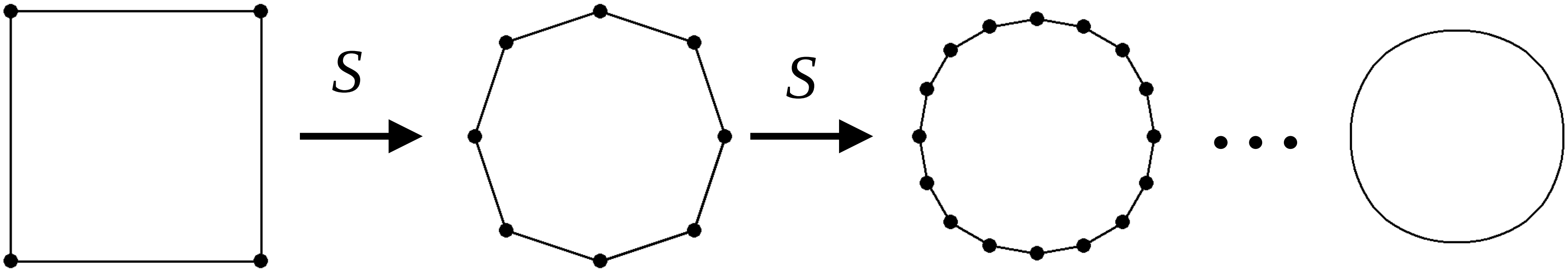
# Subdivision Curves

# What is subdivision?

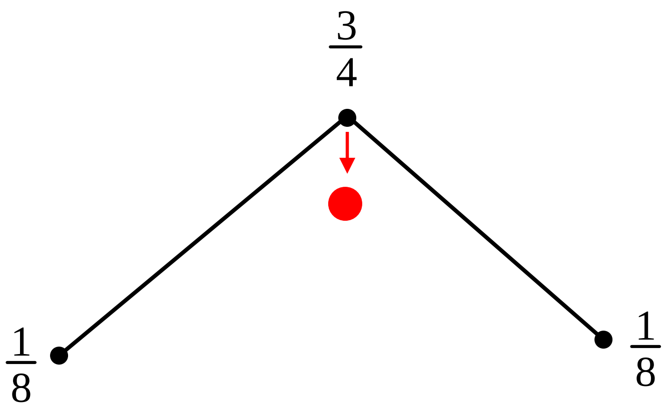
- Set of rules *S* that take a curve as input and produce a more highly refined curve as output
- Recursively applying *S* yields a sequence of curves which should converge to some limit shape



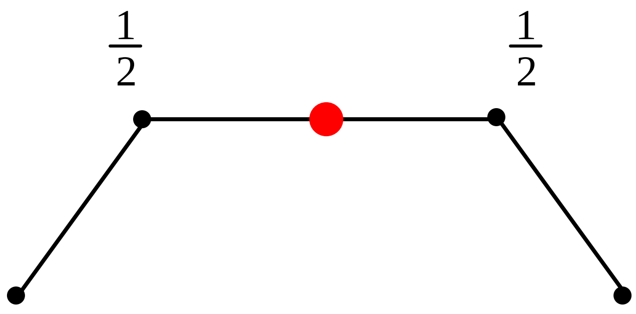
# Subdivision Rules

- Typically chosen to be linear combinations of neighboring vertices
- Rules usually depend only on local topology of shape

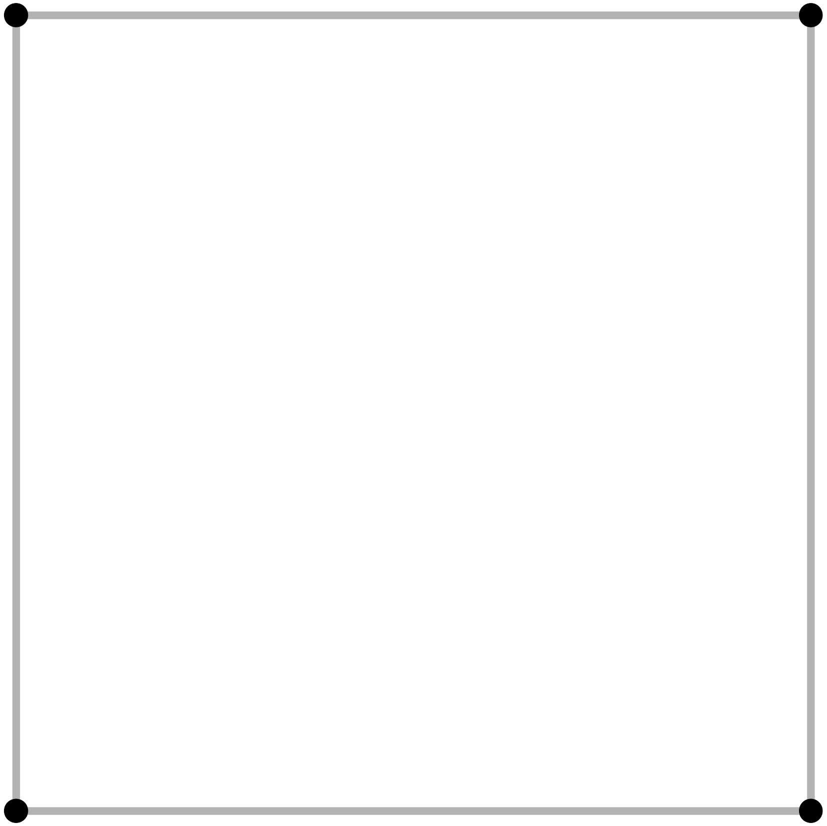
Reposition Old Vertices




Insert New Vertices



# Applying Subdivision Rules



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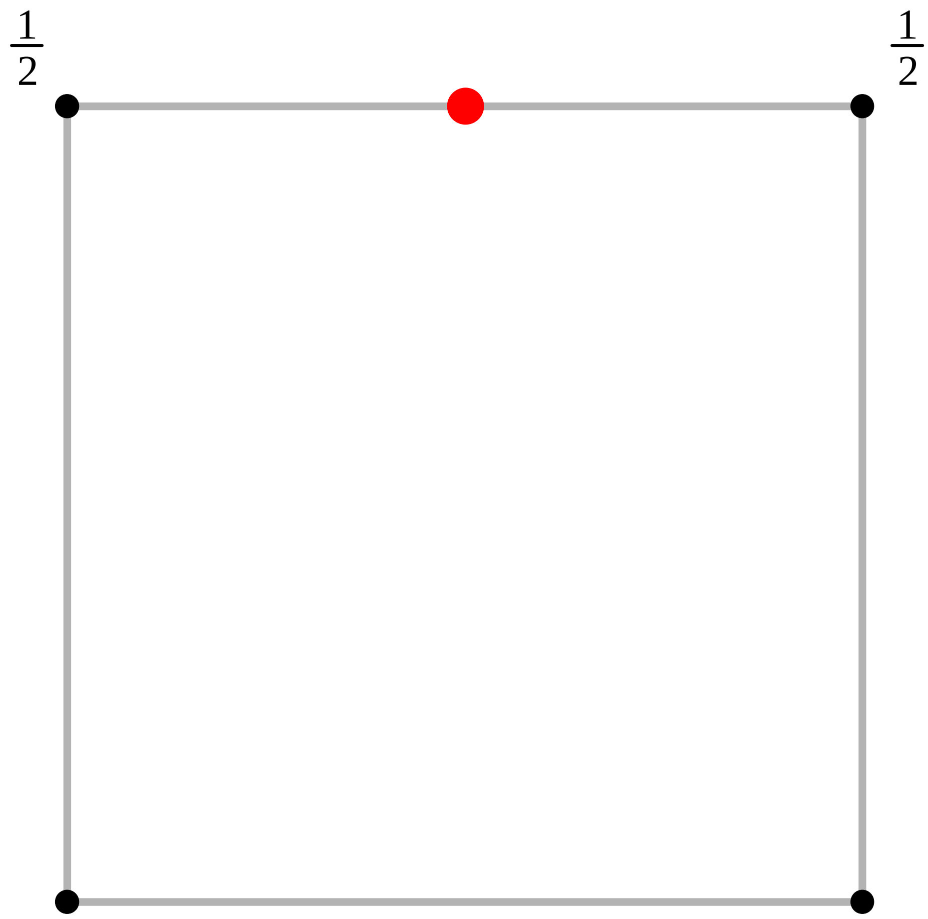

