Supplementary Material

In this supplementary material, we show how the covariance matrix can be efficiently initialized and updated from merging and swapping operations.

1 Initialization of Covariance Matrix for a Triangle

Given a triangle $\mathbf{T}$ of the input mesh with three vertices $\mathbf{v}_i, i = 1, 2, 3$, the covariance matrix $\mathbf{U}(\mathbf{T})$ can be efficiently represented using the coordinates of three vertices. We denote the centroid of $\mathbf{T}$ as $\mathbf{x}_c : \mathbf{x}_c = \frac{1}{3}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)$. $\mathbf{U}(\mathbf{T})$ can be represented:

$$\begin{aligned}\mathbf{U}(\mathbf{T}) &= \iint\limits_{\mathbf{p}\in\mathbf{T}} (\mathbf{p} - \mathbf{x}_c)(\mathbf{p} - \mathbf{x}_c)^\top \, d\mathbf{p} \\ &= \iint\limits_{\mathbf{p}\in\mathbf{T}} \mathbf{p}\mathbf{p}^\top \, d\mathbf{p} - A \cdot \mathbf{x}_c\mathbf{x}_c^\top,\end{aligned}$$

where A is the area of triangle $\mathbf{T}$:

$$A = \frac{1}{2} \left|\left|(\mathbf{v}_2 - \mathbf{v}_1) \times (\mathbf{v}_3 - \mathbf{v}_1)\right|\right| .$$

$\mathbf{p}$ can be parameterized to compute $\iint\limits_{\mathbf{p}\in\mathbf{T}} \mathbf{p}\mathbf{p}^\top \, d\mathbf{p}$:

$$\mathbf{p} = \mathbf{v}_1 + s(\mathbf{v}_2 - \mathbf{v}_1) + t(\mathbf{v}_3 - \mathbf{v}_1) \tag{1}$$

where $0 < s, t, s + t < 1$ must be satisfied. We denote this parameterization domain (i.e., a right triangle) as $\mathbf{D}$.

Thus,

$$\begin{aligned}\iint\limits_{\mathbf{p}\in\mathbf{T}} \mathbf{p}\mathbf{p}^\top \, d\mathbf{p} &= \iint\limits_{\mathbf{D}} \mathbf{p}\mathbf{p}^\top \left|\left|\frac{dr}{ds} \times \frac{dr}{dt}\right|\right| \, dsdt \\ &= \iint\limits_{\mathbf{D}} \mathbf{p}\mathbf{p}^\top \left|\left|(\mathbf{v}_2 - \mathbf{v}_1) \times (\mathbf{v}_3 - \mathbf{v}_1)\right|\right| \, dsdt \\ &= 2A \iint\limits_{\mathbf{D}} \mathbf{p}\mathbf{p}^\top \, dsdt.\end{aligned} \tag{2}$$

Substituting Eq. (1) into Eq. (2), we can get:

$$\begin{aligned}\iint\limits_{\mathbf{p}\in\mathbf{T}} \mathbf{p}\mathbf{p}^\top \, d\mathbf{p} = \frac{A}{6}(\mathbf{v}_1\mathbf{v}_1^\top + \mathbf{v}_2\mathbf{v}_2^\top + \mathbf{v}_3\mathbf{v}_3^\top) + \frac{A}{12}(\mathbf{v}_1\mathbf{v}_2^\top \\ + \mathbf{v}_1\mathbf{v}_3^\top + \mathbf{v}_2\mathbf{v}_3^\top + \mathbf{v}_2\mathbf{v}_1^\top + \mathbf{v}_3\mathbf{v}_1^\top + \mathbf{v}_3\mathbf{v}_2^\top).\end{aligned} \tag{3}$$

Note that :

$$A \cdot \mathbf{x}_c\mathbf{x}_c^\top = \frac{A}{9}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)^\top. \tag{4}$$

Combining Eq. (3) and Eq. (4), we have:

$$\mathbf{U}(\mathbf{T}) = \frac{A}{36} \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}^\top .$$

2 Efficient Update of Covariance Matrix for Merging Operation

We denote its surface area as $A(\mathbf{C}_i)$ and its centroid as $\mathbf{x}_c(\mathbf{C}_i)$:

$$A(\mathbf{C}_i) = \iint\limits_{\mathbf{p}\in\mathbf{C}_i} d\mathbf{p},$$

$$\mathbf{x}_c(\mathbf{C}_i) = \frac{\iint\limits_{\mathbf{p}\in\mathbf{C}_i} \mathbf{p} \, d\mathbf{p}}{A(\mathbf{C}_i)}.$$

Considering the merging operation $(\mathbf{C}_i, \mathbf{C}_j) \to \mathbf{C}_k$, it is obvious the surface area $A(\mathbf{C}_k)$ and the centroid $\mathbf{x}_c(\mathbf{C}_k)$ can be updated easily:

$$A(\mathbf{C}_k) = A(\mathbf{C}_i) + A(\mathbf{C}_j).$$

$$\mathbf{x}_c(\mathbf{C}_k) = \frac{A(\mathbf{C}_i)\mathbf{x}_c(\mathbf{C}_i) + A(\mathbf{C}_j)\mathbf{x}_c(\mathbf{C}_j)}{A(\mathbf{C}_i) + A(\mathbf{C}_j)}.$$

We consider the update of the covariance matrix $\mathbf{U}(\mathbf{C}_k)$, which is:

$$\mathbf{U}(\mathbf{C}_k) = \iint\limits_{\mathbf{p}\in\mathbf{C}_k} (\mathbf{p} - \mathbf{x}_c(\mathbf{C}_k))(\mathbf{p} - \mathbf{x}_c(\mathbf{C}_k))^\top \, d\mathbf{p}.$$

