Quadratic Two-Team Games

Marcello Colombino, Tyler H. Summers and Roy S. Smith

***Abstract*— We consider stochastic quadratic two-player games where each player represents a team of agents subject to information constraints. We present conditions that guarantee the existence and uniqueness of a Nash equilibrium in the space of linear decentralized policies and we provide an iterative algorithm to compute such an equilibrium. The results are illustrated on a numerical example inspired from power systems security.**

I. Introduction

Team theory is concerned with optimal decision making with a decentralized information structure. Such problems have been widely studied in the past decades both in the static and in the dynamical setting. The field was pioneered by Radner who showed in his 1962 paper [1] that for a class of stochastic quadratic decision problems the optimal decentralized decision is linear and easy to compute. This result has been embraced by the control community initially by Ho and Chu [2] who showed that under specific assumptions on the plant and the controller known as "partially nested information structure", the result by Radner can be applied and the optimal structured controller is linear. The field of decentralized control has matured since then and in [3] the authors present "quadratic invariance" as a characterization of all convex structured control problems. Partially nested structures are a significant portion of quadratically invariant structures and recently there has been significant progress in the computation of the optimal controllers [4], [5] for such structures. In [6] and [7] it is shown that the result from Radner can be obtained in a game theoretical setting by considering the Nash equilibrium of players with the same payoff.

In this paper we consider a different game theoretical setting: we study "two-team quadratic games", such games can be viewed as two-player games where each player is composed of a team of agents. Every agent has partial information on the state of the game and can only partially influence the decision of the team. We consider minimax zero sum games where, in contrast to [8], both players are subject to decentralized information structures. The traditional context for the early theory developed by Radner et al. was economic interactions of firms, but a timely modern application context is in cyber-physical network security, in which both the attackers and network operators act subject to a distributed information structure. In this work we only consider the static case for two-team quadratic games but we believe that, as for the single player counterpart, our results could be generalized to control in dynamical games where both players are subject to partially nested information structures. Early work considering two team quadratic games can be found in [9] for the zero-sum case.

The authors are with the Automatic Control Laboratory, ETH Zurich, Switzerland. E-mail addresses: {mcolombi, tsummers, rsmith}@control.ee.ethz.ch. Research partially supported by the Swiss National Science Foundation grant 2–773337–12.

Notation

We denote the real numbers by $\mathbb{R}$ and the integers by $\mathbb{Z}$. With $\mathbb{Z}_{[a,b]}$ we denote $[a,b]\cap\mathbb{Z}$. Given a matrix $Q\in\mathbb{R}^{n\times n}$ we denote by $\sigma_{\max}(Q)$ and $\sigma_{\min}(Q)$ its maximum and minimum singular value respectively. With $Q\succ 0$ ($Q\prec 0$) we denote that $Q=Q^\top$ is positive (negative) definite. Let $\mathscr{B}$ denote the Borel sigma algebra on $\mathbb{R}^n$, given a measure space $(\mathbb{R}^n,\mathscr{B},\mu)$ we denote by $\mathcal{L}^2_\mu$ the Hilbert space of square integrable measurable functions (i.e. the functions $f(\cdot)$ for which $\int_{\mathbb{R}^n}\|f(x)\|^2\mu(\mathrm{d}\,x)<\infty$). For simplicity, if μ is Gaussian, we denote the respective space by $\mathcal{L}^2$. We denote by $L(\cdot)$ the Lebesgue measure. Given $F\in\mathbb{R}^{m\times n}$, we denote by $F\cdot$ a linear function $f(\cdot)\in\mathcal{L}^2_\mu$ such that $f(x)=Fx$. Given a matrix $Q\succ 0$ of appropriate dimensions we denote the norm $\|F\|_Q:=\sqrt{\mathrm{tr}(F^\top QF)}$. Given $\Sigma\succ 0$ and $m\in\mathbb{R}^n$ we define by $\phi(x;m,\Sigma)$ the density function of the Gaussian distribution with mean m and variance Σ. The symbol $\otimes$ denotes the Kronecker product for matrices.

II. A short review on team decision theory

In this section we give a brief review on quadratic team decision theory. We first consider the stochastic case introduced in [1], and we present a proof that will give some useful insight on the structure of the optimal solution that will be exploited for the main result of the paper in Section III. We then show that this approach can be generalized to the non-Gaussian case provided the probability measure is sufficiently regular.

A. Stochastic Gaussian-quadratic team decision theory

We consider the simplest case covered in [1]. Given $m_w\in\mathbb{R}^n$ and $\Sigma_w\succ 0$, let $w\sim\mathcal{N}(m_w,\Sigma_w)$, the decision vector $u=[u_1^\top,\ldots,u_N^\top]^\top\in\mathbb{R}^m$, $u_i\in\mathbb{R}^{m_i}$, output matrices $C_i\in\mathbb{R}^{q_i\times n}$, for $i\in\mathbb{Z}_{[1,N]}$ and $Q_{uu}\succ 0$. We seek measurable functions that map $\mathbb{R}^n\to\mathbb{R}^{m_i}$ of the form $\nu_i(C_i\,\cdot)$ where $\nu_i(\cdot)\in\mathcal{L}^2$ to solve the following problem

$$\begin{aligned}\{\nu_i^\star(C_i\,\cdot)\}_{i=1}^N = \\ \arg\min_{\nu_i(\cdot)} \mathbb{E}_w\left(\begin{bmatrix} w\\ u\end{bmatrix}^\top\begin{bmatrix} Q_{ww} & Q_{wu}\\ Q_{wu}^\top & Q_{uu}\end{bmatrix}\begin{bmatrix} w\\ u\end{bmatrix}\right)\\ \text{subject to: } u_i=\nu_i(C_iw)\end{aligned} \tag{1}$$

