Optimal Placements in Ring Network for Data Replicas in Distributed Database with Majority Voting Protocol

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Abstract

In a distributed database system, data replicas are placed at different locations to achieve high data availability in the presence of link failures. With majority voting protocol, a location is survived for read/write operations if and only if it is accessible to more than half of the replicas. The problem is to find out the optimal placements for a given number of data replicas in a ring network. When the number of replicas is odd, it was conjectured by Hu et al. that every uniform placement is optimal, which is proved by Shekhar and Wu later. However, when the number of replicas is even, it was pointed out by Hu et al. that uniform placements are not optimal and the optimal placement problem may be very complicated. In this paper, we study the optimal placement problem in a ring network with majority voting protocol and even number of replicas, and give a complete characterization of optimal placements when the number of replicas is not too large compared with the number of locations.

Key words: *distributed database, data replica, majority voting protocol, ring network.*

1. Introduction

The distributed database systems [4, 8, 11, 12] are usually built in a network with certain topological structures. Multiple copies of data are distributed at different locations to increase the data availability. Such copies are called *data replicas*. In presence of link failures, optimal replica placement problem (ORP) is an important research problem in distributed data systems. In ORP, one is to find a placement of given replicas that maximizes the expected number of locations that can access the data when some links fail. Its solution depends on the data manage protocol and the distribution of link failures.

The ORP is proved to be very difficult for general networks. Nel and Colbourn [9] and Johnson and Taad [7] independently showed that the ORP is #P-complete. Stephens et al. [11] showed that the ORPs for read-only system and write-dominant system are both NP-complete.

Suppose each link can fail independently with a same probability ρ. Some works concentrate on the case when ρ is small. For tree networks, Stephens et al. [12] derived some necessary conditions for optimal placements. For ring networks, Stephens et al. [11] showed that equally spacing placements are optimal for read-only system, and grouping placements are optimal for write-dominant system. In the case of majority voting protocol, Hu et al. [6] conjectured that every uniform placement is optimal for odd k. The conjecture was proved by Shekhar and Wu [10]. However, when the number of replicas is even, it was pointed out by Hu et al. with counterexample that the uniform placement may not be optimal and the ORP may be very complicated.

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In this paper, we study the ORP in a ring network with majority voting protocol and even number of replicas, and give a complete characterization of optimal placements when the number of replicas is not too large compared with the number of locations. When the number of replicas is large, it is easy to see that the placement is dense. In this case, the structure is not so concise as our main result (Theorem 3.1).

Some other related works can be found in [2, 3, 5, 6] etc.

2. Preliminary and Terminology

Consider a ring network $R(V, L)$ with n locations and k data replicas. To control concurrency, suppose the system employs the majority voting protocol [1, 13], that is, a read/write operation must be performed on more than a half of the k replicas. Thus, data can be 'successfully' accessed at a location in this protocol if and only if the location is accessible to more than half of the replicas. Such a location is called a *survived location*.

Label the locations by $1, 2, ..., n$ clockwise. A *placement* of a set of k $(1 < k < n)$ replicas in $R(V, L)$ is described by a k-tuple $[c_1, c_2, ..., c_k]$ where the ith component c_i is the location of the ith replica.

Suppose each link can fail independently with a same probability ρ. For a placement C, denote by $E(C)$ the expected number of survived locations and by $N_C(F)$ the number of survived locations when F is the set of failure links. Then

$$E(C) = \sum_{i=0}^{n} \left(\sum_{F \subseteq L, |F|=i} N_C(F) \right) \rho^i (1-\rho)^{n-i}.$$

For any placement C,

$$\sum_{F \subseteq L, |F|=0} N_C(F) = n, \qquad \sum_{F \subseteq L, |F|=1} N_C(F) = n^2,$$

which do not depend on C. When ρ is sufficiently small,

$$\begin{aligned} &\left(\textstyle\sum_{F \subseteq L, |F|=2} N_C(F)\right) \rho^2 (1-\rho)^{n-2} \\ \gg \quad &\textstyle\sum_{i=3}^{n} \left(\sum_{F \subseteq L, |F|=i} N_C(F)\right) \rho^i (1-\rho)^{n-i}. \end{aligned}$$

Therefore, for sufficiently small ρ, to find a placement to maximize $E(C)$, it suffices to maximize

$$f(C) = \sum_{F \subseteq L, |F|=2} N_C(F).$$

So, we shall call a placement C which maximize $f(C)$ to be *optimal*.

Next, we introduce some terminologies used in this paper. For two locations u and v, we use (u, v) to denote the path between u and v clockwise, and call (u, v) a *segment*. The *length of* (u, v) is the number of edges in (u, v), denoted by $d(u, v)$. A segment containing at least $t + 1$ replicas is called a *survived segment*. For two replicas c_i and c_j, the segment (c_i, c_j) is called an *interval*, c_i and c_j are the *left- and right-bounds* of the interval, respectively. An interval bounded by two consecutive replicas such as (c_i, c_{i+1}) is call a *gap* of C. If a gap (c_i, c_{i+1}) is of length 1, then call it a *small gap*, otherwise call it a *big gap*. A small gap (c_i, c_{i+1}) is a *max-small-gap* if (c_{i-1}, c_i) and (c_{i+1}, c_{i+2}) are both big gaps. An interval consisting of consecutive small gaps is called a *1-interval*. A maximal 1-interval (c_p, c_q) (with respect to inclusion) is said to be a *max-1-interval* if (c_{p+t}, c_{q+t}) is not 'properly' contained in another 1-interval, that is, there does not exist a 1-interval $(c_{p'}, c_{q'})$ such that $p' < p + t$ and $q' > q + t$.

3 Main Results

In this section, we study ring network $R = (V, L)$ with majority voting protocol and even number of replicas. Denote by $n = |V|$, and $2t$ the number of replicas in R. Our main result is as follows.