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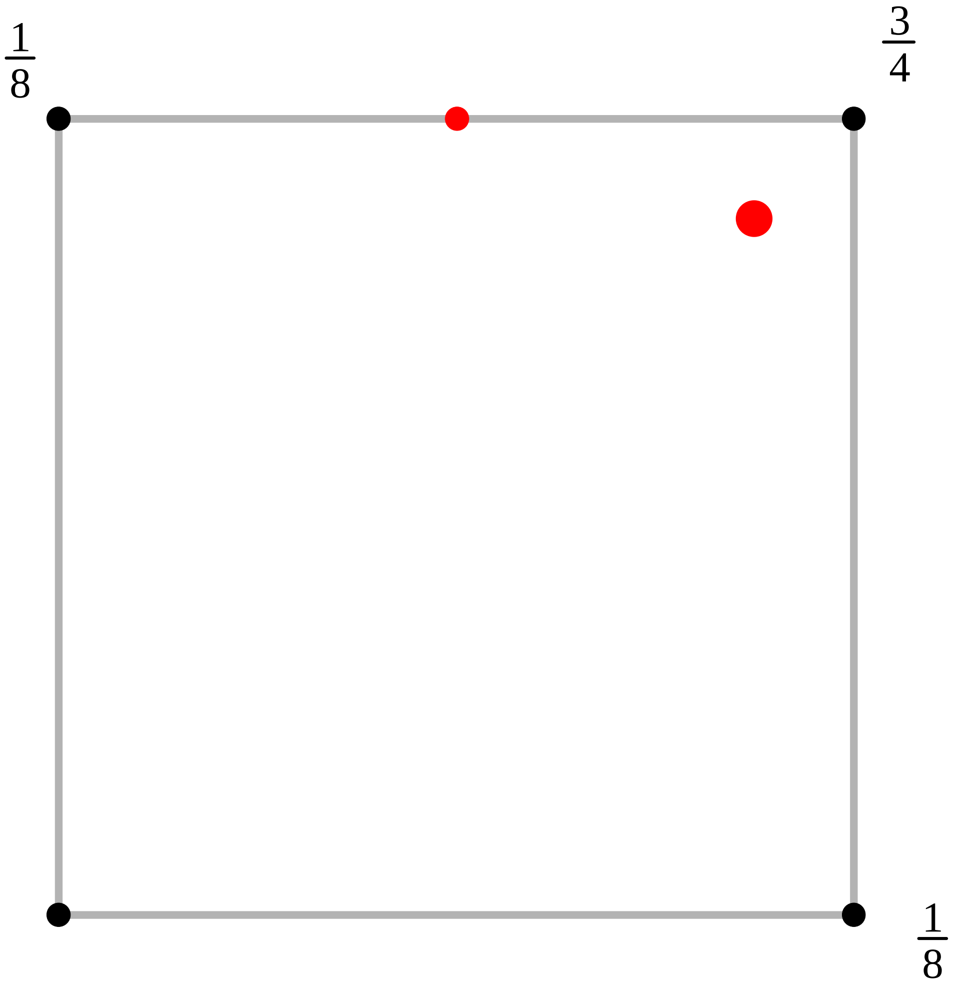

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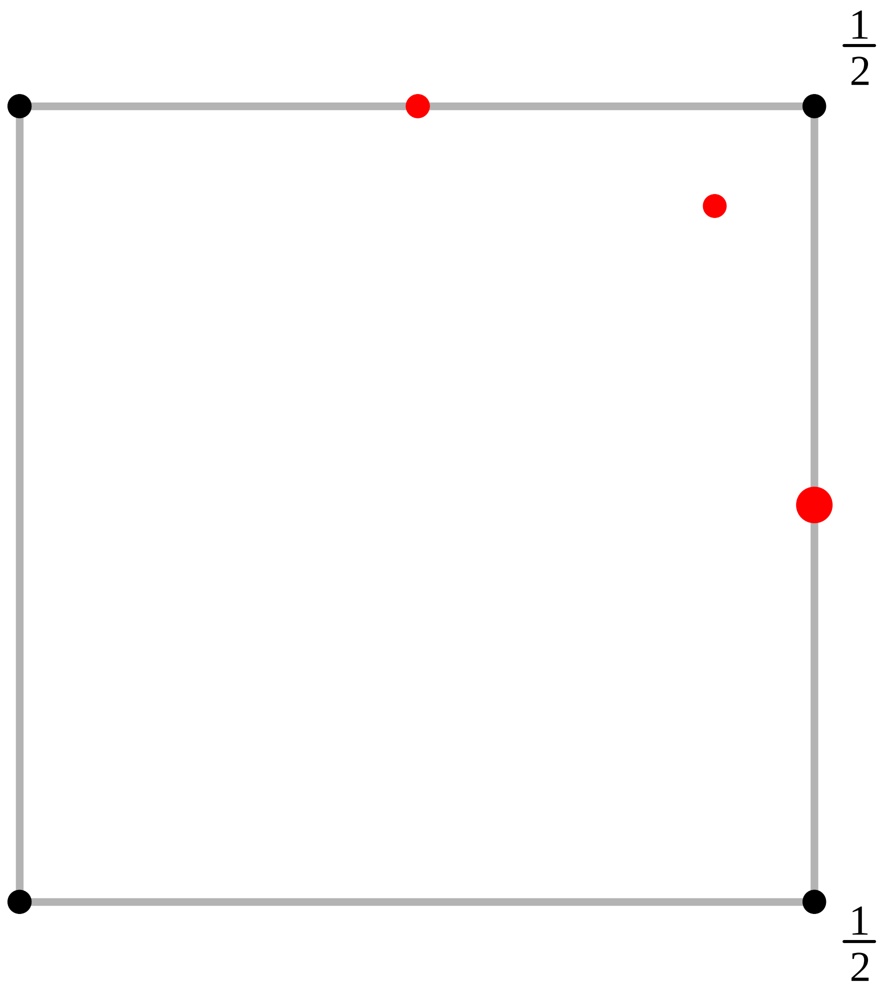

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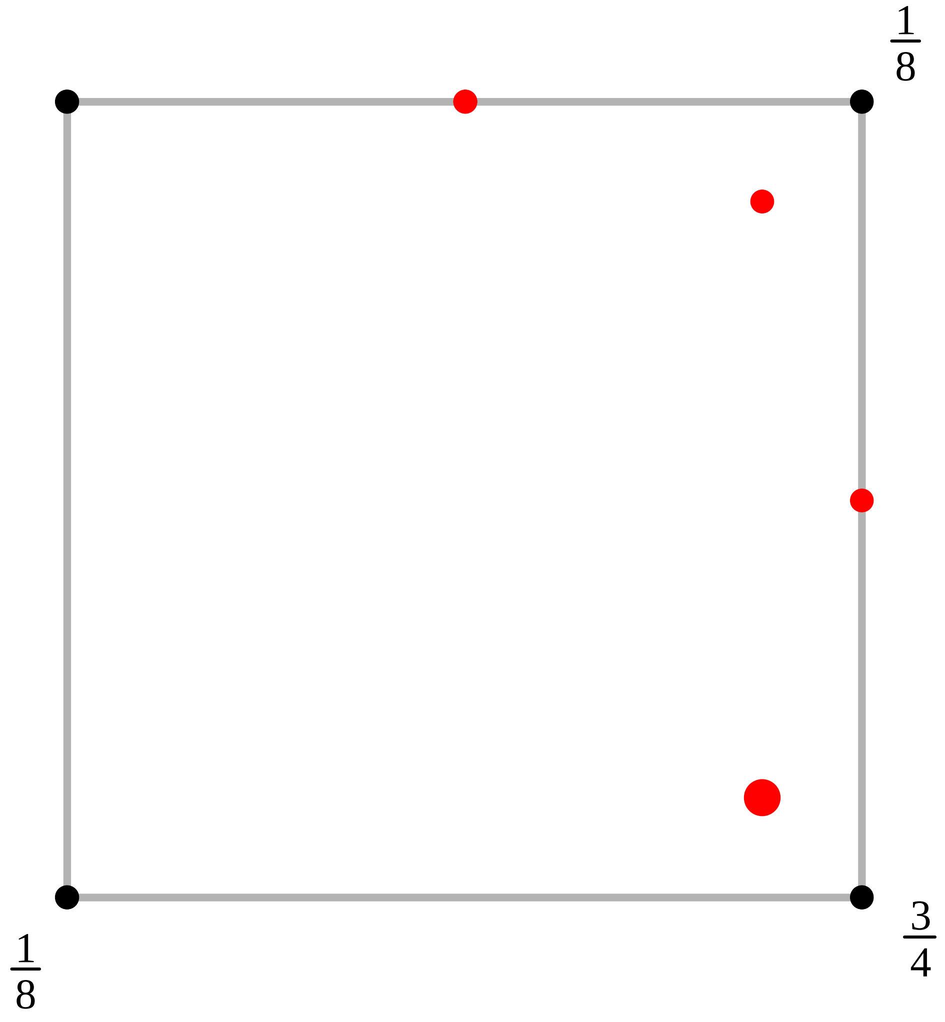

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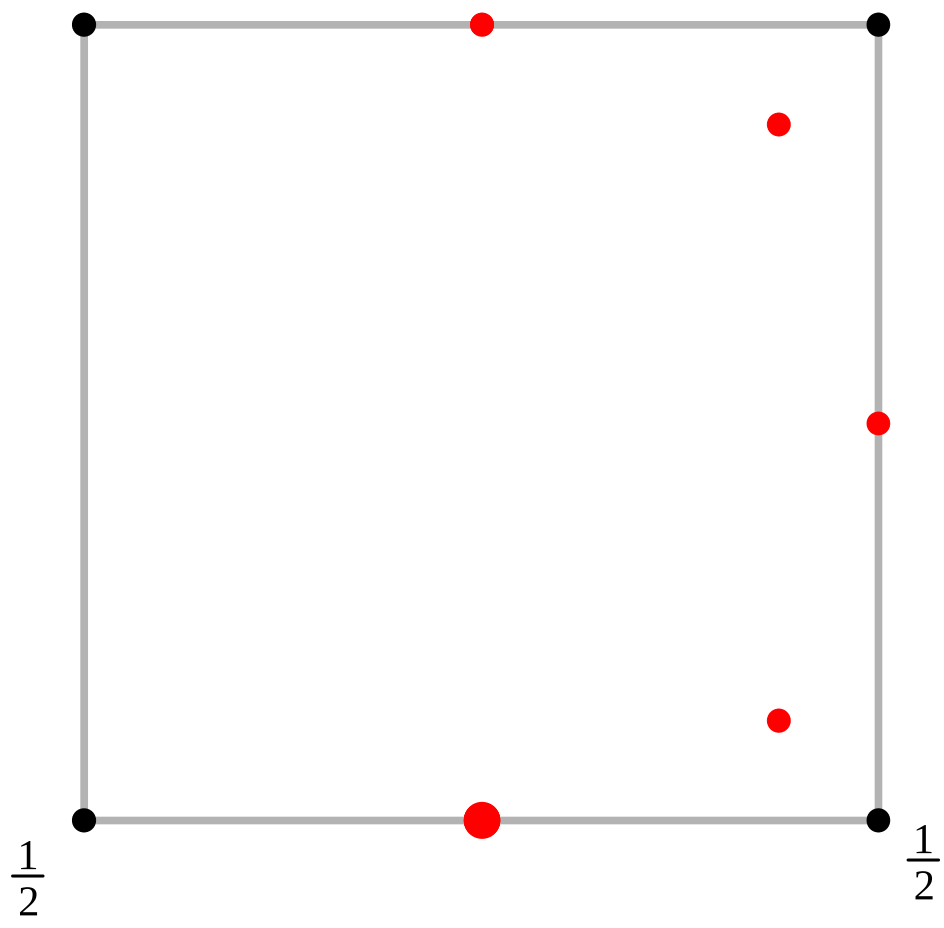

