Supplemental Appendices

This document provides information about convergence of $\hat{L}_P^t$.

In Belkin *et al.*'s work [1], the convergence of $\hat{L}_P^t$, means $L\mathbf{f}$ is converging to $\Delta_{\mathcal{M}} f$ point-wisely where $\mathbf{f}$ is the discrete form of function f. This is different from the definition of convergence in works related to finite element method [2], [3]. We take the definition in Belkin *et al.*'s work [1].

In our work, the discretization of LBO $\Delta_{\mathcal{M}}$ is different from finite element method. With Lemma 2.5 in this document, we know that it is possible to approximate $\Delta_{\mathcal{M}} f(p)$ using integration. So we discretize the integration to approximate $\Delta_{\mathcal{M}} f(p)$ for each vertex p in point cloud P. The matrix form of this discretization is our discrete LBO $\hat{L}_P^t$. The convergence of $\hat{L}_P^t$ is shown in Theorem 4.2. Theorem 4.1 is about the convergence of Voronoi cell approximation. It is essential to the proof of Theorem 4.2. Proofs for these theorems are presented in section 1 of this document. Lemmas referred from existing works are presented in section 2 of this document. Miscellaneous lemmas are presented in section 3 and 4 of this document.

1 Convergence Proof

In our construction of PB-MHB, the assumption is we have a continuous differentiable Riemannian manifold $\mathcal{M}$ on which the sample set P lies. f is a C^2 continuous function defined over $\mathcal{M}$. We are going to prove that the result of our discrete LBO applied on the function $\hat{L}_P^t\mathbf{f}$ converges to the continuous result $\Delta_M f$ point-wisely.

To show the convergence of $\hat{L}_P^t$, first we are proving that our estimation of the Voronoi cell area is converging to the real Voronoi cell area as point clouds get denser. This part is proved in appendix section 1.1. After having the Voronoi cell area convergence result, we prove that $\hat{L}_P^t\mathbf{f}$ converges to $\Delta_M f$ point-wisely in appendix section 1.2.

1.1 Proof of Theorem 4.1: Convergence of Estimated Voronoi Cell Area

As shown in figure 1, the proof consists of two steps: (1) we prove that the projection of $Vr_{\mathcal{M}}(p)$ on the estimated tangent plane $\hat{T}_p$, $\hat{\Pi}(Vr_{\mathcal{M}}(p))$, has converging area to $Vr_{\mathcal{M}}(p)$, as shown in Lemma 1.5; (2) we build the upper bound and the lower

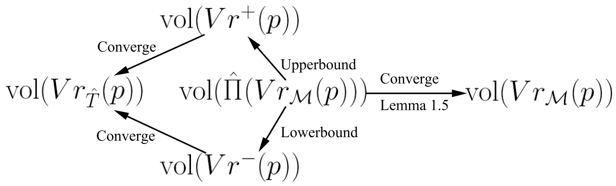

Fig. 1. Proof structure of Theorem 4.1.

bound of $\mathrm{vol}(\hat{\Pi}(Vr_{\mathcal{M}}(p)))$ that are both converging to $\mathrm{vol}(Vr_{\hat{T}}(p))$ so we know that $\mathrm{vol}(\hat{\Pi}(Vr_{\mathcal{M}}(p)))$ is converging to $\mathrm{vol}(Vr_{\hat{T}}(p))$. By combining this result with Lemma 1.5, we have Theorem 4.1 proved.

To prove Lemma 1.5 we need to prove that there are some bounds on the sizes of the Voronoi cells $Vr_{\mathcal{M}}(p)$ and $Vr_{\hat{T}}(p)$ (Lemma 1.1 and Lemma 1.3), and there are some bounds on the set of neighboring points that may influence these Voronoi cells (Lemma 1.2 and Lemma 1.4).

***Lemma** 1.1 (Bound of $Vr_{\mathcal{M}}(p)$):* Consider the underlying manifold $\mathcal{M}$ and its ε-sampling P, $\forall p \in P$:

$$Vr_{\mathcal{M}}(p) \subseteq B(p, \varepsilon) \tag{1.1}$$

holds, i.e., its Voronoi cell on the manifold is bounded by a ball with radius ε.

Proof: Suppose $\exists q \in Vr_{\mathcal{M}}(p) \subseteq \mathcal{M}$, that satisfies $\|p-q\| > \varepsilon$.

$\because$ P is ε-sampling,

$\therefore$ There is another point $p' \in P$ that satisfies $\|p' - q\| \le \varepsilon < \|p-q\|$, which means q is closer to p' instead of p.