Theorem 3.1. *Let $R = (V, L)$ be a ring network with n locations and $2t$ replicas, $n > 5t$ for odd n and $n > 4t$ for even n. When t is odd, write $t = 2m - 1$, when t is even, write $t = 2m$. Let $T = (m - 1)n + t$ and $a = \lfloor T/t \rfloor$ when t is odd, $a = \lfloor T/(t - 1) \rfloor$ when t is even. For a placement $C = (c_1, c_2, ..., c_{2t})$, write $\ell_j = d(c_j, c_{j+t})$ $(j = 1, 2, ..., 2t)$. Then C is optimal if and only if the following conditions are satisfied:*

For odd t (see Fig. 1 (a)), there is a pairing of replicas $\{c_1, c_2\}, \{c_3, c_4\}, ..., \{c_{2t-1}, c_{2t}\}$ such that

(a1) for $j = 1, 2, ..., t$, (c_{2j-1}, c_{2j}) are small gaps and (c_{2j}, c_{2j+1}) are big gaps, where '$+$' is taken as modulo $2t$;

(a2) elements in $\{\ell_1, \ell_3, ..., \ell_{2t-1}\}$ can only take the values of a and/or $a + 1$;

For even t (see Fig.1(b)), except for four replicas, say $c_{2t}, c_1, c_2, c_{t+1}$, there is a pairing for all the other replicas as $\{c_3, c_4\}, \{c_5, c_6\}, ..., \{c_{t-1}, c_t\}, \{c_{t+2}, c_{t+3}\}, ..., \{c_{2t-2}, c_{2t-1}\}$ such that

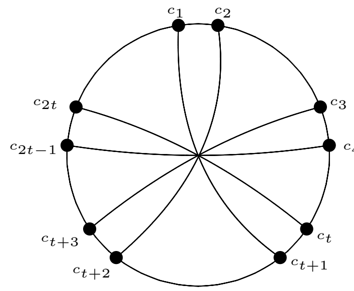
(b1) the above pairs form small gaps, $(c_{2t}, c_1), (c_1, c_2)$ are small gaps, and all the other gaps are big;

(b2) elements in $\{\ell_3, \ell_5, ..., \ell_{t-1}, \ell_{t+2}, ..., \ell_{2t-2}, \ell_{2t}\}$ can only take the values of a and/or $a + 1$;

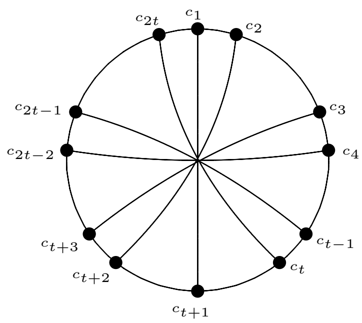
(b3) $\ell_1 = n/2$.

For any two locations x and y, define

$$s(x, y) = \sum_{|F|=2, F \subseteq (x,y)} N_C(F).$$



(a)



(b)

Figure 1. Illustration of optimal placements.

Lemma 3.2. *Let $C = [c_1, c_2, ..., c_{2t}]$ be a placement, and x, y be two locations. If there are less than t replicas in the segment (x, y), then*

$$s(x, y) = \frac{(\ell - 1)\ell(3n - \ell - 1)}{6},$$

where $\ell = d(x, y)$.

Proof. Consider an $F \subseteq (x, y)$. Suppose $F = \{(u, v), (u', v')\}$ with $v \leq u'$. Then the survived segment is (v', u). So $N_C(F) = d(v', u) + 1 = d(y, x) + \ell_1 + \ell_2 + 1 = n - \ell + \ell_1 + \ell_2 + 1$, where $\ell_1 = d(x, u)$ and $\ell_2 = d(v', y)$. Note that $\ell_2 + \ell_1 + 2 \leq \ell$, the result follows from

$$s(x, y) = \sum_{\ell_1=0}^{\ell-2} \sum_{\ell_2=0}^{\ell-2-\ell_1} (n - \ell + 1 + \ell_1 + \ell_2)$$

□

Lemma 3.3. *Let $C = [c_1, c_2, ..., c_{2t}]$ be a placement. Then*

$$f(C) = \sum_{i=1}^{2t} s(c_i, c_{i+t}) - \sum_{i=1}^{2t} s(c_i, c_{i+t-1}).$$

Proof. This follows from the observation that each F with $|F| = 2$ is contained in k of (c_i, c_{i+t})'s if and only if it is contained in $k - 1$ of (c_i, c_{i+t-1})'s. □

Lemma 3.4. *Let $C = [c_1, c_2, ..., c_{2t}]$ be an optimal placement. Then for any $i = 1, 2, ..., 2t$, either (c_i, c_{i+1}) or (c_{i+t}, c_{i+t+1}) is a small gap.*

Proof. Denote by $x_j = d(c_j, c_{j+1})$. Then $\sum_{j=1}^{2t} x_j = n$. Fix $x_1, ..., x_{i-1}, x_{i+1}, ..., x_{i+t-1}, x_{i+t+1}, ..., x_{2t}$ and let x_i, x_{i+t} vary with the restriction that $x_i + x_{i+t} = n - \sum_{j=1, j \neq i, i+t}^{2t} x_j \triangleq x$ (note that x is a constant). Then $f(C)$ as formulated in Lemma 3.3 can be viewed as a function on variable x_i. By Lemma 3.2, it can be calculated that

$$\frac{d^2}{dx_i^2} f(C) = x_i + x_{i+t} > 0.$$

Hence, $f(C)$ is a strictly convex function on x_i. Since $1 \leq x_i \leq x - 1$, $f(C)$ takes the maximum value at $x_i = 1$ or $x_i = x - 1$. In the latter case, $x_{i+t} = 1$. □

Lemma 3.5. *Let $C = [c_1, c_2, ..., c_{2t}]$ and $C' = [c_1, ..., c_{i+t-1}, c_{i+t} - 1, c_{i+t+1}, ..., c_{2t}]$ be two placements. Suppose $d(c_{i-1}, c_i) \geq d(c_i, c_{i+1})$ and $\ell_i - \ell_{i+t} \geq 1$. Then $f(C') \geq f(C)$. Furthermore, equality holds if and only if $d(c_{i-1}, c_i) = d(c_i, c_{i+1})$ and $\ell_i - \ell_{i+t} = 1$.*

Proof. It should be noted that the existence of C' implies that (c_{i+t-1}, c_{i+t}) is a big gap. To compare $f(C)$ and $f(C')$, we only need to consider those F's which has different values of $N_C(F)$'s and $N_{C'}(F)$'s.