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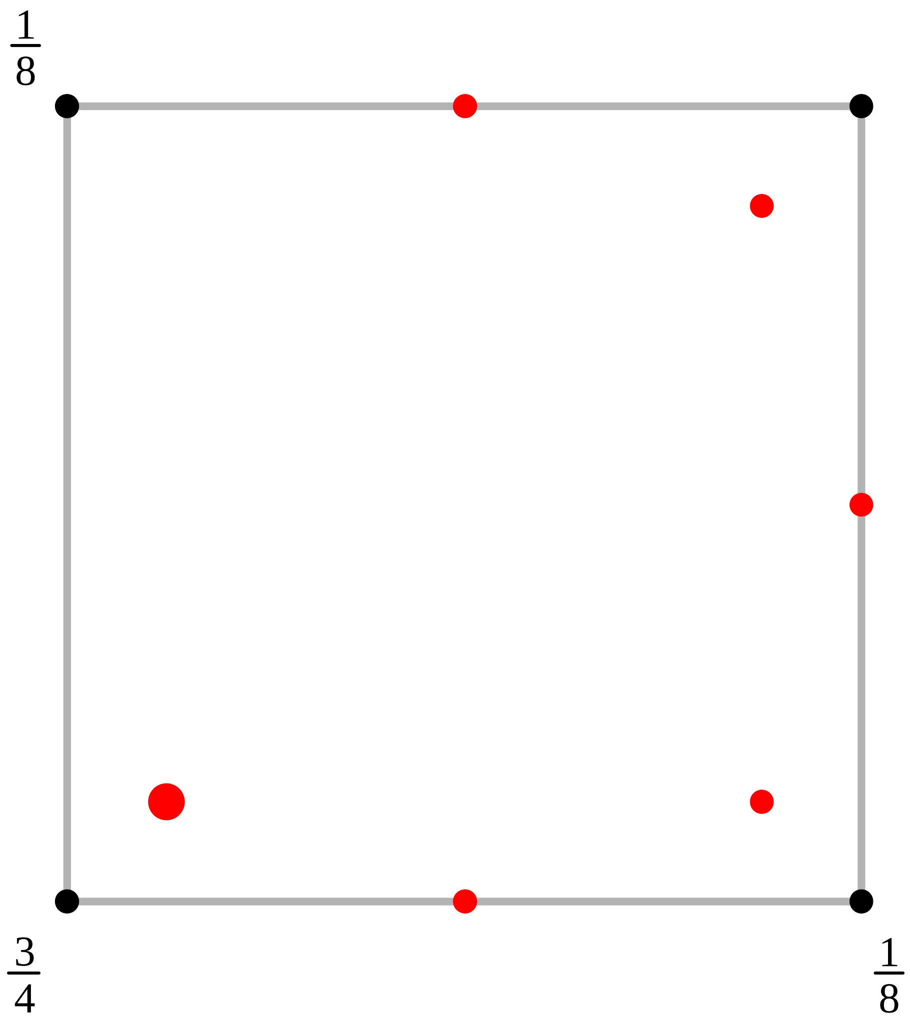

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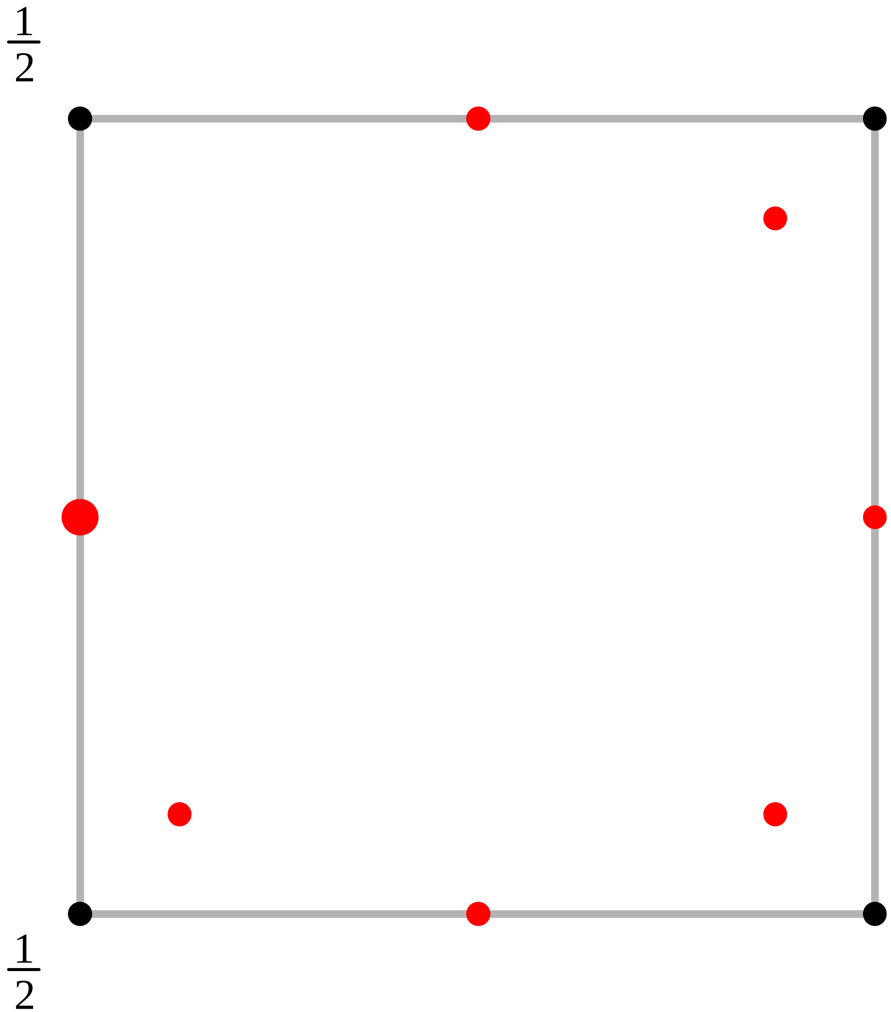

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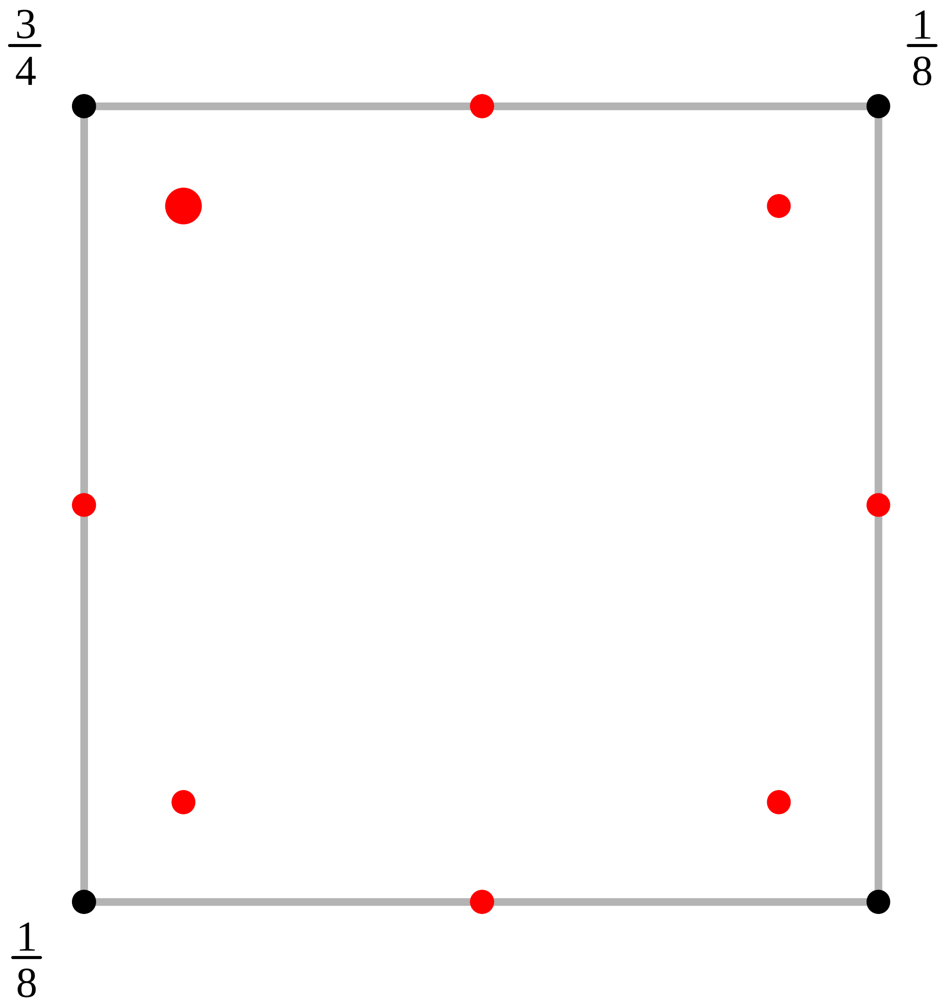