In [1] it is proven that the optimal solution $\nu_1^\star(C_1\,\cdot),\dots,\nu_N^\star(C_N\,\cdot)$ to problem (1) is given by linear decision functions, that is $u_i=\nu_i^\star(C_iw)=K_iC_iw$. We present a simple proof of this result. As already noted in [8], problem (1) can be reformulated as

$$\{\nu_i^\star(C_i\,\cdot)\}_{i=1}^N=\arg\min_{\nu_i(\cdot)}\mathbb{E}_w\left(\|u-K^cx\|^2_{Q_{uu}}\right)$$
$$\text{subject to: } u_i=\nu_i(C_iw),\quad \forall i\in\mathbb{Z}_{[1,N]},\tag{2}$$

where $K^c=-Q_{uu}^{-1}Q_{wu}^\top$ is the optimal "centralized" decision, which can be easily obtained by setting the derivative of the cost function with respect to u to zero. Problem (2) can be equivalently reformulated as

$$[\nu_1^{\star\top}(C_1\,\cdot),\dots,\nu_N^{\star\top}(C_n\,\cdot)]^\top=$$
$$=\arg\min_{\nu(\cdot)\in\mathbf{S}}\mathbb{E}_w\left(\|\nu(w)-K^cx\|_{Q_{uu}}\right),\tag{3}$$

where $\mathbf{S}$ is a linear subspace of $\mathcal{L}^2$ defined as

$$\mathbf{S}:=\left\{\nu(\cdot)\in\mathcal{L}^2\;\middle|\;\nu(\cdot)=\begin{bmatrix}\nu_1(C_1\,\cdot)\\ \\ \nu_N(C_N\,\cdot)\end{bmatrix}\right\}$$

If we define with the following norm for the space $\mathcal{L}^2$ parametrized by $m\in\mathbb{R}_n,\Sigma\succ0$ and $Q\succ0$

$$\|M(\cdot)\|_{Q,\Sigma,m}:=$$
$$\left(\int_{\mathbb{R}_n}\|M(w)\|^2_Q\phi(w;m,\Sigma)\,\mathrm{d}\,w\right)^{\frac{1}{2}}=\left(\mathbb{E}_w\left(\|M(w)\|^2_Q\right)\right)^{\frac{1}{2}},\tag{4}$$

where $w\sim\mathcal{N}(m,\Sigma)$, we notice that problem (3) is simply the orthogonal projection in $\mathcal{L}^2$ endowed with the norm $\|\cdot\|_{Q_{uu},\Sigma_w,m_w}$ onto the subspace $\mathbf{S}$. That is

$$[\nu_1^{\star\top}(C_1\,\cdot),\dots,\nu_N^{\star\top}(C_n\,\cdot)]^\top=\mathrm{Proj}_{\mathbf{S}}(K^c\,\cdot)\tag{5}$$

Since $(\mathcal{L}^2,\|\cdot\|_{Q_{uu},\Sigma_w,m_w})$ is a Hilbert space, the projection onto a linear subspace is a linear operator [10, Corollary 3.22]. The composition of linear operators is linear therefore the optimal decentralized decisions $\nu_i^\star(C_i\,\cdot)$ must also be linear.

B. Generalization to the non-Gaussian case

The approach of Section II-A can be generalized to different quadratic team decision problems whose solution is given by the subspace projection in $\mathcal{L}^2_\mu$ of the decision function $K_c\,\cdot$, where μ is any nonnegative measure on $(\mathbb{R}^n,\mathscr{B})$ with finite first and second moments that is absolutely continuous with respect to the Lebesgue measure (i.e. $L(A)=0\implies\mu(A)=0$ for all $A\in\mathscr{B}$). Since such a measure μ induces a norm on $\mathcal{L}^2_\mu$ parametrized by $Q\succ0$ defined as

$$\|M\|_{Q,\mu}=\left(\int_{\mathbb{R}_n}\|M(w)\|^2_Q\,\mu(\mathrm{d}\,w)\right)^{\frac{1}{2}}.$$

We can define the equivalent of problem (1) in a much more general setting:

$$\{\nu_i^\star(C_i\,\cdot)\}_{i=1}^N=$$
$$\arg\min_{\nu_i(\cdot)}\int_{\mathbb{R}_n}\begin{bmatrix}w\\u\end{bmatrix}^\top\begin{bmatrix}Q_{ww}&Q_{wu}\\Q_{wu}^\top&Q_{uu}\end{bmatrix}\begin{bmatrix}w\\u\end{bmatrix}\mu(\mathrm{d}\,w)$$
$$\text{subject to: } u_i=\nu_i(C_iw).\tag{6}$$

The optimal decisions $\nu_i^\star(C_i,\,\cdot)$ in (6) will be linear and unique as they are the projection of $K^c\,\cdot$ onto the subspace $\mathbf{S}_\mu$ of "structured" measurable functions in the Hilbert space $\mathcal{L}^2_\mu$ endowed with the norm $\|\cdot\|_{Q_{uu},\mu}$. Note that this means that linear decentralized decisions are optimal for all absolutely continuous probability measures.