If $(c_{i+t} - 1, c_{i+t}) \notin F$, then $c_{i+t} - 1$ plays the same role as c_{i+t} in determining the survived segment for both C and C'. So, we may assume $(c_{i+t} - 1, c_{i+t}) \in F$. Let $(j, j+1)$ be another edge in F. If $(j, j+1) \subseteq (c_{i+t}, c_{i-1})$, then the survived segment is $(j+1, c_{i+t} - 1)$ for both C and C'. If $(j, j+1) \subseteq (c_{i+1}, c_{i+t} - 1)$, then the survived segment is (c_{i+t}, j) for both C and C'. So, we are left with two cases.

(1) $(j, j+1) \subseteq (c_{i-1}, c_i)$. Then the survived segment is $\emptyset$ for C and $(j+1, c_{i+t} - 1)$ for C'. So, $N_C(F) = 0$ and $N_{C'}(F) = \ell_i + \ell$, where $\ell = d(j+1, c_i)$.

(2) $(j, j+1) \subseteq (c_i, c_{i+1})$. Then the survived segment is (c_{i+t}, j) for C and $\emptyset$ for C'. So, $N_C(F) = \ell_{i+t} + 1 + \ell$ and $N_{C'}(F) = 0$, where $\ell = d(c_i, j)$.

Then $f(C') - f(C)$ equals to

$$\sum_{\ell=0}^{d(c_{i-1},c_i)-1} [\ell_i + \ell] - \sum_{\ell=0}^{d(c_i,c_{i+1})-1} [\ell_{i+t} + 1 + \ell] =$$
$$\frac{1}{2}\{n[d(c_{i-1}, c_i) - d(c_i, c_{i+1})] + [\ell_i - \ell_{i+t}]d(c_{i-1}, c_{i+1})$$
$$+ d(c_{i-1}, c_i)[d(c_{i-1}, c_i) - 1] - d(c_i, c_{i+1})[d(c_i, c_{i+1}) + 1]\},$$

where the observation that $\ell_i + \ell_{i+t} = n$ is used. By the assumption $d(c_{i-1}, c_i) \geq d(c_i, c_{i+1})$ and $\ell_i - \ell_{i+t} \geq 1$, the result follows. □

Remark 3.6. Lemma 3.5 can be used with a 'flipping', that is, if $d(c_i, c_{i+1}) \geq d(c_{i-1}, c_i)$ and $\ell_{i+t} - \ell_i \geq 1$, then $C'' = [c_1, ..., c_{i+t-1}, c_{i+t} + 1, c_{i+t+1}, ..., c_{2t}]$ satisfies $f(C'') \geq f(C)$, and equality holds if and only if $d(c_i, c_{i+1}) = d(c_{i-1}, c_i)$ and $\ell_{i+t} - \ell_i = 1$. Such a 'flipping' is applicable to other results in the following.

Corollary 3.7. *Let $C = [c_1, c_2..., c_{2t}]$ be an optimal placement with (c_{i-1}, c_{i+1}) being a 1-interval.*

(a) If (c_{i+t-1}, c_{i+t}) is a big gap, then $\ell_i - \ell_{i+t} \le 1$.

(b) If (c_{i+t}, c_{i+t+1}) is a big gap, then $\ell_{i+t} - \ell_i \le 1$.

(c) If both (c_{i+t-1}, c_{i+t}) and (c_{i+t}, c_{i+t+1}) are big gaps, then $|\ell_i - \ell_{i+t}| \le 1$.

Proof. (a) is a direct consequence of Lemma 3.5 and the assumption that C is optimal, note that the 'big gap' requirement guarantees the existence of C'. (b) can be obtained from (a) by a flipping. (c) is a combination of (a) and (b). □

Lemma 3.8. *Let $C = [c_1, c_2..., c_{2t}]$ be a placement with $(c_i, c_{i+1}), (c_{i+t-1}, c_{i+t}), (c_{i+t+1}, c_{i+t+2})$ being small gaps, (c_{i+1}, c_{i+2}) being a big gap, and $\ell_i - \ell_{i+t+1} \ge 1$. Then $C' = [c_1, ..., c_{i-1}, c_i + 1, c_{i+1} + 1, c_{i+2}, ..., c_{2t}]$ is a placement with $f(C') \ge f(C)$. If furthermore $\ell_i = \ell_{i+t+1} + 1$, then $f(C') = f(C)$ and C' satisfies $\ell'_i - \ell'_{i+t+1} = -1$.*

Proof. Note that since (c_{i+1}, c_{i+2}) is a big gap, C' exists. Similar to the proof of Lemma 3.5, it can be calculated that

$$f(C') - f(C) = [\ell_i - \ell_{i+t+1} - 1] \cdot [1 + d(c_{i+t}, c_{i+t+1})].$$

Then it follows from $\ell_i - \ell_{i+t+1} \ge 1$ that $f(C') \ge f(C)$. If furthermore, $\ell_i = \ell_{i+t+1} + 1$, then $\ell'_i - \ell'_{i+t+1} = d(c'_i, c_{i+t}) - d(c_{i+t+1}, c'_{i+1}) = \ell_i - \ell_{i+t+1} - 2 = -1$. □

By Lemma 3.8 and a flipping use of it, we have

Corollary 3.9. *Let $C = [c_1, c_2..., c_{2t}]$ be an optimal placement with $(c_i, c_{i+1}), (c_{i+t-1}, c_{i+t}), (c_{i+t+1}, c_{i+t+2})$ being small gaps. If both (c_{i+1}, c_{i+2}) and (c_{i-1}, c_i) are big gaps, then $|\ell_i - \ell_{i+t+1}| \le 1$, and in the case $|\ell_i - \ell_{i+t+1}| = 1$, there is an optimal placement C' with $\ell'_i = \ell'_{i+t+1} - 1$, also an optimal placement C'' with $\ell''_i = \ell''_{i+t+1} + 1$.*

Lemma 3.10. *Let $C = [c_1, c_2..., c_{2t}]$ be an optimal placement with (c_{i-1}, c_{i+1}) being a 1-interval, then $|\ell_i - \ell_{i+t}| \le 1$.*

Proof. This result is an improvement on Corollary 3.7 since there is no 'big gap' requirement here.