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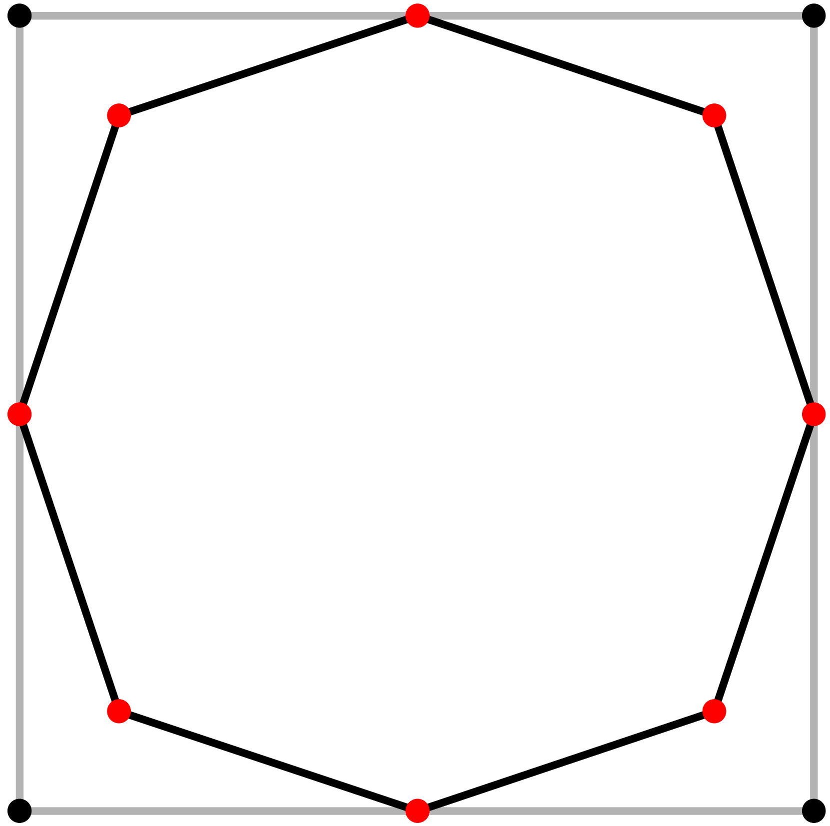

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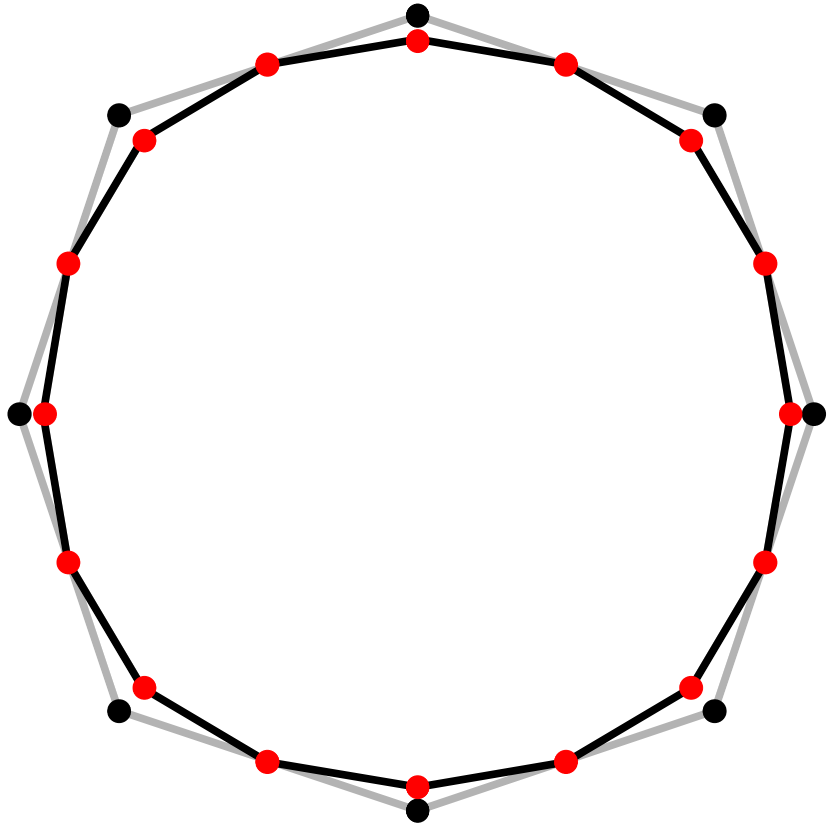

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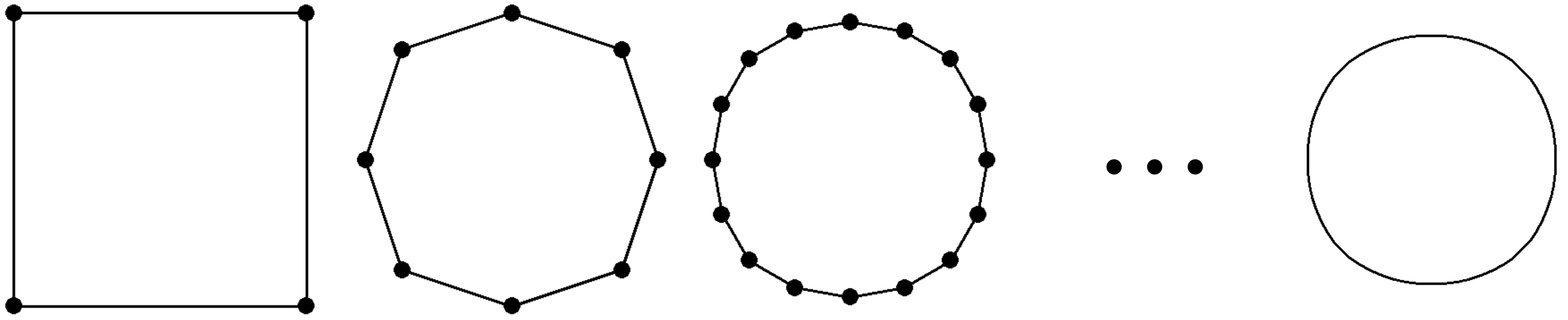

# Subdivision Rules Via Blossoming

- Assume knot-spacing uniform 0,2,4,6,8,…
- Find control points for refined knots 0,1,2,3,…

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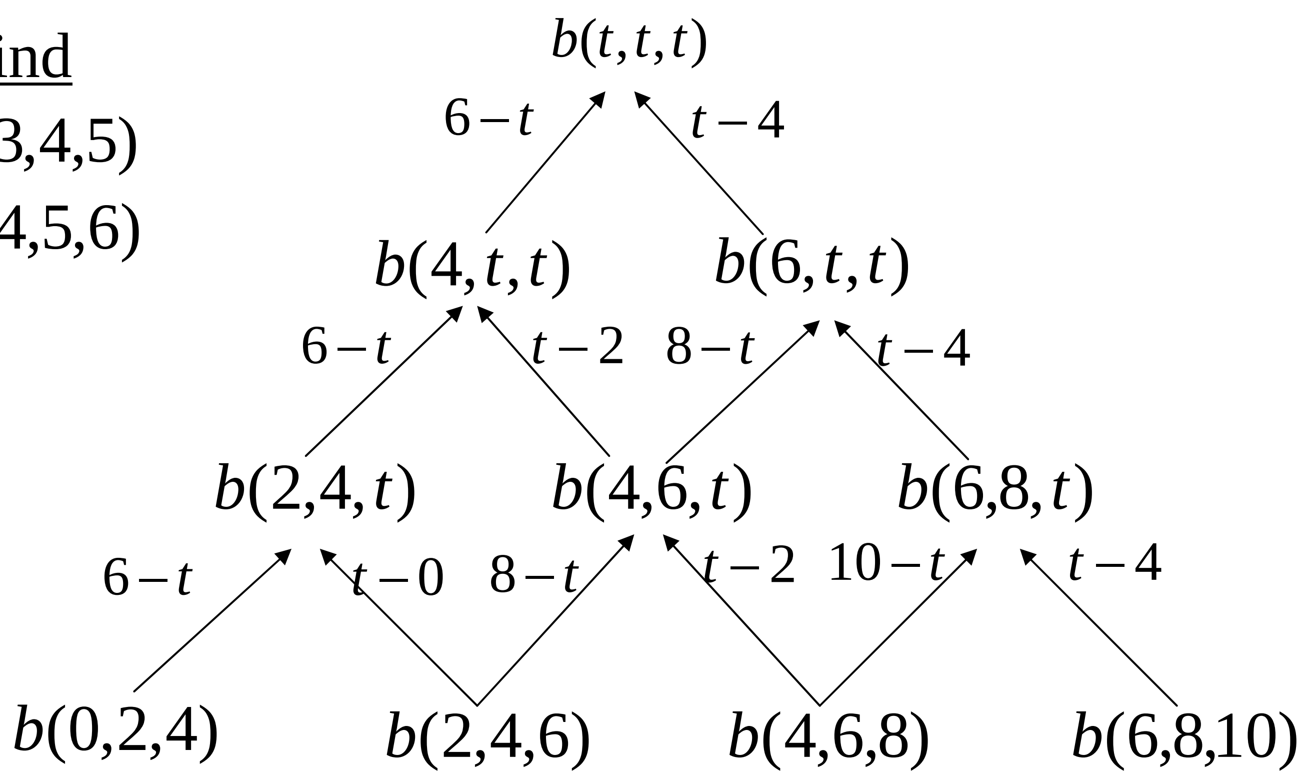
|---|---|
| $b(0,2,4)$ | $b(3,4,5)$ |
| $b(2,4,6)$ | $b(4,5,6)$ |
| $b(4,6,8)$ | |
| $b(6,8,10)$ | |



