35), $\mathbb{E}[r_k]\to 0$ as $k\to\infty$. That is,

$$\mathbb{E}[e_\infty]=0, \qquad \mathbb{E}[r_\infty]=0$$

In steady-state, the left- and right-hand sides of (42) converge identically to eachother, i.e.

$$\mathcal{X}_\infty = H\mathcal{X}_\infty + \Phi\mathcal{V}.$$

Rearranging, we obtain

$$\mathcal{X}_\infty = (I_{4n^2}-H)^{-1}\Phi\mathcal{V}$$

where $(I_{4n^2}-H)^{-1}$ exists by the mean-square stability assumption of the compensator gains (K,L). This amounts to solving a (generalized) Lyapunov equation. Such an equation can be solved more efficiently by specialized solvers which do not require the inverse to be computed explicitly; for simplicity we present the equation and its solution in this form. After solving for $\mathcal{X}_\infty$, the steady-state second moments of the state- and output-residuals can be computed from (43) and (44) as

$$\begin{aligned}E_\infty &= X_\infty - \tilde{X}_\infty - \breve{X}_\infty + \hat{X}_\infty\\ R_\infty &= (\bar{C}\otimes\bar{C})E_\infty + \Sigma_C' X_\infty + \operatorname{vec}(\Sigma_v)\end{aligned}$$

and reshaped into matrices using the $\operatorname{mat}(\cdot)$ operator, i.e. $\Sigma_{x_\infty} = \operatorname{mat}(E_\infty, n, n)$ and $\Sigma_r = \operatorname{mat}(R_\infty, p, p)$.

Appendix II
Coupled Riccati Equations

The coupled Riccati equations used for solving the MLQG problem are

$$\begin{aligned}P_1 = Q + \bar{A}^\top P_1\bar{A} + \sum_{i=1}^{n^2}\sigma_{a,i}^2 A_i^\top P_1 A_i - K^\top\left(R + \bar{B}^\top P_1\bar{B} + \sum_{j=1}^{nm}\sigma_{b,j}^2\bar{B}_j^\top P_1\bar{B}_j + \sum_{j=1}^{nm}\sigma_{b,j}^2\bar{B}_j^\top P_2\bar{B}_j\right)K\\ + \sum_{i=1}^{n^2}\sigma_{a,i}^2 A_i^\top P_2 A_i + \sum_{i=1}^{pn}\lambda_i C_i^\top L^\top P_2 L C_i,\end{aligned} \tag{45}$$

$$P_2 = (\bar{A}-L\bar{C})^\top P_2(\bar{A}-L\bar{C}) + K^\top\left(R + \bar{B}^\top P_1\bar{B} + \sum_{j=1}^{nm}\sigma_{b,j}^2\bar{B}_j^\top P_1\bar{B}_j + \sum_{j=1}^{nm}\sigma_{b,j}^2\bar{B}_j^\top P_2\bar{B}_j\right)K, \tag{46}$$

$$\begin{aligned}P_3 = \Sigma_w + \bar{A}P_3\bar{A}^\top + \sum_{i=1}^{n^2}\sigma_{a,i}^2 A_i P_3 A_i^\top - L\left(\Sigma_v + \bar{C}P_3\bar{C}^\top + \sum_{j=1}^{pn}\sigma_{c,j}^2\bar{C}_j P_3\bar{C}_j^\top + \sum_{j=1}^{pn}\sigma_{c,j}^2\bar{C}_j P_4\bar{C}_j^\top\right)L^\top\\ + \sum_{i=1}^{n^2}\sigma_{a,i}^2 A_i P_4 A_i^\top + \sum_{i=1}^{nm}\beta_i B_i K P_4 K^\top B_i^\top,\end{aligned} \tag{47}$$

$$P_4 = (\bar{A}+\bar{B}K)P_4(\bar{A}+\bar{B}K)^\top + L\left(\Sigma_v + \bar{C}P_3\bar{C}^\top + \sum_{j=1}^{pn}\sigma_{c,j}^2\bar{C}_j P_3\bar{C}_j^\top + \sum_{j=1}^{pn}\sigma_{c,j}^2\bar{C}_j P_4\bar{C}_j^\top\right)L^\top. \tag{48}$$

where the associated optimal controller and estimator gains (K, L) are

$$K = -\left(R + \bar{B}^\top P_1 \bar{B} + \sum_{j=1}^{nm} \sigma_{b,j}^2 \bar{B}_j^\top P_1 \bar{B}_j + \sum_{j=1}^{nm} \sigma_{b,j}^2 \bar{B}_j^\top P_2 \bar{B}_j\right)^{-1} \bar{B}^\top P_1 \bar{A}, \tag{49}$$

$$L = \bar{A} P_3 \bar{C}^\top \left(\Sigma_v + \bar{C} P_3 \bar{C}^\top + \sum_{j=1}^{pn} \sigma_{c,j}^2 \bar{C}_j P_3 \bar{C}_j^\top + \sum_{j=1}^{pn} \sigma_{c,j}^2 \bar{C}_j P_4 \bar{C}_j^\top\right)^{-1}. \tag{50}$$