Suppose, without loss of generality, that $\ell_i - \ell_{i+t} \ge 2$. We shall derive a contradiction. By Corollary 3.7 (a), (c_{i+t-1}, c_{i+t}) is a small gap. So $\ell_{i-1} - \ell_{i+t-1} = \ell_i - \ell_{i+t} \ge 2$. If (c_{i-2}, c_{i-1}) is a big gap, then by Lemma 3.4, (c_{i+t-2}, c_{i+t-1}) is a small gap. Hence $\ell_{i-2} - \ell_{i+t-2} \ge \ell_{i-1} - \ell_{i+t-1} \ge 2$. If furthermore (c_{i+t-3}, c_{i+t-2}) is a small gap, then by taking the place of c_i by c_{i+t-2} in Lemma 3.5 (with a flipping), we have a better placement than C, a contradiction. So (c_{i+t-3}, c_{i+t-2}) is a big gap, and thus (c_{i-3}, c_{i-2}) is a small gap. Taking the place of c_i by c_{i+t-2} in Corollary 3.9 (with a flipping), it follows from the optimality of C that $\ell_{i-1} - \ell_{i+t-2} \le 1$. On the other hand, since $\ell_{i+t-2} = \ell_{i+t-1} + 1 - d(c_{i-2}, c_{i-1}) < \ell_{i+t-1}$, we have $\ell_{i-1} - \ell_{i+t-2} > \ell_{i-1} - \ell_{i+t-1} \ge 2$, a contradiction. So (c_{i-2}, c_{i-1}) is a small gap. Taking the place of c_i by c_{i-1} in the above deduction, we see that (c_{i-3}, c_{i-2}) is also a small gap. This procedure continues infinitely, which contradicts the finiteness of $R(V, L)$. □

Corollary 3.11. *Let $C = [c_1, c_2..., c_{2t}]$ be an optimal placement. If (c_{i-1}, c_{i+1}) is a 1-interval, then $\ell_i = n/2$ when n is even, and $\ell_i = (n-1)/2$ or $(n+1)/2$ when n is odd. Furthermore, in the case that n is odd, there exists an optimal placement C' with $\ell'_i = (n-1)/2$, also an optimal placement C'' with $\ell''_i = (n+1)/2$.*

Proof. The first part follows from Lemma 3.10 and the observation that $\ell_i + \ell_{i+t} = n$. The second part is the result of Lemma 3.5. □

Lemma 3.12. *Let $C = [c_1, c_2..., c_{2t}]$ be an optimal placement, and (c_p, c_q) be a max-1-interval. Then*

(a) $|\ell_p - \ell_{q+t}| \le 1$.

(b) In the case $|\ell_p - \ell_{q+t}| = 1$, there is an optimal placement C' with $\ell'_p = \ell'_{q+t} - 1$, also an optimal placement C'' with $\ell''_p = \ell''_{q+t} + 1$.

Proof. If $q = p + 1$, the result is true by Corollary 3.9. So, suppose $q > p+1$ in the following. For $i = p+1, ..., q-1$, we have $\ell_i = n/2$ when n is even, and $\ell_i = (n-1)/2$ or $(n+1)/2$ when n is odd. Since (c_{p-1}, c_p) and (c_q, c_{q+1}) are both big gaps, (c_{p+t-1}, c_{p+t}) and (c_{q+t}, c_{q+t+1}) are small gaps. For simplicity of notation, denote by $a = d(c_{q+t-1}, c_{q+t})$ and $b = d(c_{p+t}, c_{p+t+1})$. First, we consider the case that $a > 1$ and $b > 1$.

Let $C' = [c_1, ..., c_{p-1},\ c_p + 1, c_{p+1} + 1, ..., c_q + 1, c_{q+1}, ..., c_{p+t}, c_{p+t+1} + 1, ..., c_{q+t-1} + 1, c_{q+t}, ..., c_{2t}]$ be a new placement. Note that such a placement is feasible since (c_q, c_{q+1}) is a big gap and we have assumed $d(c_{q+t-1}, c_{q+t}) = a > 1$. Similar to the proof of Lemma 3.5, we have

$$\begin{aligned} & f(C') - f(C) \\ = \ & [\ell_{q-1} - \ell_{p+t+1} - 1] + a - b \\ & +[\ell_{p+1} - \ell_{p+t+1} - 1]b - b(b-1) \\ & +[\ell_{q-1} + 1 - \ell_{q+t-1}](a-1) + (a-2)(a-1). \end{aligned}$$

In the case n is even, we have $\ell_{q-1} = \ell_{q+t-1} = \ell_{p+1} = \ell_{p+t+1} = n/2$. Thus

$$f(C') - f(C) = a(a-1) - b(b+1).$$

By $f(C') \le f(C)$, we have $a \le b + 1$ and equality holds if and only if $a = b + 1$. Symmetrically, by considering $C'' = [c_1, ..., c_{p-1},\ c_p - 1, c_{p+1} - 1, ..., c_q - 1, c_{q+1}, ..., c_{p+t}, c_{p+t+1} - 1, ..., c_{q+t-1} - 1, c_{q+t}, ..., c_{2t}]$ (note that (c_{p-1}, c_p) being a big gap and $d(c_{p+t}, c_{p+t+1}) =$

$b > 1$ guarantees that such a placement is feasible), we have $b \le a+1$ and equality holds if and only if $b = a+1$. So, $|a-b| \le 1$. Since $\ell_p = \ell_{p+1} + 1 - b$ and $\ell_{q+t} = \ell_{q+t-1} + 1 - a$, we have $\ell_p - \ell_{q+t} = a - b$. Thus $|\ell_p - \ell_{q+t}| \le 1$. Furthermore, if $\ell_p - \ell_{q+t} = 1$, then $a = b+1$ and C' is an optimal placement with $\ell'_p = \ell'_{q+t} - 1$; if $\ell_p - \ell_{q+t} = -1$, then $b = a+1$ and C'' is an optimal placement with $\ell''_p = \ell''_{q+t} + 1$.

In the case n is odd, there are three cases to consider.

(1) $\ell_{p+1} = \ell_{q-1} = (n+1)/2$. Then $f(C') - f(C) = a^2 - b^2$. It follows from the optimality of C that $a \le b$, and equality holds if and only if $a = b$.

(2) $\ell_{p+1} = \ell_{q-1} = (n-1)/2$. Then $f(C') - f(C) = a(a-2) - b(b+2)$. It follows that $a \le b+2$, and equality holds if and only if $a = b+2$.

(3) $\ell_{p+1} = (n-1)/2$ and $\ell_{q-1} = (n+1)/2$. Then $f(C') - f(C) = (a-1)(a+1) - b(b+2)$. It follows that $a \le b+1$ and equality holds if and only if $a = b+1$.