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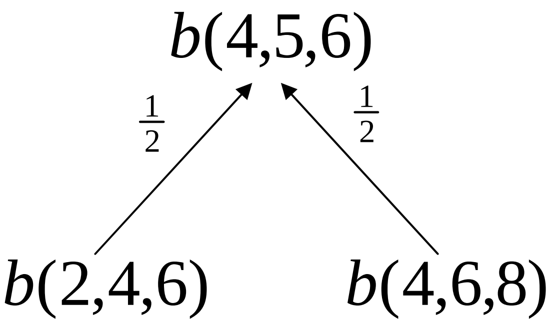

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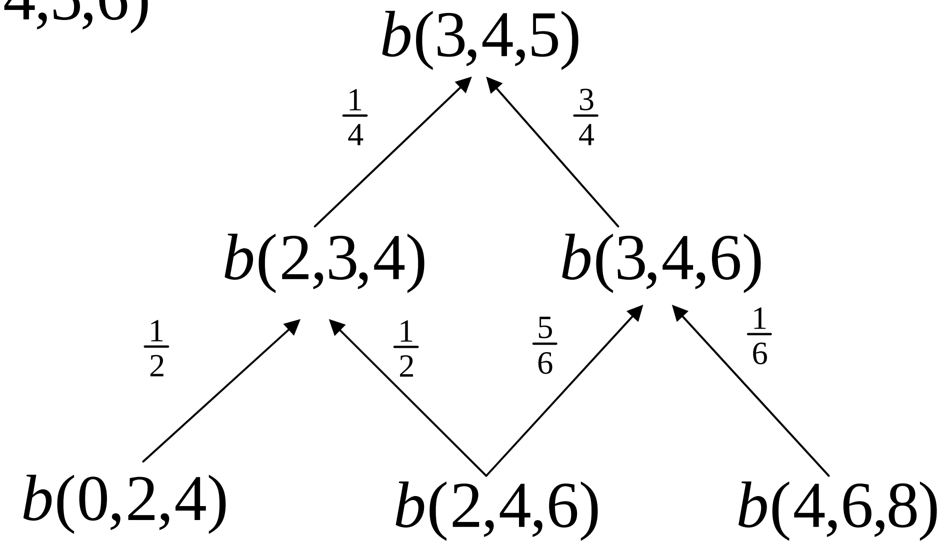


# Subdivision Rules Via Blossoming

- Assume knot-spacing uniform 0,2,4,6,8,…
- Find control points for refined knots 0,1,2,3,…

| Given | Find |
|---|---|
| $b(0,2,4)$ | $b(3,4,5) = \frac{1}{8}b(0,2,4) + \frac{3}{4}b(2,4,6) + \frac{1}{8}b(4,6,8)$ |
| $b(2,4,6)$ | $b(4,5,6) = \frac{1}{2}b(2,4,6) + \frac{1}{2}b(4,6,8)$ |
| $b(4,6,8)$ | |
| $b(6,8,10)$ | |

Works for arbitrary degree B-splines!!!

# Subdivision Convergence Analysis

- The new control points $\mathbf{d}_i^{(1)}$ are generated from the existing ones $\mathbf{d}_i$ by setting

$$\mathbf{d}_{2i}^{(1)} = \tfrac{1}{8}\mathbf{d}_{i-1} + \tfrac{3}{4}\mathbf{d}_i + \tfrac{1}{8}\mathbf{d}_{i+1}$$

$$\mathbf{d}_{2i+1}^{(1)} = \tfrac{1}{2}\mathbf{d}_i + \tfrac{1}{2}\mathbf{d}_{i+1}$$

- Investigate the limit of the sequence $\mathbf{d}_k$ , $\mathbf{d}_{2k}^{(1)}$ , $\mathbf{d}_{4k}^{(2)}$ ,...

$$\begin{bmatrix} \mathbf{d}_{2k-1}^{(1)} \\ \mathbf{d}_{2k}^{(1)} \\ \mathbf{d}_{2k+1}^{(1)} \end{bmatrix} = \tfrac{1}{8}\begin{bmatrix} 4 & 4 & 0 \\ 1 & 6 & 1 \\ 0 & 4 & 4 \end{bmatrix}\begin{bmatrix} \mathbf{d}_{k-1} \\ \mathbf{d}_k \\ \mathbf{d}_{k+1} \end{bmatrix}$$

$$D^{(1)} = AD$$

# Subdivision Convergence Analysis

- The matrix A has eigenvalues 1, ½, ¼, and can be diagonalized as:

$$A = E\Lambda E^{-1}$$

$$E = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix} \qquad E^{-1} = \frac{1}{6}\begin{bmatrix} 1 & 4 & 1 \\ 3 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$$

- For the next iteration:

$$D^{(2)} = AD^{(1)} = A^2 D$$

$$A^2 = E\Lambda E^{-1} E\Lambda E^{-1} = E\Lambda^2 E^{-1}$$

# Subdivision Convergence Analysis

$$A^r = E\Lambda^r E^{-1}$$

$$A^\infty = E\Lambda^\infty E^{-1}$$

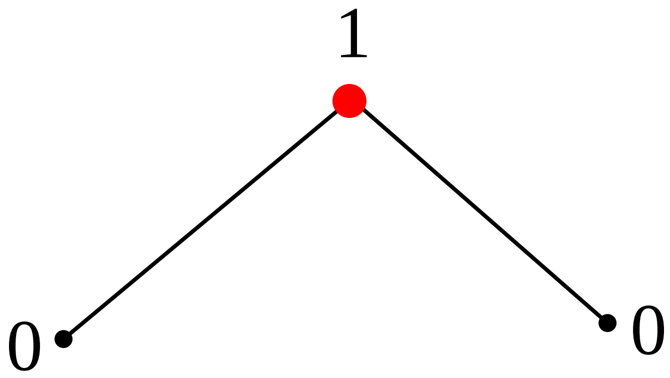
$$\Lambda^\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A^\infty = \frac{1}{6}\begin{bmatrix} 1 & 4 & 1 \\ 1 & 4 & 1 \\ 1 & 4 & 1 \end{bmatrix}$$

Implying that all three points $\mathbf{d}_{k-1}$, $\mathbf{d}_k$ , $\mathbf{d}_{k+1}$ converge to the same point $(\mathbf{d}_{k-1} + 4\mathbf{d}_k + \mathbf{d}_{k+1})/6$ .