III. Quadratic two-team games

In this section we consider a game theoretical version of the problem in (1) where two players, both knowing the distribution of w, need to decide policies to compute vectors $u\in\mathbb{R}^m$ and $v\in\mathbb{R}^q$ as a function of the realization $w\in\mathbb{R}^n$ in order to minimize the expectation of *different* quadratic forms in w,u,v. Each player is composed of a "team" of agents, each of which observes a different linear function of w and decides a portion of the vectors u or v. For clarity of exposition we will present the Gaussian case. More formally, given a vector $w\sim\mathcal{N}(m_w,\Sigma_w)$, where $\Sigma_w\succ0$, consider the following game

$$P_1:\begin{cases}\min\limits_{\kappa_i(\cdot)}\mathbb{E}_w\left(J_1(w,u,v)\right)\\ \text{s. t. } u_i=\kappa_i(C_iw)\\ \qquad\forall i\in\mathbb{Z}_{[1,N]}.\end{cases},P_2:\begin{cases}\min\limits_{\lambda_i(\cdot)}\mathbb{E}_w\left(J_2(w,u,v)\right)\\ \text{s. t. } v_i=\lambda_i(\Gamma_iw)\\ \qquad\forall j\in\mathbb{Z}_{[1,M]}.\end{cases}\tag{7}$$

where $J_i(w,u,v):=$

$$\begin{bmatrix}w\\u\\v\end{bmatrix}^\top\begin{bmatrix}Q_{i\,ww}&Q_{i\,wu}&Q_{i\,wv}\\Q_{i\,wu}^\top&Q_{i\,uu}&Q_{i\,uv}\\Q_{i\,wv}^\top&Q_{i\,uv}^\top&Q_{i\,vv}\end{bmatrix}\begin{bmatrix}w\\u\\v\end{bmatrix},i\in\{1,2\}$$

We make the following assumption.

Assumption 1:

$$\begin{bmatrix}Q_{1\,uu}&Q_{1\,uv}\\Q_{1\,uv}^\top&Q_{2\,vv}\end{bmatrix}\succ0,\begin{bmatrix}Q_{1\,uu}&Q_{2\,uv}\\Q_{2\,uv}^\top&Q_{2\,vv}\end{bmatrix}\succ0.$$

note that in the case of zero-sum game ($J_1=-J_2$), Assumption 1 is standard to guarantee the existence of a saddle point equilibrium to the game without decentralized information structure [11, condition 6.3.9]. If $J_1=J_2$, Assumption 1 reduces to the standard positive definite assumption of team theory.

We now define the set of Nash optimal policies for the game in (7).

Definition 1: A pair of policies $(\kappa^\star(\cdot),\lambda^\star(\cdot))$ of the form $[\kappa_1^{\star\top}(C_1\,\cdot),\dots,\kappa_N^{\star\top}(C_n\,\cdot)]^\top$ and $[\lambda_1^{\star\top}(\Gamma_1\,\cdot),\dots,\lambda_M^{\star\top}(\Gamma_n\,\cdot)]^\top$ is Nash optimal for the game in (7) if

$$\begin{cases}\kappa^\star(\cdot)\in\arg\min\limits_{\kappa(\cdot)}\mathbb{E}_wJ_1(w,\lambda^\star(w),\kappa(w))\\ \lambda^\star(\cdot)\in\arg\min\limits_{\lambda(\cdot)}\mathbb{E}_wJ_2(w,\lambda(w),\kappa^\star(w)),\end{cases}\tag{8}$$

We are now ready to state the main result of the paper.

Theorem 1: Under Assumption 1, the game in (7) always admits a unique set of linear Nash optimal policies. □

Before proving this result we review some important results from operator theory in Hilbert spaces.

Definition 2 (Contraction mapping): Given a Hilbert space $\mathcal{H}$, a mapping $M : \mathcal{H} \to \mathcal{H}$ is called a *contraction mapping* if there exists an $\varepsilon > 0$ such that

$$\|M(x) - M(y)\| \leq (1-\varepsilon)\,\|(x-y)\|\,,\ \ \forall x, y \in \mathcal{H}. \quad (9)$$

or equivalently there exists $\eta > 0$ such that

$$\|M(x) - M(y)\|^2 \leq (1-\eta)\,\|(x-y)\|^2\,,\ \ \forall x, y \in \mathcal{H}. \quad (10)$$

Definition 3 (Non expansive mapping): Given a Hilbert space $\mathcal{H}$, a mapping $M : \mathcal{H} \to \mathcal{H}$ is called a *non expansive mapping* if (9) or (10) holds for $\varepsilon = 0$ or $\eta = 0$ respectively.

Proposition 1 ([10] Proposition 4.8): Given a Hilbert space $\mathcal{H}$, the projection map Proj_C onto a closed convex set $C \subset \mathcal{H}$ is non expansive.

Proposition 2: Given a Hilbert space $\mathcal{H}$, mappings $M_1, M_2 : \mathcal{H} \to \mathcal{H}$ such that M_1 is non expansive and M_2 is a contraction then the composition mapping $M_1 \circ M_2$ is a contraction. □

Proposition 3 (Banach's Fixed Point Theorem [12]): Given a Hilbert space $\mathcal{H}$ and a contraction mapping $M : \mathcal{H} \to \mathcal{H}$ and ε as in (9), M admits a unique fixed point, that is $\exists! \bar{x} \in \mathcal{H}$ such that $\bar{x} = M(\bar{x})$ and the Picard–Banach iteration $x_{k+1} = M(x_k)$ converges to $\bar{x}$ as $k \to \infty$ for any initial guess $x_0 \in \mathcal{H}$. Furthermore, for any k, $\|x_k - \bar{x}\| \leq (1-\varepsilon)^k \|x_0 - \bar{x}\|$. □