By analyzing the placement C'' symmetrically, we see that $b-2 \le a \le b$ in the first case, $a-2 \le b \le a$ in the second case, and $b-1 \le a \le b+1$ in the third case. Hence, $\ell_p - \ell_{q+t} = \ell_{p+1} - \ell_{q+t-1} + a - b \in [-1, 1]$ in any case. Furthermore, if $\ell_p - \ell_{q+t} = 1$, then $a = b$ in the first case, $a = b+2$ in the second case, and $a = b+1$ in the third case. In any case, C' is also optimal. By a symmetric analysis, if $\ell_p - \ell_{q+t} = -1$, then C'' is optimal.

It can be seen from the above analysis that if $|\ell_p - \ell_{q+t}| = 1$, then either C or C' is an optimal placement satisfying $\ell_p = \ell_{q+t} - 1$, either C or C'' is an optimal placement satisfying $\ell_p = \ell_{q+t} + 1$

Next, suppose $a > 1$ but $b = 1$. In this case, C'' does not exist, but the analysis for C' is still valid. Furthermore, $\ell_p = \ell_{p+1}$. So, when n is even, we have $1 < a \le b+1 = 2$, which implies that $\ell_p - \ell_{q+t} = a - b = 1$ and C' is also optimal. When n is odd, case (1) does not occur. If case (2) occurs, then $2 \le a \le b+2$, and thus $\ell_p - \ell_{q+t} = -1 + a - b \in \{0, 1\}$. Furthermore, when $\ell_p - \ell_{q+t} = 1$, we have $a = b+2$ and C' is also optimal. If case (3) occurs, then it follows from $2 \le a \le b+1$ that $a = b+1 = 2$. Hence, $\ell_p - \ell_{q+t} = a - b = 1$, and C' is also optimal.

The case $b > 1$ but $a = 1$ can be analyzed symmetrically, and C'' is also an optimal placement.

Now, suppose $a = b = 1$. Since (c_{p-1}, c_p) and (c_q, c_{q+1}) are big gaps, we see that (c_{q+t}, c_{q+t+1}) and (c_{p+t-1}, c_{p+t}) are small gaps. If n is even, then $\ell_p = \ell_{q+t} = n/2$, and we are done. So, suppose n is odd. Then ℓ_p and ℓ_{q+t} take the values of $(n-1)/2$ or $(n+1)/2$, and thus $|\ell_p - \ell_{q+t}| \le 1$. Furthermore, if $\ell_p - \ell_{q+t} = 1$, then $\ell_p = (n+1)/2$ and $\ell_{q+t} = (n-1)/2$. It follows that $\ell_i = (n+1)/2$ for $i = p, p+1, ..., q$, and thus $d(c_j, c_{j+1}) = 1$ for $j = p+t, p+t+1, ..., q+t-1$. Furthermore, since (c_{p-1}, c_p) and (c_q, c_{q+1}) are big gaps, we have $d(c_j, c_{j+1}) = 1$ also holds for $j = p+t-1$ and $q+t$. But then (c_{p+t}, c_{q+t}) is properly contained in a 1-interval (c_{p+t-1}, c_{q+t+1}), contradicting that (c_p, c_q) is a max-1-interval. $\square$

In proving the following lemma, our notation of ℓ's is a little different from before, for the sake of easier statement.

Lemma 3.13. *Let $\{(c_{i_1}, c_{i_2}), ..., (c_{i_{2k-1}}, c_{i_{2k}})\}$ be the set of all max-1-intervals of C. For $j = 1, 2, ..., k$, denote by $\ell^{(l)}_j = d(c_{i_{2j-1}}, c_{i_{2j-1}+t})$ and $\ell^{(r)}_j = d(c_{i_{2j}+t}, c_{i_{2j}})$. Then elements in $\mathcal{L} = \{\ell^{(l)}_j, \ell^{(r)}_j\}_{j=1,...,k}$ differ by at most 1.*

Proof. We shall show that there exists an optimal placement $\tilde{C}$ such that elements in $\tilde{\mathcal{L}}$ differ by at most 1, and $\tilde{\mathcal{L}} = \mathcal{L}$ setwisely. The placement $\tilde{C}$ is obtained by applying adjustments as in Lemma 3.12 step by step to the sequence $(c_{p_1}, c_{q_1}), (c_{p_2}, c_{q_2}), ..., (c_{p_s}, c_{q_s})$ of max-1-intervals constructed as follows (see Fig. 2).

(1) Let (c_{p_1}, c_{q_1}) be a max-1-interval. Furthermore, if there exists an index j such that $\ell^{(l)}_j \ne \ell^{(r)}_j$, suppose, without loss of generality, that $\ell^{(l)}_j < \ell^{(r)}_j$, then choose p_1 to be such a j.

(2) For $i = 2, 3, ...$, let (c_{p_i}, c_{q_i}) be a max-1- interval containing $c_{p_{i-1}+t}$.

It should be observed that

(a) Since $(c_{p_{i-1}-1}, c_{p_{i-1}})$ is a big gap, then $(c_{p_{i-1}+t-1}, c_{p_{i-1}+t})$ is a small gap, and thus $p_i < p_{i-1} + t$.

(b) Because $(c_{p_{i-1}}, c_{q_{i-1}})$ is a max-1-gap, we have $q_i < q_{i-1} + t - 1$ (otherwise (c_{p_i+t}, c_{q_i+t}) would be properly contained in the 1-interval $(c_{p_i+t-1}, c_{q_i+t+1})$). So, (2) is realizable since now a maximal 1-interval containing $c_{p_{i-1}+t}$ must be a max-1-interval.

(c) Since (c_{q_1}, c_{q_1+1}) is a big gap, this procedure must stop with some s such that $c_{q_1+t} \in (c_{p_s}, c_{q_s})$.

(d) $\{(c_{p_1}, c_{q_1}), (c_{p_2}, c_{q_2}), ..., (c_{p_s}, c_{q_s})\}$ consists all max-1-intervals (so $s = k$).

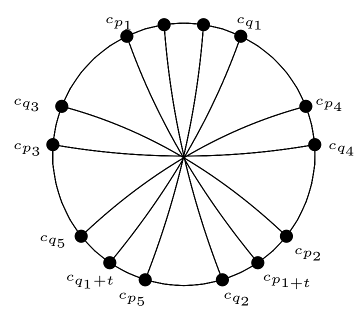


Figure 2. An illustration of how to construct the sequence.