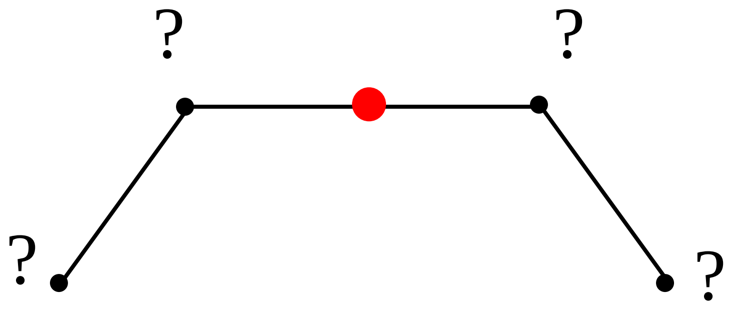
# Interpolatory Subdivision

- Interpolating control vertices may be desirable

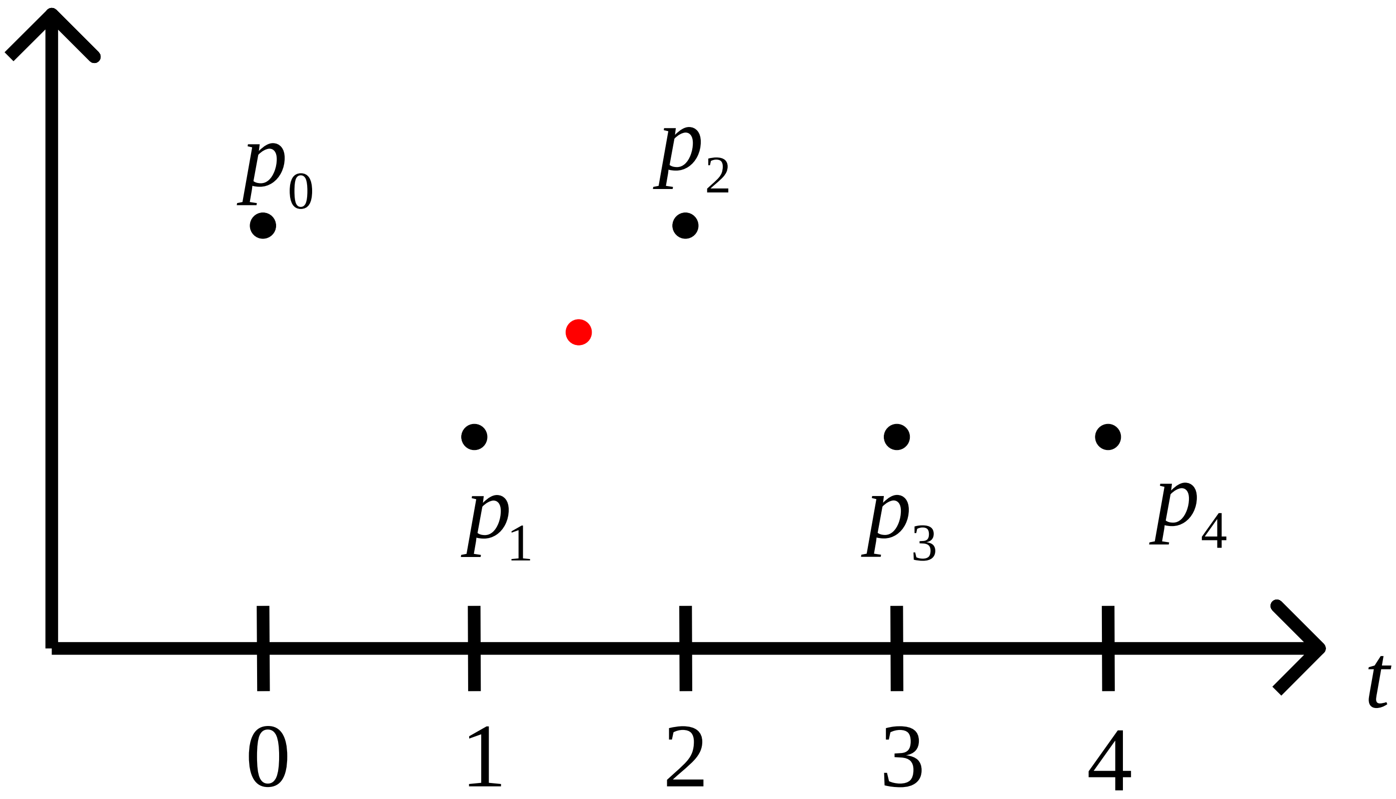
Reposition Old Vertices




Insert New Vertices



# Four-Point Subdivision



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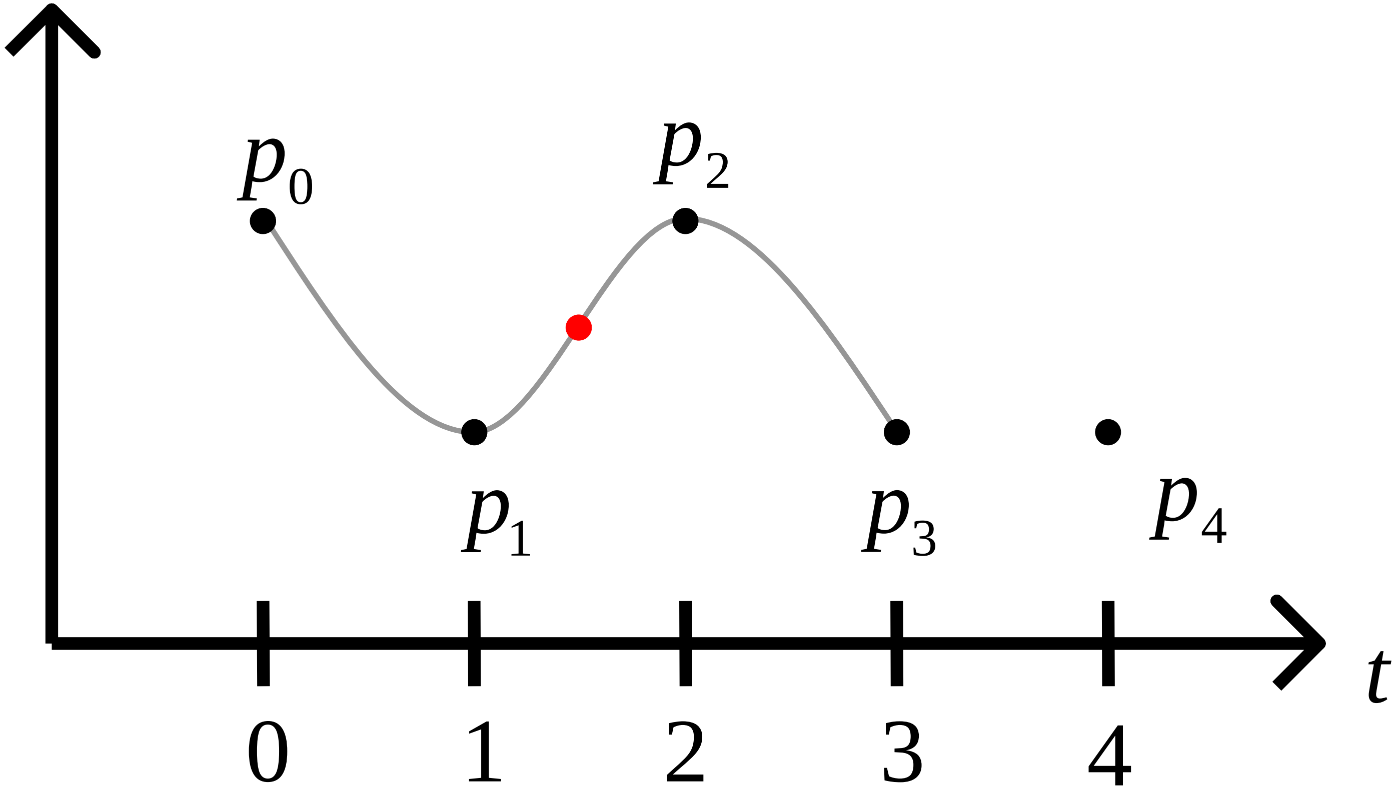

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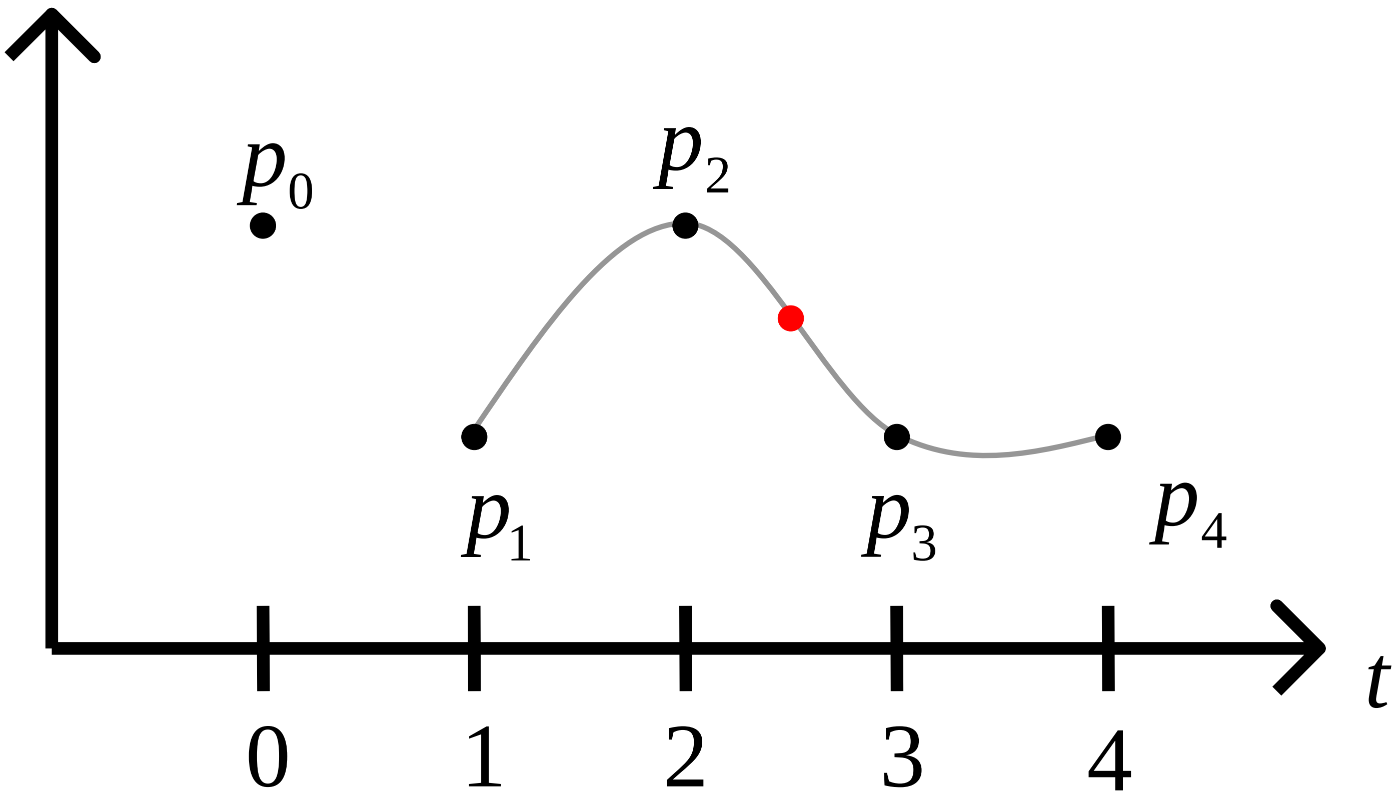