$\therefore$ $q \notin Vr_{\mathcal{M}}(p)$. This is contradictory to assumption. □

***Lemma** 1.2 (Bound of Influencing Points on $\mathcal{M}$):* Consider the boundary of the Voronoi cell: $\partial Vr_{\mathcal{M}}(p)$. Given that P is an ε-sampling of $\mathcal{M}$, $\forall q \in \partial Vr_{\mathcal{M}}(p)$, $\exists p' \in P$, $p' \ne p$ satisfies $\|q-p\| = \|q-p'\|$, then for all such kind of points p',

$$\|p - p'\| \le 2\varepsilon \tag{1.2}$$

holds. That is, only the point set in $B(p, 2\varepsilon)$ may influence the Voronoi cell of point p.

Proof: According to Lemma 1.1, we have $\|q-p\| \le \varepsilon$ and $\|q-p'\| \le \varepsilon$ hold for $\forall q \in \partial Vr_{\mathcal{M}}(p)$. Thus we have

$$\|p - p'\| \le \|q-p\| + \|q-p'\| \le \varepsilon + \varepsilon = 2\varepsilon. \tag{1.3}$$

□

As described in the paper, we project a local neighborhood of points $P_\delta = P \cap B(p,\delta)$, $\delta \geq 10\varepsilon$ onto the estimated tangent plane $\hat{T}_p$. When δ, ε and $r/\rho = 10\varepsilon/\rho$ are small enough, the projection from the local patch $\mathcal{M} \cap B(p,\delta)$ to $\hat{T}_p$, denoted as $\hat{\Pi}$, is bijective. Let $\hat{\Phi} = \hat{\Pi}^{-1}$.

***Lemma** 1.3 (Bound of $Vr_{\hat{T}}(p)$):* Consider the Voronoi diagram of $p \cup \{\hat{\Pi}(P_\delta - p)\}$ on $\hat{T}_p$, where $p \in P$ is a sample point and P is an ε-sampling of $\mathcal{M}$. Denote the Voronoi cell of p on $\hat{T}_p$ as $Vr_{\hat{T}}(p)$, then

$$Vr_{\hat{T}}(p) \subseteq B(p,\varepsilon) \tag{1.4}$$

holds. That is, $Vr_{\hat{T}}(p)$ is bounded by a ball with radius ε.

Proof:

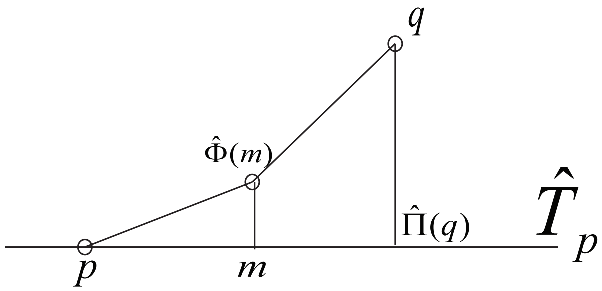

Fig. 2. Bound of $Vr_{\hat{T}}(p)$ for Lemma 1.3.

Suppose $\exists m \in Vr_{\hat{T}}(p)$ and $\|p-m\| > \varepsilon$, as shown in figure 2.

$\because m \in Vr_{\hat{T}}(p)$,

$\therefore \forall q \in P$ and $p \neq q$, $\|\hat{\Pi}(q) - m\| \geq \|p-m\| > \varepsilon$.

Here $p = \hat{\Pi}(p)$ since p lies on both $\mathcal{M}$ and $\hat{T}_p$.

$\because \hat{\Pi}$ is the projection from $\mathcal{M}$ to $\hat{T}_p$ and $\hat{\Phi} = \hat{\Pi}^{-1}$,

$\therefore \|p-m\| \leq \|p-\hat{\Phi}(m)\|$, $\|\hat{\Pi}(q)-m\| \leq \|q-\hat{\Phi}(m)\|$,

$\therefore \forall q \in P, \|q-\hat{\Phi}(m)\| \geq \|\hat{\Pi}(q)-m\| > \varepsilon$. This is contradictory to the assumption that P is ε-sampled. □

***Lemma** 1.4 (Bound of Influencing Points on $\hat{T}_p$):* Consider the boundary of the Voronoi cell on the estimated tangent plane: $\partial Vr_{\hat{T}}(p)$. Given that P is an ε-sampling of $\mathcal{M}$, $\forall q \in \partial Vr_{\hat{T}}(p)$, $\exists p' \in P$, $p' \neq p$ satisfies $\|q-p\| = \|q-\hat{\Pi}(p')\|$, then for all such kind of points p',

$$\|p-\hat{\Pi}(p')\| \leq 2\varepsilon \tag{1.5}$$

holds. That is, only the projected sample points in $B(p,2\varepsilon)$ may influence the Voronoi cell $Vr_{\hat{T}}(p)$.