In the following, we change the notation ℓ's to $\ell^{(l)}_j = d(c_{p_j}, c_{p_j+t})$ and $\ell^{(r)}_j = d(c_{q_j+t}, c_{q_j})$. First, note that for any $j = 1, 2, ..., s$, $|\ell^{(l)}_j - \ell^{(r)}_j| \le 1$ by Lemma 3.12. Furthermore, since the 1-intervals under consideration are all max-1-intervals, we see that $\ell^{(l)}_j = \ell^{(r)}_{j+1}$.

Suppose, without loss of generality that $\ell_1^{(l)} \leq \ell_1^{(r)}$. We do as follows. For $j = 1, 2, ...s$, if $\ell_{j+1}^{(l)} \leq \ell_{j+1}^{(r)}$ (then $\ell_{j+1}^{(l)} \leq \ell_{j+1}^{(r)} = \ell_j^{(l)}$), we move on to the next j. If $\ell_{j+1}^{(l)} > \ell_{j+1}^{(r)}$, then by Lemma 3.12, there is an optimal placement with $\tilde{\ell}_{j+1}^{(l)} = \tilde{\ell}_{j+1}^{(r)} - 1$. In this case, $\tilde{\ell}_{j+1}^{(r)} = \tilde{\ell}_j^{(l)} = \ell_j^{(l)} + 1$ and $\tilde{\ell}_{j+1}^{(l)} = \tilde{\ell}_{j+1}^{(r)} - 1 = \ell_j^{(l)}$. Note that in the next step, $\tilde{\ell}_{j+1}^{(l)}$ may increase, but by at most 1, and once for all. The idea in the above process is to make the final sequence $(\tilde{\ell}_1^{(l)}, \tilde{\ell}_2^{(l)}, ..., \tilde{\ell}_s^{(l)})$ a 'nearly' decreasing one, that is, for any $j = 1, 2, ..., s-1$, $\tilde{\ell}_m^{(l)} \leq \tilde{\ell}_j^{(l)} + 1$ holds for any $m \geq j+1$. In fact, if in the final placement, there is an index j such that $\tilde{\ell}_{j+1}^{(l)} > \tilde{\ell}_j^{(l)}$, then $\tilde{\ell}_{j+1}^{(l)}$ is obtained by being increased by 1 in the $(j+2)$'s step, and thus $\tilde{\ell}_{j+1}^{(l)} = \tilde{\ell}_j^{(l)} + 1$ and $\tilde{\ell}_{j+2}^{(l)} = \tilde{\ell}_{j+1}^{(l)} - 1 = \tilde{\ell}_j^{(l)}$. So, even $\tilde{\ell}_{j+2}^{(l)}$ is increased in the $(j+3)$'s step, it can not exceed $\tilde{\ell}_j^{(l)} + 1$. As a result of the above 'nearly monotonicity', $\tilde{\ell}_j^{(l)} \leq \ell_1^{(l)} + 1$ and $\tilde{\ell}_s^{(l)} \leq \ell_j^{(l)}$ for any $j = 1, 2, ..., s-1$ ($\tilde{\ell}_s^{(l)}$ can not be increased since it is the last step). Note that $\tilde{\ell}_1^{(r)} = \tilde{\ell}_s^{(l)}$, and $\tilde{\ell}_1^{(r)} \geq \ell_1^{(r)} - 1$ (since $\ell_1^{(l)} \leq \ell_1^{(r)}$ and $\ell_1^{(l)}$ can be increased in the adjustment by at most once), we see that in the case $\ell_1^{(l)} < \ell_1^{(r)}$, $\tilde{\ell}_s^{(l)} \leq \tilde{\ell}_j^{(l)} \leq \ell_1^{(l)} + 1 \leq \ell_1^{(r)} \leq \tilde{\ell}_1^{(r)} + 1 = \tilde{\ell}_s^{(l)} + 1$ for any $j = 1, 2, ..., s-1$. In the case $\ell_1^{(l)} = \ell_1^{(r)}$, by the choice of (c_{p_1}, c_{q_2}), we see that $\ell_j^{(l)} = \ell_j^{(r)}$ holds for any j. By the observation $\ell_j^{(l)} = \ell_{j+1}^{(r)}$, we see that all elements in $\mathcal{L}$ have the same value. In any case, elements in $\{\tilde{\ell}_1^{(l)}, \tilde{\ell}_2^{(l)}, ..., \tilde{\ell}_s^{(l)}\}$ differ by at most 1. A careful review of the above process shows that the values of $\{\tilde{\ell}_1^{(r)}, \tilde{\ell}_2^{(r)}, ..., \tilde{\ell}_s^{(r)}\}$ also fall into $\{\tilde{\ell}_s^{(l)}, \tilde{\ell}_s^{(l)} + 1\}$. Furthermore, in the above process, we merely exchanges some consecutive values of ℓ's, and hence $\tilde{\mathcal{L}} = \mathcal{L}$. □

Call the values in $\mathcal{L}$ the *essential-bound-values*.

Lemma 3.14. *Let $C = [c_1, c_2..., c_{2t}]$ be an optimal placement, $n > 4t$ for even n and $n > 5t$ for odd n. Then the essential-bound-value $a < n/2 - 1$ when n is even and $a < (n-1)/2 - 1$ when n is odd.*

Proof. Let a sequence of 1-intervals $(c_{p_1}, c_{q_1}), (c_{p_2}, c_{q_2})$, ..., (c_{p_s}, c_{q_s}) be constructed as follows: Let (c_p, c_q) be a max-1-interval. Set $p_1 = p$, and q_1 the first replica in (c_p, c_q) with $d(c_{q_1+t-1}, c_{q_1+t}) > 1$ (note that such q_1 exists since (c_p, c_q) is a max-1-interval). For $j = 2, 3, ...$, let (c_{p_j}, c_{q_j}) be a 1-interval with $q_j = p_{j-1} + t$ and $d(c_{p_j - 1}, c_{p_j}) > 1$. An illustration of the construction is shown in Fig 3. Note that this construction is different from that in Lemma 3.13, (c_{p_j}, c_{q_j}) is not required to be a max-1-interval. But it is easy to see that $d(c_{p_j}, c_{p_j+t})$'s and $d(c_{q_j+t}, c_{q_j})$'s are also a or $a+1$. Furthermore, $p_s = q_1 + t$ since $d(c_{q_1+t-1}, c_{q_1+t}) > 1$. Denote by $N_j = \{c_{p_j}, ..., c_{q_j}, c_{p_j+t} + 1, ..., c_{q_j+t} - 1\}$. Then N_j's are mutually disjoint and $\bigcup_{j=1}^{s} N_j = V$.