# Four-Point Subdivision

- LaGrange Polynomials

$$L_i(t) = \prod_{j \neq i} \frac{(t - t_j)}{(t_i - t_j)}$$

$$p(t) = \sum_{i=0}^{n} p_i L_i(t)$$

# Four-Point Subdivision

$$\begin{pmatrix} \frac{-1}{16} & \frac{9}{16} & \frac{9}{16} & \frac{-1}{16} \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

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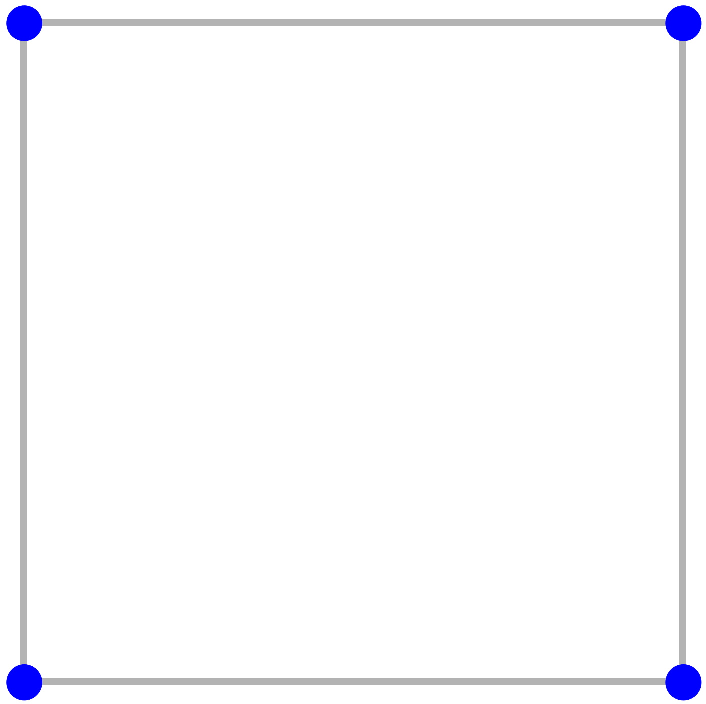

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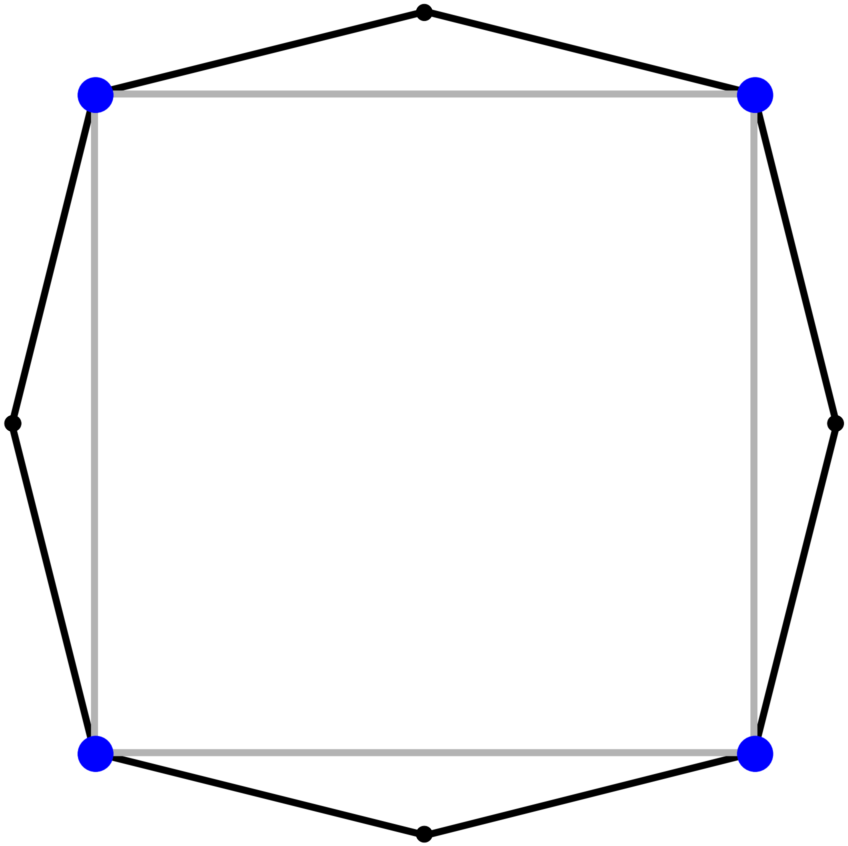

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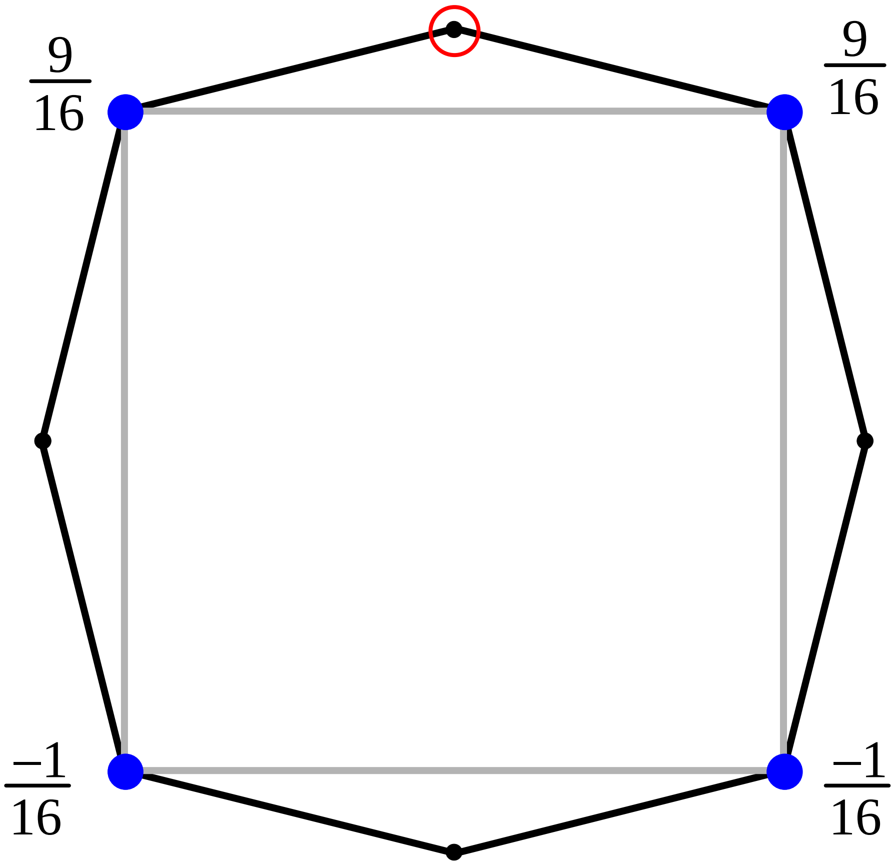

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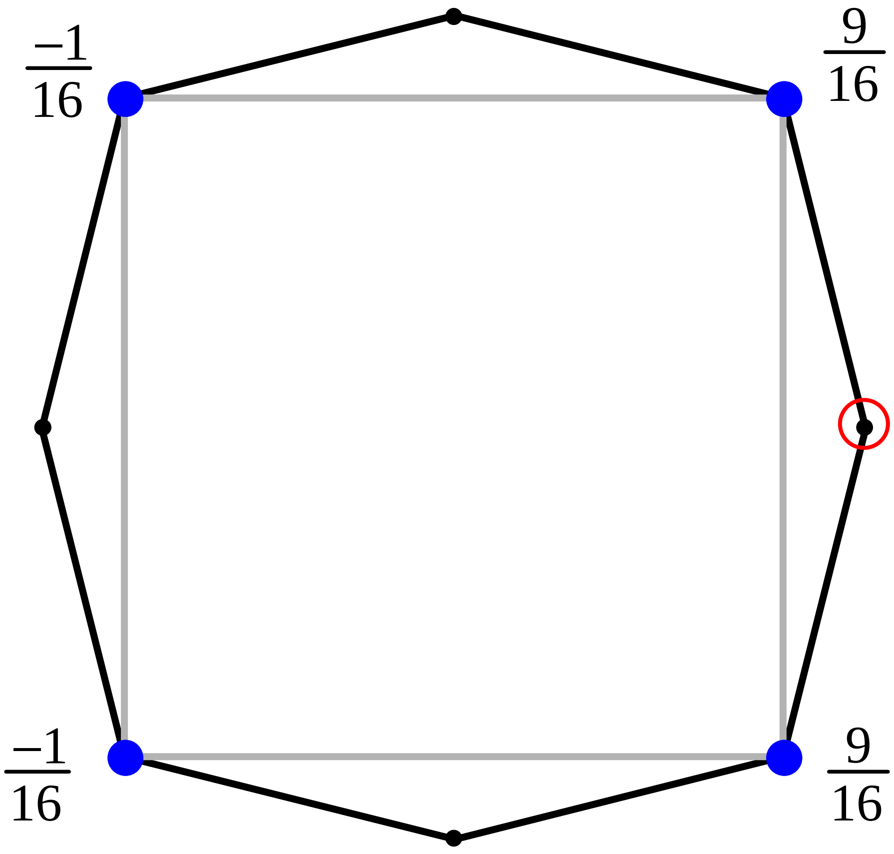

