Supplementary Material

In this supplementary material, we show how the covariance matrix can be efficiently initialized and updated from merging and swapping operations.

1 Initialization of Covariance Matrix for a Triangle

Given a triangle $T$ of the input mesh with three vertices $v_i, i = 1, 2, 3$, the covariance matrix $U(T)$ can be efficiently represented using the coordinates of three vertices. We denote the centroid of $T$ as $x_c: x_c = \frac{1}{3}(v_1 + v_2 + v_3)$.

$U(T)$ can be represented:

$$U(T) = \int_{\mathcal{T}} (p - x_c)(p - x_c)^T dp$$

where $\mathcal{T}$ is the parameterization domain (i.e., a right triangle) as $D$.

$p$ can be parameterized to compute $\int_{\mathcal{T}} pp^T dp$:

$$p = v_1 + s(v_2 - v_1) + t(v_3 - v_1)$$

where $0 < s,t,s + t < 1$ must be satisfied. We denote this parameterization domain (i.e., a right triangle) as $D$.

Thus,

$$\int_{\mathcal{T}} pp^T dp = \int_D pp^T \frac{dr}{ds} \frac{dr}{dt} ds dt$$

$$= \int_D pp^T ||(v_2 - v_1) \times (v_3 - v_1)|| ds dt$$

$$= 2A \int_D pp^T ds dt.$$

Substituting Eq. (1) into Eq. (2), we get:

$$\int_{\mathcal{T}} pp^T dp = \frac{A}{6}(v_1v_1^T + v_2v_2^T + v_3v_3^T) + \frac{A}{12}(v_1v_2^T)$$

$$+ v_1v_3^T + v_2v_3^T + v_2v_1^T + v_3v_1^T + v_3v_2^T).$$

Note that:

$$A \cdot x_c x_c^T = \frac{A}{9}(v_1 + v_2 + v_3)(v_1 + v_2 + v_3)^T.$$

Combining Eq. (3) and Eq. (4), we have:

$$U(T) = \frac{A}{36}(v_1,v_2,v_3) \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}^T.$$ 

2 Efficient Update of Covariance Matrix for Merging Operation

We denote its surface area as $A(C_i)$ and its centroid as $x_c(C_i)$:

$$A(C_i) = \int_{C_i} dp,$$

$$x_c(C_i) = \frac{\int_{C_i} p dp}{\int_{C_i} dp}.$$

Considering the merging operation $(C_i, C_j) \rightarrow C_k$, it is obvious the surface area $A(C_k)$ and the centroid $x_c(C_k)$ can be updated easily:

$$A(C_k) = A(C_i) + A(C_j),$$

$$x_c(C_k) = \frac{A(C_i)x_c(C_i) + A(C_j)x_c(C_j)}{A(C_i) + A(C_j)}.$$

We consider the update of the covariance matrix $U(C_k)$, which is:

$$U(C_k) = \int_{C_k} (p - x_c(C_k))(p - x_c(C_k))^T dp.$$ 

The integral domain $C_k$ can be split into two parts: $C_k = C_i \cup C_j$. For the first part $p \in C_i$, substituting $p - x_c(C_k)$ with $p - x_c(C_i) + x_c(C_i) - x_c(C_k)$, it becomes:

$$\int_{p \in C_i} (p - x_c(C_k))(p - x_c(C_k))^T dp$$

$$= U(C_i) + A(C_i)(x_c(C_i) - x_c(C_k))(x_c(C_i) - x_c(C_k))^T.$$

The second part is alike, so we can update the covariance matrix $U(C_k)$ as:

$$U(C_k) = U(C_i) + U(C_j)$$

$$+ A(C_i)(x_c(C_i) - x_c(C_k))(x_c(C_i) - x_c(C_k))^T + A(C_j)(x_c(C_j) - x_c(C_k))(x_c(C_j) - x_c(C_k))^T.$$

3 Efficient Update of Covariance Matrix for Swapping Operation

For a swapping operation which swaps a triangle face $T$ from $C_i$ to $C_j$. Suppose after swapping, $C_i$ becomes $C_i'$ and $C_j$ becomes $C_j'$, i.e., $C_i = C_i' \cup T$, and $C_j = C_j' \cup T$.

It is obvious the surface areas and the centroids can be updated by:

$$A(C_i') = A(C_i) - A(T),$$

$$A(C_j') = A(C_j) + A(T).$$
\[ x_c(C_i') = \frac{A(C_i)x_c(C_i) - A(T)x_c(T)}{A(C_i) - A(T)}, \]
\[ x_c(C_j') = \frac{A(C_j)x_c(C_j) + A(T)x_c(T)}{A(C_j) + A(T)}. \]

This swapping operation can be interpreted as two merging operations: \( C_i \) is the cluster merged from \( C_i' \) and \( T \), and \( C_j' \) is the cluster merged from \( C_j \) and \( T \):

\[
U(C_i') = U(C_i) - U(T) - A(C_i')(x_c(C_i') - x_c(C_i))(x_c(C_i') - x_c(C_i))^T - A(T)(x_c(T) - x_c(C_i))(x_c(T) - x_c(C_i))^T.
\]
\[
U(C_j') = U(C_j) + U(T) + A(C_j)(x_c(C_j) - x_c(C_j'))(x_c(C_j) - x_c(C_j'))^T + A(T)(x_c(T) - x_c(C_j'))(x_c(T) - x_c(C_j'))^T.
\]