Proof of Theorem 1: We start by noticing that, if one team plays a linear strategy the optimal response of the other team is also linear. For example, assume that the second team plays $v_i = \tilde{L}_i \Gamma_i w,\ \forall i \in \mathbb{Z}_{[1,M]}$, if we define $\tilde{L} := \mathrm{blkdiag}(\tilde{L}_1, ..., \tilde{L}_M)$ and $\Gamma = [\Gamma_1^\top, ..., \Gamma_M^\top]^\top$, the optimal strategy for the first team is given by

$$\begin{aligned}\{\kappa_i^\star(C_i\,\cdot)\}_{i=1}^N \in \\ &\arg\min_{\kappa_i(\cdot)} \mathbb{E}_w \left(\begin{bmatrix} w \\ u \end{bmatrix}^\top \begin{bmatrix} \tilde{Q}_{1\,ww} & \tilde{Q}_{1\,wu} \\ \tilde{Q}_{1\,wu}^\top & Q_{1\,uu} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \right) \\ &\text{subject to: } \ u_i = \kappa_i(C_i w)\end{aligned} \quad (11)$$

where $\tilde{Q}_{1\,ww}$ is irrelevant and

$$\tilde{Q}_{1\,wu} = Q_{1\,wu} + \Gamma^\top \tilde{L}^\top Q_{1\,uv}^\top.$$

Following the same procedure as in Section II-A, we know that the optimal policy for the first team (in response of the linear policy of the second team) is linear, unique and is given by:

$$\begin{aligned}[\kappa_1^{\star\top}(C_1\,\cdot), \ldots, \kappa_N^{\star\top}(C_n\,\cdot)]^\top = K^\star C\,\cdot = \\ \mathrm{Proj}_{\mathbf{S}_\kappa}(-Q_{1\,uu}^{-1}(Q_{1\,wu}^\top + Q_{1\,uv}\tilde{L}\Gamma)\,\cdot),\end{aligned} \quad (12)$$

where $K^\star = \mathrm{blkdiag}(K_1^\star, ..., K_2^\star)$ and

$$\mathbf{S}_\kappa := \left\{ \kappa(\cdot) \in \mathcal{L}^2 \;\middle|\; \kappa(\cdot) = \begin{bmatrix} \kappa_1(C_1\,\cdot) \\ \\ \kappa_N(C_N\,\cdot) \end{bmatrix} \right\}.$$

and the projection is performed in $\mathcal{L}^2$ with respect to the norm $\|\cdot\|_{Q_{1\,uu}, m_w, \Sigma_w}$ defined in (4).

With identical reasoning we can conclude that if the first team plays a linear policy of the form $u_i = \tilde{K}_i C_i w,\ \forall i \in \mathbb{Z}_{[1,N]}$ and we define $\tilde{K} := \mathrm{blkdiag}(\tilde{K}_1, ..., \tilde{K}_N)$ and $C = [C_1^\top, ..., C_N^\top]^\top$, the optimal response for the second team is again linear and is given by

$$\begin{aligned}[\lambda_1^{\star\top}(\Gamma_1\,\cdot), \ldots, \lambda_M^{\star\top}(\Gamma_n\,\cdot)]^\top = L^\star \Gamma\,\cdot = \\ \mathrm{Proj}_{\mathbf{S}_\lambda}(-Q_{2\,vv}^{-1}(Q_{2\,wv}^\top + Q_{2\,uv}^\top \tilde{K} C)\,\cdot),\end{aligned} \quad (13)$$

where $L^\star = \mathrm{blkdiag}(L_1^\star, ..., L_2^\star)$ and

$$\mathbf{S}_\lambda := \left\{ \lambda(\cdot) \in \mathcal{L}^2 \;\middle|\; \lambda(\cdot) = \begin{bmatrix} \lambda_1(\Gamma_1\,\cdot) \\ \\ \lambda_M(\Gamma_M\,\cdot) \end{bmatrix} \right\}.$$

and the projection is performed in $\mathcal{L}^2$ with respect to the norm $\|\cdot\|_{Q_{2\,vv}, m_w, \Sigma_w}$.

Now consider (12) and (13), since we know that the optimal response functions are linear, we can search for the matrices $K^\star = \mathrm{blkdiag}(K_1^\star, ..., K_N^\star)$ and $L^\star = \mathrm{blkdiag}(L_1^\star, ..., L_M^\star)$. We make the following substitutions

$$F_K^\star = K^\star C, \quad F_L^\star = L^\star \Gamma, \quad \tilde{F}_K = \tilde{K} C, \quad \tilde{F}_L = \tilde{L}\Gamma$$

and we get:

$$\begin{aligned}F_K^\star \in \arg\min_{F_K, K} \ & \mathbb{E}_w \|F_K w - Q_{1uu}^{-1}(Q_{1wu}^\top + Q_{1uv}\tilde{F}_L) w\|_{Q_{1uu}}^2 \\ \text{s.t. } \ & F_K = KC \\ & K = \mathrm{blkdiag}(K_1, ..., K_N) \\ F_L^\star \in \arg\min_{F_L, L} \ & \mathbb{E}_w \|F_L w - Q_{2vv}^{-1}(Q_{2wv}^\top + Q_{2uv}^\top \tilde{F}_K) w\|_{Q_{2vv}}^2 \\ \text{s.t. } \ & F_L = L\Gamma \\ & L = \mathrm{blkdiag}(L_1, ..., L_M).\end{aligned} \quad (14)$$

We exploit the following: if $w \sim \mathcal{N}(m, \Sigma) \in \mathbb{R}^n$, $Q \succ 0$ and $F \in \mathbb{R}^{m \times n}$ then

$$\begin{aligned}\|F\,\cdot\|_{Q,\Sigma,m}^2 &= \mathbb{E}_w \|Fx\|_Q^2 \\ &= \mathrm{tr}(\Sigma^{\frac{1}{2}} F^\top Q F \Sigma^{\frac{1}{2}}) + m^\top F^\top Q F m \\ &= \|F\Sigma^{\frac{1}{2}}\|_Q^2 + \|Fm\|_Q^2\,.\end{aligned} \quad (15)$$