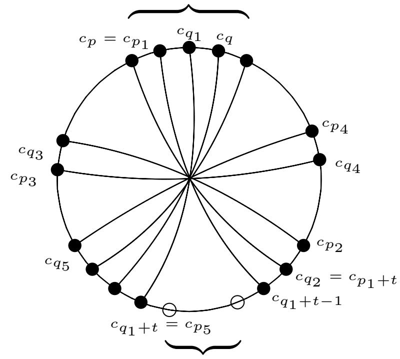


Figure 3. The locations in the braces constitute N_1.

If $a \geq n/2 - 1$ when n is even and $a \geq (n-1)/2 - 1$ when n is odd, then for each j, N_j contains at most 2 non-replicas when n is even, and at most 3 non-replicas when n is odd. In the case n is even, the number of replicas $\geq |N_j|/2$ for each j. So, $2t \geq n/2$. When n is odd, the number of replicas $\geq 2|N_j|/5$ for each j. So, $2t \geq 2n/5$. These contradict our order assumption. □

Corollary 3.15. *Let $C = [c_1, c_2..., c_{2t}]$ be an optimal placement with $n > 4t$ for even n, and $n > 5t$ for odd n. Then, any max-1-interval has at most 3 replicas.*

Proof. Suppose (c_p, c_q) is a max-1-interval containing at least 4 replicas. By corollary 3.11 and Lemma 3.5, we may assume that $\ell_{p+1} = \ell_{p+2} = n/2$ when n is even and $\ell_{p+1} = \ell_{p+2} = (n-1)/2$ when n is odd. It follows that (c_{p+t+1}, c_{p+t+2}) is a small gap. Combining this with the observation that (c_p, c_{p+1}) and (c_{p+t-1}, c_{p+t}) are both small gaps, it follows from Lemma 3.8 that $\ell_{p+t+1} \leq \ell_p + 1$. In the case that n is odd, since $\ell_{p+t+1} = n - \ell_{p+1} = (n+1)/2$, we have $\ell_p \geq (n-1)/2$. So, $a \geq (n-1)/2 - 1$, and thus $n \leq 5t$ by Lemma 3.14, a contradiction. The case when n is even can be shown similarly. □

Call a max-1-interval with 3 replicas a *big-1-interval*, and a max-1-interval with 2 replicas a *small-1-interval*. A replica which is the intersection of two big gaps is called a *singleton*.

For an edge $e \in L$, the *contribution of* e *to* $f(C)$ is

$$ctr(e) = \sum_{F=\{e,g\} \subset L} N_C(F).$$

For a gap (c_i, c_{i+1}), its *contribution* to $f(C)$ is

$$ctr(c_i, c_{i+1}) = \sum_{e \in (c_i, c_{i+1})} ctr(e).$$

Lemma 3.16. *Let $C = [c_1, c_2..., c_{2t}]$ be a placement. For each gap (c_{i-1}, c_i), its contribution to $f(C)$ only depends on ℓ_{i-1} and ℓ_i.*

Proof. Write $a = d(c_{i-1}, c_i)$. If (c_{i-1}, c_i) is a small gap, then $e = (c_{i-1}, c_i)$ is an edge, and

$$ctr(e) = [(2n - \ell_{i-1})(\ell_{i-1} - 1) + (n + \ell_i)(n - 1 - \ell_i)]/2.$$

Next, suppose (c_{i-1}, c_i) is a big gap. Let $e_j = (j, j+1) \in (c_{i-1}, c_i)$. Then

$$\begin{aligned} ctr(e_j) = \; & [(2n - \ell_{i-1} + k_1)(\ell_{i-1} - k_1 - 1) \\ & +(n + \ell_i + k_2)(n - \ell_i - k_2 - 1)]/2, \end{aligned}$$

where $k_1 = d(c_{i-1}, j)$ and $k_2 = d(j+1, c_i) = a - 1 - k_1$. Then

$$\begin{aligned} & ctr(c_{i-1}, c_i) \\ = \; & \frac{a}{2}[(n + \ell_i)(n - 1 - \ell_i) + (2n - \ell_{i-1})(\ell_{i-1} - 1) \\ & +(2\ell_{i-1} - 2\ell_i - 1 - 2n)(a - 1) - \frac{(a-1)(2a-1)}{3}] \end{aligned}$$

Note that $a = \ell_{i-1} - \ell_i + 1$, we are done. □

Denote by $r_i = d(c_{i+t}, c_i)$. By symmetry, it is easy to see the following corollary:

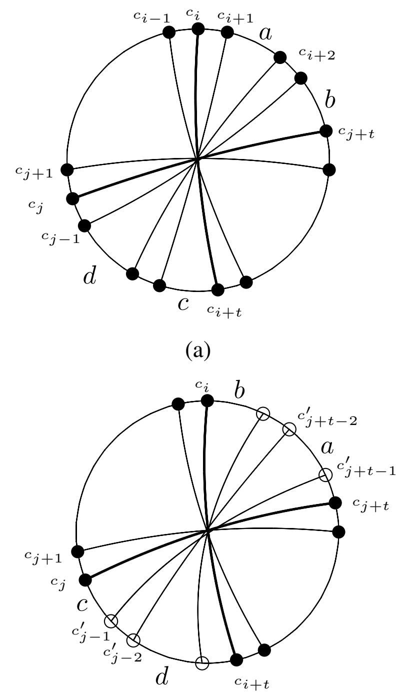
Corollary 3.17. *Let $C = [c_1, c_2..., c_{2t}]$ be a placement, $(c_{i-1}, c_i), (c_{j-1}, c_j)$ be two gaps. If $\ell_i = r_{j-1}$ and $\ell_{i-1} = r_j$, then $ctr(c_{i-1}, c_i) = ctr(c_{j-1}, c_j)$.*