Proof: According to Lemma 1.3, we know that $\forall q \in \partial Vr_{\hat{T}}(p)$, $\|p-q\| \leq \varepsilon$.

Thus for the influencing projected point $\hat{\Pi}(p')$ we have

$$\|p-q\| = \|q-\hat{\Pi}(p')\| \tag{1.6}$$

$$\|p-\hat{\Pi}(p')\| \leq \|p-q\| + \|q-\hat{\Pi}(p')\| \tag{1.7}$$

$$\leq \varepsilon + \varepsilon = 2\varepsilon \tag{1.8}$$

□

***Lemma** 1.5 (Convergence of $\hat{\Pi}(Vr_{\mathcal{M}}(p))$ to $Vr_{\mathcal{M}}(p)$):* Consider projecting $Vr_{\mathcal{M}}(p)$ to $\hat{T}_p$. P is an ε-sampling of $\mathcal{M}$, and ρ is the local feature size of point $p \in P$. Then

$$\left\| \frac{\mathrm{vol}(Vr_{\mathcal{M}}(p))}{\mathrm{vol}(\hat{\Pi}(Vr_{\mathcal{M}}(p)))} - 1 \right\| = O(\frac{\varepsilon^2}{\rho^2}) \tag{1.9}$$

holds.

Proof:

$\forall q \in Vr_{\mathcal{M}}(p)$, consider the angle $\angle(T_q, \hat{T}_p)$ between the two planes T_q and $\hat{T}_p$, where T_p and T_q are the real tangent planes of $\mathcal{M}$ at points p and q, $\hat{T}_p$ is the estimated tangent plane at point p, as described in the paper.

According to Lemma 2.1 (in appendix section 2), when $\|p-q\| < \rho/3$ we have

$$\angle(T_p, T_q) \leq \frac{\|p-q\|}{\rho - \|p-q\|} \leq O(\varepsilon/\rho). \tag{1.10}$$

We get the last inequality by applying Lemma 1.1.

Now we have three planes here: T_p, T_q and $\hat{T}_p$. According to Lemma 4.1 (in appendix section 4), when all angles are small,

$$\angle(T_q,\hat{T}_p) \leq \angle(T_p,T_q) + \angle(T_p,\hat{T}_p)$$
$$\leq O(\varepsilon/\rho) \tag{1.11}$$

holds, where we get the second inequality by applying Lemma 2.3 (in appendix section 2). Thus we have:

$$\cos\angle(T_q,\hat{T}_p) = \sqrt{1-\sin^2\angle(T_q,\hat{T}_p)} \tag{1.12}$$

$$\geq \sqrt{1-(\angle(T_q,\hat{T}_p))^2} \tag{1.13}$$

$$\geq \sqrt{1-O(\varepsilon^2/\rho^2)} \tag{1.14}$$

$$\geq 1-O(\varepsilon^2/\rho^2). \tag{1.15}$$

When δ, ε and $r/\rho = 10\varepsilon/\rho$ are small enough, the projection from the local patch $\mathcal{M} \cap B(p,\delta)$ to $\hat{T}_p$, denoted as $\hat{\Pi}$, is bijective. Denote $\gamma = \angle(T_q,\hat{T}_p)$, then we have

$$\mathrm{vol}(Vr_{\mathcal{M}}(p)) = \int_{q\in\hat{\Pi}(Vr_{\mathcal{M}}(p))} \frac{1}{\cos\gamma} ds \tag{1.16}$$

$$\leq \max(\frac{1}{\cos\gamma}) \int_{q\in\hat{\Pi}(Vr_{\mathcal{M}}(p))} ds \tag{1.17}$$

$$= \max(\frac{1}{\cos\gamma})\, \mathrm{vol}(\hat{\Pi}(Vr_{\mathcal{M}}(p))). \tag{1.18}$$

Thus by combining (1.18) with (1.15), we can have

$$1 \leq \left\| \frac{\mathrm{vol}(Vr_{\mathcal{M}}(p))}{\mathrm{vol}(\hat{\Pi}(Vr_{\mathcal{M}}(p)))} \right\| \leq \frac{1}{\min(\cos\gamma)} \leq 1 + O(\varepsilon^2/\rho^2). \tag{1.19}$$

□

With these results we can prove Theorem 4.1 as follows:

Proof:

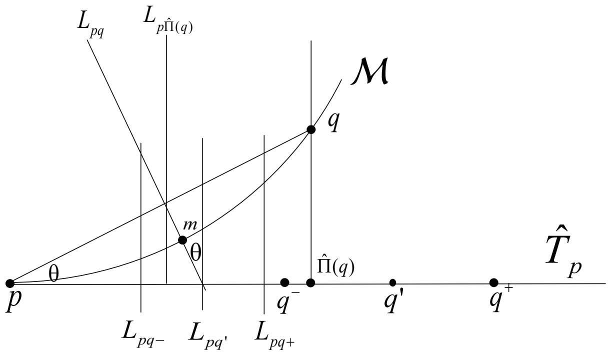

Fig. 3. Parallel bisecting planes.