We define the following norm for $\mathbb{R}^{m \times n}$

$$\|F\|_{Q,\Sigma,m} := \sqrt{\|F\Sigma^{\frac{1}{2}}\|_Q^2 + \|Fm\|_Q^2}, \quad (16)$$

and from (15) we get that $\|F\,\cdot\|_{Q,\Sigma,m} = \|F\|_{Q,\Sigma,m}$ In other words, the norm in $\mathcal{L}^2$ defined in (4), parametrized by Q, Σ and m of a linear function $F\,\cdot$ is equal to the norm in $\mathbb{R}^{m \times n}$ of F, also parametrized by Q, Σ and m and defined in (16). In view of (15) and (16) we can then see (14) as projection operators on the space of matrices endowed with such norm. That is

$$\left\{ \begin{array}{ll} F_K^\star = & \mathrm{Proj}_{\mathbf{S}_K}(-Q_{1\,uu}^{-1}(Q_{1\,wu}^\top + Q_{1\,uv}\tilde{F}_L)) \\ F_L^\star = & \mathrm{Proj}_{\mathbf{S}_L}(-Q_{2\,vv}^{-1}(Q_{2\,wv}^\top + Q_{2\,uv}^\top \tilde{F}_K)), \end{array} \right. \quad (17)$$

where

$$\mathbf{S}_K = \left\{ F \in \mathbb{R}^{m \times n} \mid F = KC, K = \mathrm{blkdiag}(K_1, ..., K_N) \right\}$$

and

$$\mathbf{S}_L = \left\{ F \in \mathbb{R}^{q\times n} \,|\, F = L\Gamma, L = \mathrm{blkdiag}(L_1, ..., L_M) \right\}.$$

The projections above are performed with respect to the norms $\|\cdot\|_{Q_{1\,uu},\Sigma_w,m_w}$ and $\|\cdot\|_{Q_{2\,vv},\Sigma_w,m_w}$ respectively. The formulation in (17) can be written compactly as

$$\begin{bmatrix} F_K^\star \\ F_L^\star \end{bmatrix} = \mathrm{Proj}_{\mathbf{S}} \left(\hat{Q} \begin{bmatrix} \tilde{F}_K \\ \tilde{F}_L \end{bmatrix} - \begin{bmatrix} Q_{1\,uu}^{-1} Q_{1\,wu}^\top \\ Q_{2\,vv}^{-1} Q_{2\,wv}^\top \end{bmatrix} \right) \quad (18)$$

where

$$\hat{Q} := \begin{bmatrix} 0 & -Q_{1\,uu}^{-1} Q_{1\,uv} \\ -Q_{2\,vv}^{-1} Q_{2\,uv}^\top & 0 \end{bmatrix}, \quad (19)$$

and $\mathbf{S} = \mathbf{S}_K \times \mathbf{S}_L$. The last projection is in the space $\mathbb{R}^{m\times n} \times \mathbb{R}^{q\times n}$ equipped with the norm $\|\cdot\|_{Q,\Sigma_w,m_w}$ where

$$Q = \begin{bmatrix} Q_{1\,uu} & 0 \\ 0 & Q_{2\,vv} \end{bmatrix}. \quad (20)$$

Note that $Q \succ 0$ by Assumption 1. Equation (18) allows us to compute the optimal response of each player to any linear strategy adopted by the other player. Now suppose we could find matrices $F_K^\star, F_L^\star$ such that

$$\begin{bmatrix} F_K^\star \\ F_L^\star \end{bmatrix} = \mathrm{Proj}_{\mathbf{S}} \left(\hat{Q} \begin{bmatrix} F_K^\star \\ F_L^\star \end{bmatrix} - \begin{bmatrix} Q_{1\,uu}^{-1} Q_{1\,wu}^\top \\ Q_{2\,vv}^{-1} Q_{2\,wv}^\top \end{bmatrix} \right), \quad (21)$$

then, according to (18), we would have that $F_K^\star\,\cdot$ is the optimal policy of the first team if the second team plays $F_L^\star\,\cdot$, but also that $F_L^\star\,\cdot$ is the optimal policy for the second team if the first plays $F_K^\star\,\cdot$. Then, according to Definition 1, the pair $(\kappa^\star(\cdot), \lambda^\star(\cdot)) = (F_K^\star\,\cdot, F_L^\star\,\cdot)$ would be Nash optimal. Conversely, if a set of linear policies is Nash Optimal according to Definition 1, then the corresponding matrices must satisfy the fixed point equation (21). The set of linear Nash policies can therefore be characterized as the set of fixed points of the map $\mathcal{M} : \mathbb{R}^{m\times n} \times \mathbb{R}^{q\times n} \to \mathbb{R}^{m\times n} \times \mathbb{R}^{q\times n}$

$$\mathcal{M}\left(\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \right) := \mathrm{Proj}_{\mathbf{S}} \left(\hat{Q} \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} - \begin{bmatrix} Q_{1\,uu}^{-1} Q_{1\,wu}^\top \\ Q_{2\,vv}^{-1} Q_{2\,wv}^\top \end{bmatrix} \right).$$