The integral domain $\mathbf{C}_k$ can be split into two parts : $\mathbf{C}_k = \mathbf{C}_i \cup \mathbf{C}_j$. For the first part $\mathbf{p} \in \mathbf{C}_i$, substituting $\mathbf{p} - \mathbf{x}_c(\mathbf{C}_k)$ with $\mathbf{p} - \mathbf{x}_c(\mathbf{C}_i) + \mathbf{x}_c(\mathbf{C}_i) - \mathbf{x}_c(\mathbf{C}_k)$, it becomes:

$$\begin{aligned}&\iint\limits_{\mathbf{p}\in\mathbf{C}_i} (\mathbf{p} - \mathbf{x}_c(\mathbf{C}_k))(\mathbf{p} - \mathbf{x}_c(\mathbf{C}_k))^\top \, d\mathbf{p} \\ &= \mathbf{U}(\mathbf{C}_i) + A(\mathbf{C}_i)(\mathbf{x}_c(\mathbf{C}_i) - \mathbf{x}_c(\mathbf{C}_k))(\mathbf{x}_c(\mathbf{C}_i) - \mathbf{x}_c(\mathbf{C}_k))^\top.\end{aligned}$$

The second part is alike, so we can update the covariance matrix $\mathbf{U}(\mathbf{C}_k)$ as:

$$\begin{aligned}\mathbf{U}(\mathbf{C}_k) = \mathbf{U}(\mathbf{C}_i) &+ \mathbf{U}(\mathbf{C}_j) \\ &+ A(\mathbf{C}_i)(\mathbf{x}_c(\mathbf{C}_i) - \mathbf{x}_c(\mathbf{C}_k))(\mathbf{x}_c(\mathbf{C}_i) - \mathbf{x}_c(\mathbf{C}_k))^\top \\ &+ A(\mathbf{C}_j)(\mathbf{x}_c(\mathbf{C}_j) - \mathbf{x}_c(\mathbf{C}_k))(\mathbf{x}_c(\mathbf{C}_j) - \mathbf{x}_c(\mathbf{C}_k))^\top.\end{aligned}$$

3 Efficient Update of Covariance Matrix for Swapping Operation

For a swapping operation which swaps a triangle face $\mathbf{T}$ from $\mathbf{C}_i$ to $\mathbf{C}_j$. Suppose after swapping, $\mathbf{C}_i$ becomes $\mathbf{C}_{i'}$ and $\mathbf{C}_j$ becomes $\mathbf{C}_{j'}$, i.e., $\mathbf{C}_i = \mathbf{C}_{i'} \cup \mathbf{T}$, and $\mathbf{C}_{j'} = \mathbf{C}_j \cup \mathbf{T}$.

It is obvious the surface areas and the centroids can be updated by:

$$A(\mathbf{C}_{i'}) = A(\mathbf{C}_i) - A(\mathbf{T}),$$

$$A(\mathbf{C}_{j'}) = A(\mathbf{C}_j) + A(\mathbf{T}),$$

$$\mathbf{x}_c(\mathbf{C}_{i'}) = \frac{A(\mathbf{C}_i)\mathbf{x}_c(\mathbf{C}_i) - A(\mathbf{T})\mathbf{x}_c(\mathbf{T})}{A(\mathbf{C}_i) - A(\mathbf{T})},$$

$$\mathbf{x}_c(\mathbf{C}_{j'}) = \frac{A(\mathbf{C}_j)\mathbf{x}_c(\mathbf{C}_j) + A(\mathbf{T})\mathbf{x}_c(\mathbf{T})}{A(\mathbf{C}_i) + A(\mathbf{T})}.$$

This swapping operation can be interpreted as two merging operations: $\mathbf{C}_i$ is the cluster merged from $\mathbf{C}_{i'}$ and $\mathbf{T}$, and $\mathbf{C}_{j'}$ is the cluster merged from $\mathbf{C}_j$ and $\mathbf{T}$:

$$\begin{aligned}
\mathbf{U}(\mathbf{C}_{i'}) &= \mathbf{U}(\mathbf{C}_i) - \mathbf{U}(\mathbf{T}) \\
&- A(\mathbf{C}_{i'})(\mathbf{x}_c(\mathbf{C}_{i'}) - \mathbf{x}_c(\mathbf{C}_i))(\mathbf{x}_c(\mathbf{C}_{i'}) - \mathbf{x}_c(\mathbf{C}_i))^\top \\
&- A(\mathbf{T})(\mathbf{x}_c(\mathbf{T}) - \mathbf{x}_c(\mathbf{C}_i))(\mathbf{x}_c(\mathbf{T}) - \mathbf{x}_c(\mathbf{C}_i))^\top. \\
\mathbf{U}(\mathbf{C}_{j'}) &= \mathbf{U}(\mathbf{C}_j) + \mathbf{U}(\mathbf{T}) \\
&+ A(\mathbf{C}_j)(\mathbf{x}_c(\mathbf{C}_j) - \mathbf{x}_c(\mathbf{C}_{j'}))(\mathbf{x}_c(\mathbf{C}_j) - \mathbf{x}_c(\mathbf{C}_{j'}))^\top \\
&+ A(\mathbf{T})(\mathbf{x}_c(\mathbf{T}) - \mathbf{x}_c(\mathbf{C}_{j'}))(\mathbf{x}_c(\mathbf{T}) - \mathbf{x}_c(\mathbf{C}_{j'}))^\top.
\end{aligned}$$