Lemma 3.18. *Let $C = [c_1, c_2..., c_{2t}]$ be an optimal placement, $n > 4t$ for even n and $n > 5t$ for odd n. Then there are at most one big-1-interval in C.*

Proof. Suppose this is not true. Let (c_{i-1}, c_{i+1}) and (c_{j-1}, c_{j+1}) be two big-1-intervals such that c_{j+t} is a singleton nearest to c_{i+1}. By Corollary 3.11 and Lemma 3.12, we may assume that $\ell_i = r_{j+t}$. By 'flipping' the placement in the interval $(c_i, c_{j+t}) \cup (c_{i+t}, c_j)$ as shown in Fig. 4, we obtain a placement C' (the literals a, b, c, d in the figure indicates that the corresponding lengths are the same). For each gap in $(c_i, c_{j+t}) \cup (c_{i+t}, c_j)$, the bound values exchange their 'left' and 'right' positions in C'. As to other gaps outside of $(c_i, c_{j+t}) \cup (c_{i+t}, c_j)$, their contributions are clearly kept. Since

$$f(C) = \frac{1}{2}\sum_{i=1}^{2t} ctr(c_i, c_{i+1}),$$

we see that $f(C') = f(C)$, and thus C' is also an optimal placement. In C', there is a small-1-interval (c'_{j+t-1}, c_{j+t}) with a bound value $\ell_{j+t} \geq (n-1)/2$. So the essential bound value $a \geq (n-1)/2 - 1$, which leads to a contradiction to Lemma 3.14. □



(a)

(b)

Figure 4. (b) is obtained from (a) by flipping locations in $(c_i, c_{j+t}) \cup (c_{i+t}, c_j)$.

Corollary 3.19. *Let $C = [c_1, c_2..., c_{2t}]$ be an optimal placement, $n > 4t$ for even n and $n > 5t$ for odd n. When t is odd, every 1-interval is small. When t is even, there is exactly one big-1-interval.*

Proof. Let $(c_{p_1}, c_{q_1}), (c_{p_2}, c_{q_2}), ..., (c_{p_s}, c_{q_s})$ be a sequence of max-1-intervals constructed as in Lemma 3.13. Note that if there is a big-1-interval, then a placement as described by the above lemmas is feasible only when t is even. The result follows. □

Proof of Theorem 3.1

To prove the necessity, we see from Lemma 3.13 and Corollary 3.19 that it is sufficient to show that the essential-bound-value $a = \lfloor T/t \rfloor$ when t is odd and $a = \lfloor T/(t-1) \rfloor$ when t is even. Suppose the pairing is as described in Theorem 3.1.

First, consider the case that t is odd, write $t = 2m - 1$. Each edge in a small gap is covered by the set of intervals $I = \{(c_1, c_{t+1}), (c_3, c_{t+3}), ..., (c_{2t-1}, c_{t+2t-1})\}$ exactly m times. For each edge in a big gap, I covers it exactly $m-1$ times. So

$$\ell_1 + \ell_3 + ... + \ell_{2t-1} = (m-1)n + t = T.$$

Suppose that the remainder of T/t is x. Then by Lemma 3.13, among the t integers $\ell_1, \ell_3, ..., \ell_{2t-1}$, x of them equal to $a+1$, and $t - x$ of them equal to a.

Next, suppose t is even, write $t = 2m$. Each edge in a small gap is covered by the set of intervals $I' = \{(c_3, c_{t+3}), (c_5, c_{t+5}), ..., (c_{t-1}, c_{2t-1}), (c_{t+2}, c_2), ..., (c_{2t}, c_t)\}$ exactly m times. For each edge in a big gap, I' covers it exactly $m-1$ times. So

$$\ell_3+\ell_5+...+\ell_{t-1}+\ell_{t+2}+...+\ell_{2t-2}+\ell_{2t} = (m-1)n+t = T.$$

Suppose that the remainder of $T/(t-1)$ is x. Then among the $t-1$ integers $\ell_3, \ell_5, ..., \ell_{t-1}, \ell_{t+2}, ..., \ell_{2t-2}, \ell_{2t}$, x of them equal to $a+1$, and $t-2-x$ of them equal to a.

In the following, we show the sufficiency. The idea is to show that for any placement C satisfying the conditions in Theorem 3.1, $f(C)$ is completely determined by the parameters n and t, which implies that every placement C satisfying the conditions in Theorem 3.1 has the same objective function $f(C)$, and thus the sufficiency follows from the necessity.

First, suppose t is odd. Consider the sequence $S = (\ell_1, \ell_{1+(t-1)}, \ell_{1+2(t-1)}, ..., \ell_{1+(t-1)^2})$. Then each of $\ell_1, \ell_3, ..., \ell_{2t-1}$ appears exactly once in S. By the above analysis, the number of a's and the number of $(a+1)$'s are the same for any placement satisfying condition (a2).

Suppose, without loss of generality, that the number of a's is smaller than that of $a+1$'s. We are to modify C to a 'standard' placement C' with $f(C') = f(C)$, where the term standard means that there is no consecutive a's in the sequence S. The modification can be done by induction on the number of consecutive a's. First, suppose there exists an index i such that $\ell_{1+(i-1)(t-1)} = \ell_{1+i(t-1)} = a$ and $\ell_{1+(i+1)(t-1)} = \ell_{1+(i+2)(t-1)} = a+1$, that is, there is a sub-sequence of S with the form $(a, a, a+1, a+1)$. By Lemma 3.12, we can change C to C' with $f(C') = f(C)$ and $\ell'_{1+i(t-1)} = a+1$, $\ell'_{1+(i+1)(t-1)} = a$. Since $\ell_{1+(i+2)(t-1)} = a+1$, such an adjustment decreases the number of consecutive a's. Symmetrically, if there exists a sub-sequence of S with the form $(a+1, a+1, a, a)$, we can also decrease the number of consecutive a's. Since the number of $(a+1)$'s is at least that of a, we see that one of the above two cases must occur.

In C', the bound values of each gap is either $(a, a+1)$, or $(a+1, a+1)$. Furthermore, the number of $(a, a+1)$'s equals to $t-x$ (the number of a's in C'). By Lemma 3.16 and Corollary 3.17, $f(C')$ is completely determined by n and t, and thus so is $f(C)$.

The case that t is even can be considered similarly. $\square$

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