According to Proposition 3 if we can show that the map $\mathcal{M}$ is a contraction mapping (Definition 2), then it always admits a unique fixed point and we have proven the theorem. From Proposition 1 we know that the projection operator is non expansive. Since $\mathcal{M}$ is the composition of the projection operator and a linear mapping, by Proposition 2, in order to show contractiveness of $\mathcal{M}$, it is enough show that the affine map

$$\hat{Q} \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} - \begin{bmatrix} Q_{1\,uu}^{-1} Q_{1\,wu}^\top \\ Q_{2\,vv}^{-1} Q_{2\,wv}^\top \end{bmatrix} \quad (22)$$

is a contraction in the norm parametrized by Q, Σ_w and m_w. By definition, the affine map in (22) is contractive if for all $\begin{bmatrix} F_{K1} \\ F_{L1} \end{bmatrix}$ and $\begin{bmatrix} F_{K2} \\ F_{L2} \end{bmatrix}$ if there exist $\eta > 0$ such that

$$\left\| \hat{Q} \begin{bmatrix} F_{K1} - F_{K2} \\ F_{L1} - F_{L2} \end{bmatrix} \right\|_{Q,\Sigma_w,m_w}^2 \leq (1-\eta) \left\| \begin{bmatrix} F_{K1} - F_{K2} \\ F_{L1} - F_{L2} \end{bmatrix} \right\|_{Q,\Sigma_w,m_w}^2. \quad (23)$$

We can expand (23) and we get that it is equivalent to the existence of $\eta > 0$ such that for all F_{K1}, F_{K2}, F_{L1} and F_{L2} of appropriate dimensions

$$\mathrm{tr}\left(\begin{bmatrix} (F_{K1} - F_{K2})\Sigma^{\frac{1}{2}} \\ (F_{L1} - F_{L2})\Sigma^{\frac{1}{2}} \end{bmatrix}^\top \bar{Q} \begin{bmatrix} (F_{K1} - F_{K2})\Sigma^{\frac{1}{2}} \\ (F_{L1} - F_{L2})\Sigma^{\frac{1}{2}} \end{bmatrix} \right) + \left(\begin{bmatrix} (F_{K1} - F_{K2})m_w \\ (F_{L1} - F_{L2})m_w \end{bmatrix}^\top \bar{Q} \begin{bmatrix} (F_{K1} - F_{K2})m_w \\ (F_{L1} - F_{L2})m_w \end{bmatrix} \right) \leq -\eta \left\| \begin{bmatrix} F_{K1} - F_{K2} \\ F_{L1} - F_{L2} \end{bmatrix} \right\|_{Q,\Sigma_w,m_w}^2, \quad (24)$$

where $\bar{Q} = \mathrm{blkdiag}(\bar{Q}_{11}, \bar{Q}_{22})$ and

$$\bar{Q}_{11} := Q_{2\,uv} Q_{2\,vv}^{-1} Q_{2\,uv}^\top - Q_{1\,uu}$$
$$\bar{Q}_{22} := Q_{1\,uv}^\top Q_{1\,uu}^{-1} Q_{1\,uv} - Q_{2\,vv}$$

From Assumption 1 and the Schur complement, $\bar{Q} \prec 0$ therefore (24) always holds (take $\eta \leq \sigma_{\min}(\bar{Q})/\sigma_{\max}(Q)$) and the map $\mathcal{M}$ is a contraction. This implies there always exist a unique fixed point and therefore a unique set of linear Nash optimal policies for the game in (7) and the proof is complete. ■

IV. Computation of the linear Nash optimal strategies

In this section we show how to compute the unique Nash optimal linear strategy, whose existence was proven in Theorem 1. We first show a simple iterative algorithm that converges to the Nash optimal strategies and can be easily implemented using semidefinite programming. We then show that the Nash strategies can be computed directly by solving a system of linear equations.

A. Iterative algorithm for the computation of the linear Nash optimal strategies

Directly from the proof of Theorem 1, we obtain an iterative algorithm to compute the optimal Nash strategies as a corollary.

Corollary 1: The sequence

$$\begin{bmatrix} F_K \\ F_L \end{bmatrix}_{k+1} = \mathrm{Proj}_{\mathbf{S}} \left(\hat{Q} \begin{bmatrix} F_K \\ F_L \end{bmatrix}_k - \begin{bmatrix} Q_{1\,uu}^{-1} Q_{1\,wu}^\top \\ Q_{2\,vv}^{-1} Q_{2\,wv}^\top \end{bmatrix} \right) \quad (25)$$

$k = 0, 1, ...$ converges as $k \to \infty$ to the unique set of Nash optimal linear policies $\begin{bmatrix} F_K^\star \\ F_L^\star \end{bmatrix}$ for any initial condition.

Proof: This is just an application of Banach fixed point theorem (Proposition 3). ■

We will now show how the iteration in (25) can be implemented with semidefinite programming. If we expand the projection we get

$$\begin{bmatrix} F_K \\ F_L \end{bmatrix}_{k+1} \in \arg\min_{\hat{F}_K, \hat{F}_L} \left\| \begin{bmatrix} \hat{F}_K \\ \hat{F}_L \end{bmatrix} - H \right\|_{Q,\Sigma_w,m_w}^2$$

subject to: $\hat{F}_K \in \mathbf{S}_K, \hat{F}_L \in \mathbf{S}_L$,

$$H = \hat{Q} \begin{bmatrix} F_K \\ F_L \end{bmatrix}_k - \begin{bmatrix} Q_{1\,uu}^{-1} Q_{1\,wu}^\top \\ Q_{2\,vv}^{-1} Q_{2\,wv}^\top \end{bmatrix}.$$