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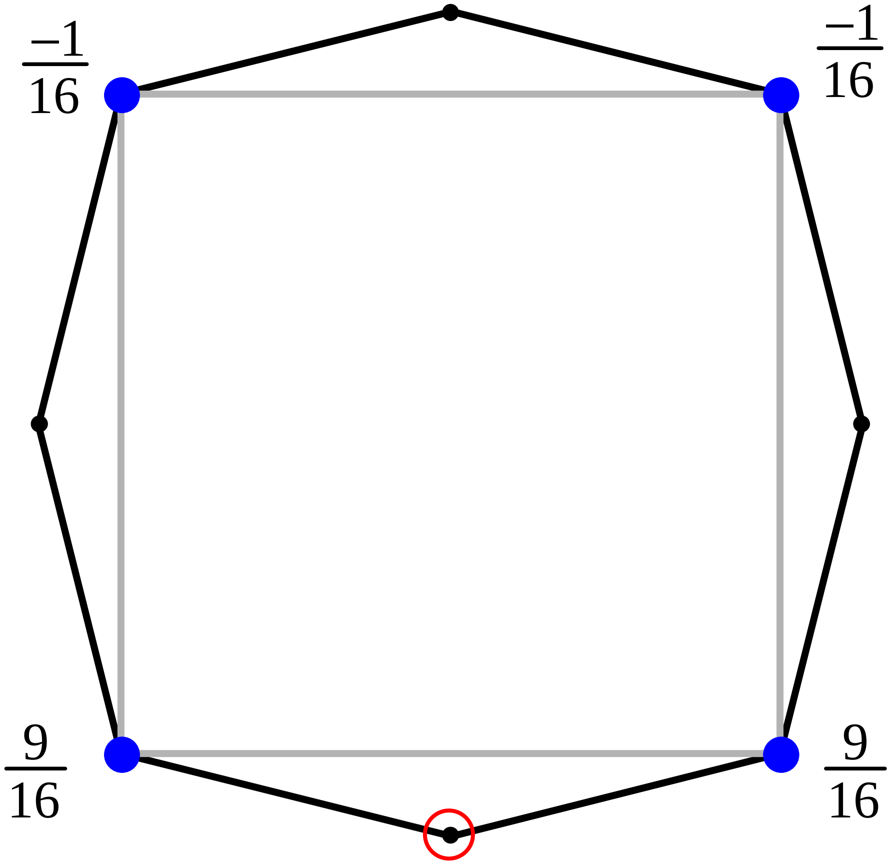

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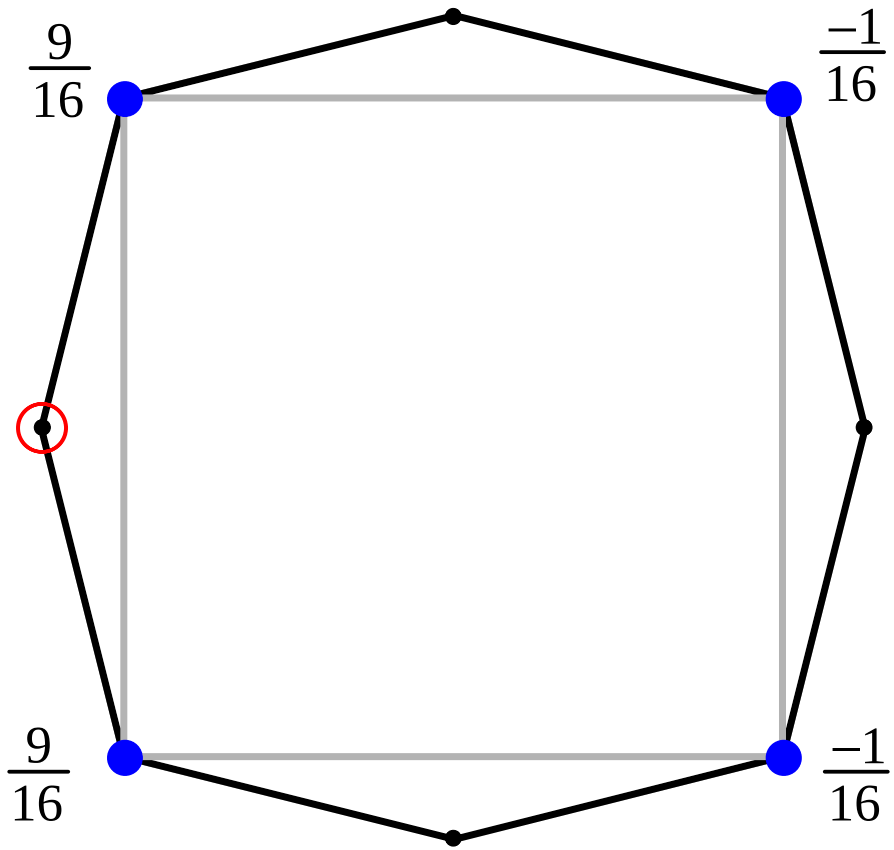

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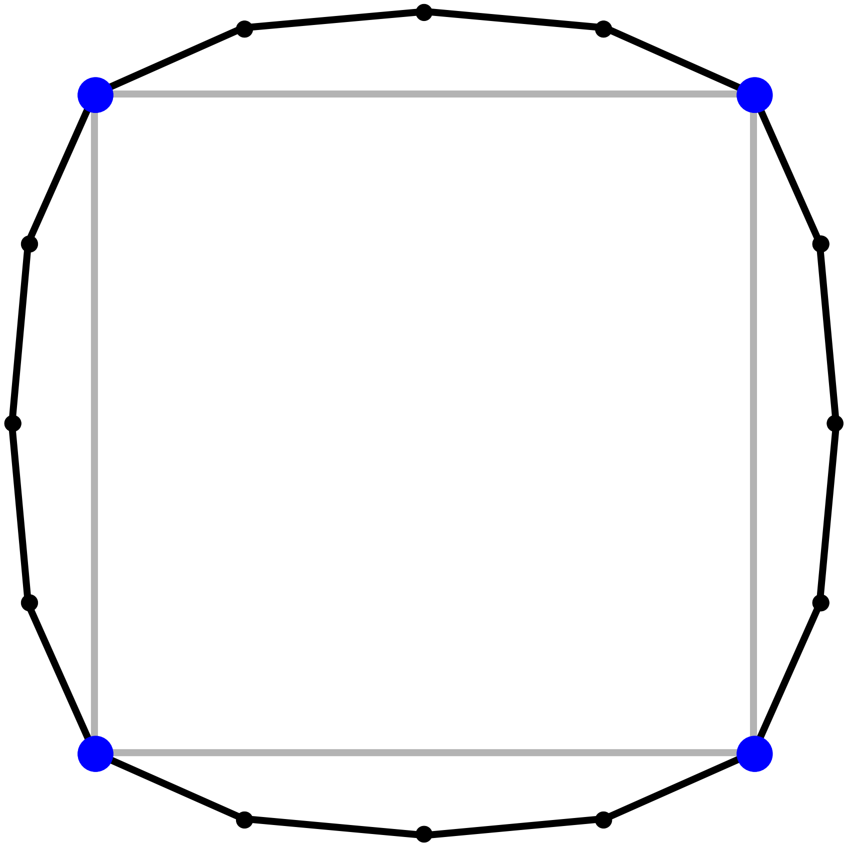

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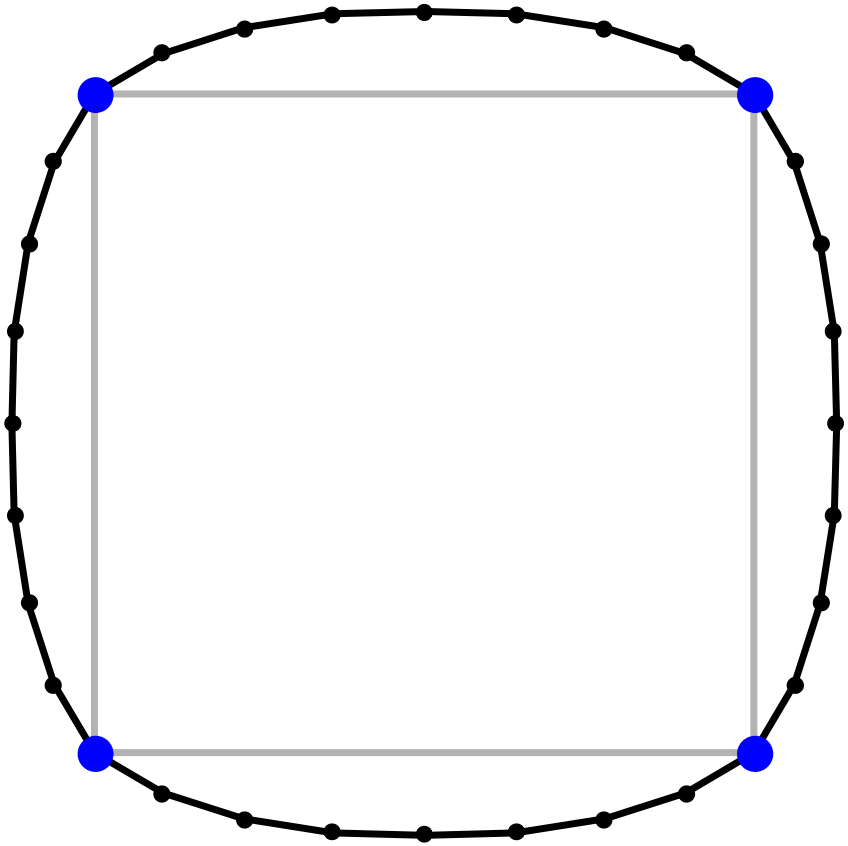

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