which can be rewritten as

$$\begin{aligned}
\arg \min_{\substack{\hat{F}_K,\hat{F}_L,K,\\ L,Z,\gamma,\delta}} \quad & \gamma+\delta \\
\text{subject to:} \quad & \begin{bmatrix} Q^{-1} & \left(\begin{bmatrix} \hat{F}_K \\ \hat{F}_L \end{bmatrix} - H\right)\Sigma_w \\ \star & Z \end{bmatrix} \succcurlyeq 0, \\
& \begin{bmatrix} Q^{-1} & \left(\begin{bmatrix} \hat{F}_K \\ \hat{F}_L \end{bmatrix} - H\right) m_w \\ \star & \delta \end{bmatrix} \succcurlyeq 0, \\
& H = \hat{Q}\begin{bmatrix} F_K \\ F_L \end{bmatrix}_k - \begin{bmatrix} Q_{1\,uu}^{-1}Q_{1\,wu}^{\top} \\ Q_{2\,vv}^{-1}Q_{2\,wv}^{\top} \end{bmatrix}, \\
& Z \succcurlyeq 0,\ \mathrm{tr}(Z) \leq \gamma, \\
& \hat{F}_K = KC, K = \mathrm{blkdiag}(K_1, ..., K_N), \\
& \hat{F}_L = L\Gamma, L = \mathrm{blkdiag}(L_1, ..., L_M),
\end{aligned}$$

where $\hat{Q}$ and Q are defined in (19) and (20) respectively.

B. Direct computation of the Nash optimal strategies from the solution of a system of linear equations

In order to derive the linear system we exploit the following matrix identity [13]. Given $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times q}$

$$\mathrm{vec}(AB) = (I_q \otimes A)\mathrm{vec}(B).$$

Since we know that $\mathrm{Proj}_{\mathbf{S}}(\cdot)$ is a linear operator, we can rewrite (21) as

$$\mathrm{vec}\left(\begin{bmatrix} \bar{F}_K \\ \bar{F}_L \end{bmatrix}\right) = \Pi\left(I\otimes\hat{Q}\right)\left(\mathrm{vec}\left(\begin{bmatrix} \bar{F}_K \\ \bar{F}_L \end{bmatrix}\right) - \mathrm{vec}\left(\begin{bmatrix} Q_{1\,uu}^{-1}Q_{1\,wu}^{\top} \\ Q_{2\,vv}^{-1}Q_{2\,wv}^{\top} \end{bmatrix}\right)\right), \tag{26}$$

where Π is the appropriate projector matrix that performs $\mathrm{Proj}_{\mathbf{S}}(\cdot)$ in the vectorized space. The solution of the system of linear equations (26) gives the optimal Nash policies directly but some effort is required in constructing the matrix Π. Note that computing the optimal Nash policies requires global information on the cost function of both players.

V. Numerical example

In this section we present a numerical example to illustrate the theoretical results. The example shows a game between the system operator and an attacker in the context of power system security. We first introduce the system model and then we provide a numerical example using the IEEE 9 bus benchmark power system [14].

A. Application to power systems security

We adopt a linearized version of the classic differential-algebraic structure-preserving power network model presented in [15]. Consider a connected power network consisting of n generators and m load buses. The interconnection structure of the power network is encoded by a connected weighted graph with $n+m$ vertices representing the generators $\{g_i\}_{i=1}^n$ and buses $\{b_i\}_{i=1}^m$. The edges of the graph represent either a transmission line (b_i, b_j) weighted by the susceptance between buses b_i and b_j, or a connection (g_i, b_i) weighted by the transient susceptance between generator g_i and its adjacent bus b_i. The Laplacian associated with the weighted graph is the symmetric admittance matrix $\mathcal{L} = \left[\begin{smallmatrix} \mathcal{L}_{\mathrm{gg}} & \mathcal{L}_{\mathrm{gl}} \\ \mathcal{L}_{\mathrm{lg}} & \mathcal{L}_{\mathrm{ll}} \end{smallmatrix}\right] \in \mathbb{R}^{(n+m)\times(n+m)}$, where the first n rows are associated with the generators and the last m rows to the loads. Since the network is connected, we can reduce the model via Kron reduction [16] and consider the dynamics of the generator nodes only. In particular, the submatrix $\mathcal{L}_{\mathrm{ll}}$ of the admittance matrix is invertible and the Kron-reduced power system model is of the form

$$\begin{bmatrix} \dot{\delta} \\ \dot{\omega} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & I \\ -M^{-1}\mathcal{L}_{\mathrm{r}} & -M^{-1}D \end{bmatrix}}_{A} \begin{bmatrix} \delta \\ \omega \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ -\mathcal{L}_{\mathrm{gl}}\mathcal{L}_{\mathrm{ll}}^{-1} & I \end{bmatrix}}_{B} w\,, \tag{27}$$

where $\delta \in \mathbb{R}^n$ and $\omega \in \mathbb{R}^n$ represent the generator angles and frequencies respectively, M and D are diagonal matrices with inertia and damping coefficients for each generator on the diagonal. The Kron-reduced admittance matrix is given by the Schur complement $\mathcal{L}_{\mathrm{r}} = \mathcal{L}_{\mathrm{gg}} - \mathcal{L}_{\mathrm{gl}}\mathcal{L}_{\mathrm{ll}}^{-1}\mathcal{L}_{\mathrm{lg}}$ and the disturbance term $w \in \mathbb{R}^{n+m}$ represents the deviation from the forecast of the power produced by the generators or consumed by the loads.

Since the Laplacian $\mathcal{L}_{\mathrm{r}}$ is singular with kernel span($\mathbb{1}$), the system matrix in (27) is marginally stable. However, if we change coordinates by considering the deviation of the angles from their mean, we obtain a reduced stable model. To do so we consider the unitary matrix $[\frac{1}{n}\mathbb{1}, V]$ where each column is an eigenvectors of the projector $\Pi_n := I - \frac{1}{n}\mathbb{1}\mathbb{1}^{\top}$. We then construct the state transformation matrix $T = \mathrm{blkdiag}(V^{\top}, I)$ and change the state as: $[\delta_r^{\top}, \omega^{\top}]^{\top} = T[\delta^{\top}, \omega^{\top}]^{\top}$. The dynamics in the new variables become

$$\begin{bmatrix} \dot{\delta}_r \\ \dot{\omega} \end{bmatrix} = A_r \begin{bmatrix} \delta_r \\ \omega \end{bmatrix} + B_r w,$$

where $A_r := TAT^{\top}$ and $B_r := TB$.

We assume that w is a constant disturbance (over time) whose value is drawn from the normal distribution $\mathcal{N}(0, I)$. The value of w can be measured both by the system operator and an attacker. The goal of the network operator is to reduce the effect of the disturbance w on the steady-state, while operating the local controllers efficiently. The goal of the attacker is to exacerbate the effect of the disturbance w on the steady-state, while also operating the local controllers to maintain stealth. Both the attacker and the system operators are subject to information constraints and can manipulate the generator powers and the loads, i.e.

$$\begin{bmatrix} \dot{\delta}_r \\ \dot{\omega} \end{bmatrix} = A_r \begin{bmatrix} \delta_r \\ \omega \end{bmatrix} + B_r w + B_r u + B_r v.$$

In particular, the game between the system operator and the attacker becomes

$$\min_u \max_w \quad \mathbb{E}_w\left(\begin{bmatrix} \bar{\delta}_r \\ \bar{\omega} \end{bmatrix}^{\top} \begin{bmatrix} \bar{\delta}_r \\ \bar{\omega} \end{bmatrix} + u^{\top}Ru - v^{\top}Sv\right), \tag{28}$$

subject to the information structure.

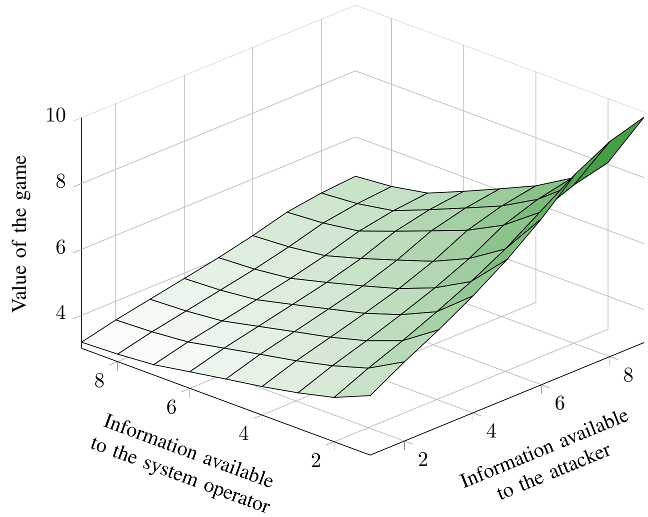

Fig. 1: The value of the game as a function of the information structure of the two players. The information structures corresponding to the axes are shown in Figure 2

The vector $[\bar{\delta}_r^\top, \bar{\omega}^\top]^\top$ in (28) is the steady-state angle and frequency deviation resulting from the constant inputs and disturbances w, u and v. $R \succ 0$ encodes the efficiency objective of the network operator and $S \succ 0$ encodes the stealth objective of the attacker. The steady-state is given by

$$\begin{bmatrix} \bar{\delta}_r \\ \bar{\omega} \end{bmatrix} = -A_r^{-1}(B_r w + B_r u + B_r v).$$

Then the team game becomes

$$\min_u \max_v \; \mathbb{E}_w \left(\begin{bmatrix} w \\ u \\ v \end{bmatrix}^\top \begin{bmatrix} Q_{ww} & Q_{wu} & Q_{wv} \\ Q_{wu}^\top & Q_{uu} & Q_{uv} \\ Q_{wv}^\top & Q_{uv}^\top & Q_{vv} \end{bmatrix} \begin{bmatrix} w \\ u \\ v \end{bmatrix} \right),$$

where $Q_{ww} = Q_{wu} = Q_{wv} = Q_{uv} = B_r^\top A_r^{-\top} A_r^{-1} B_r$ and $Q_{uu} = R + B_r^\top A_r^{-\top} A_r^{-1} B_r,\ Q_{vv} = B_r^\top A_r^{-\top} A_r^{-1} B_r - S$.

For the numerical example we used data from the IEEE 9 bus benchmark power system and we computed R and S such that Assumption 1 is satisfied. We then computed the Nash optimal strategies for the system operator and the attacker using the iterative algorithm presented in Section IV-A for different information structures varying from fully decentralized to centralized. In Figure 1 we plot the value of the game as a function of the information structures of the players. Figure 2 illustrates the different information structures used in the experiment. As expected we notice that having more information is advantageous for both players.

VI. Conclusion and Outlook

In this paper we showed that for a class of zero sum quadratic games, under suitable conditions we can guarantee the existence and uniqueness of a Nash equilibrium in the space of linear distributed policies. We present an iterative algorithm based on a fixed point iteration in order to compute this equilibrium. An interesting research direction, in view of the results of [17], is the use of more complex fixed point iterations in order to guarantee convergence to a Nash equilibrium under milder conditions and possibly under compact constraints for each player. Finally the most interesting research direction is the extension of our result to finite horizon distributed dynamic games possibly under partially nested information structures.

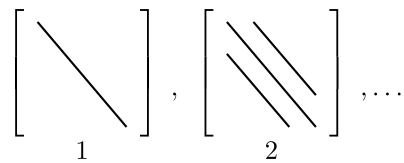

Fig. 2: Varying information structures for each player in the experiment shown in Figure 1. The structures are banded: 1 corresponds to the fully decentralized information structure while 9 to the centralized one.

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