As described in the paper, we chose the neighboring points $P_\delta = P \cap B(p,\delta)$, $\delta \geq 10\varepsilon$ for projection. According to Lemma 1.4, for $Vr_{\hat{T}}(p)$, any influencing projected point $q \in P_\delta$ satisfies $\|\hat{\Pi}(q) - p\| \leq 2\varepsilon$. Consider $\forall q \in P_\delta$, according to Lemma 2.2, 2.3 and 4.2, we have $\angle(pq, \hat{T}_p)) \leq \angle(pq, T_p) + \angle(\hat{T}_p, T_p) = O(\frac{\|p-q\|}{\rho}) + O(r/\rho)$ and $\frac{\|p-q\|}{\|p-\hat{\Pi}(q)\|} - 1 = O(\frac{\varepsilon}{\rho}) + O(\frac{\|p-q\|}{\rho})$. When ε is small enough, we have $(\hat{T}_p \cap B(p, 2\varepsilon)) \subseteq \hat{\Pi}(\mathcal{M} \cap B(p,r))$. Thus we know for $\forall q \in P_\delta$ that satisfies $\|p - \hat{\Pi}(q)\| \leq 2\varepsilon$, $\|p-q\| \leq r = 10\varepsilon$ holds. That is, all neighboring points that could influence $Vr_{\hat{T}}(p)$ are included in $P_r = P \cap B(p,r)$. According to Lemma 1.2, we know that all points which may influence $Vr_{\mathcal{M}}(p)$ are also included in P_r.

On the estimated tangent plane $\hat{T}_p$, we are building 4 sets of Voronoi diagrams to get the converging approximation of the Voronoi cell area, as shown in Fig. 3.

For each point $q \in P_\delta$, $q \neq p$, we consider the bisecting plane L_{pq} between points p and q. We also build the bisecting plane $L_{p\hat{\Pi}(q)}$ for the point-pair $\{p, \hat{\Pi}(q)\}$. As shown in Fig. 3, it is obvious that $L_{p\hat{\Pi}(q)} \perp \hat{T}_p$, and the straight line $l_{p\hat{\Pi}(q)} = L_{p\hat{\Pi}(q)} \cap \hat{T}_p$ is also the bisecting line on $\hat{T}_p$ for the point-pair $\{p, \hat{\Pi}(q)\}$. $\{l_{p\hat{\Pi}(q)}\}$ are also the lines that compose $\partial Vr_{\hat{T}}(p)$, which is the boundary of the Voronoi cell of p on $\hat{T}_p$. Notice that for some points $q \in P_\delta$, $l_{p\hat{\Pi}(q)} \cap \partial Vr_{\hat{T}}(p) = \emptyset$. That is, it is not necessary that all bisecting lines contribute to the boundary of $Vr_{\hat{T}}(p)$.

Consider the lines $l_{pq} = L_{pq} \cap \hat{T}_p$. Since we have $q\hat{\Pi}(q) \perp \hat{T}_p$ and $pq \perp L_{pq}$, we know that $l_{pq} \perp pq$ and $l_{pq} \perp q\hat{\Pi}(q)$. Thus $l_{pq} \perp p\hat{\Pi}(q)$. Because $L_{p\hat{\Pi}(q)} \perp p\hat{\Pi}(q)$, we know that $l_{pq} \parallel L_{p\hat{\Pi}(q)}$. Then for each l_{pq}, we can build the plane $L_{pq'}$ satisfying $l_{pq} \subset L_{pq'}$ and $L_{pq'} \parallel L_{p\hat{\Pi}(q)}$.

As shown in Fig. 3, we denote $\theta = \angle(L_{p\hat{\Pi}(q)}, L_{pq})$. According to Lemma 4.2 (in appendix section 4), we have

$$\begin{aligned} \theta =& \angle(L_{p\hat{\Pi}(q)}, L_{pq}) && (1.20)\\ =& \angle(p\hat{\Pi}(q), pq) && (1.21)\\ \leq& \angle(pq, T_p) + \angle(T_p, \hat{T}_p). && (1.22) \end{aligned}$$

According to lemma 2.2 (in appendix section 2), we have

$$\sin\angle(pq, T_p) \leq O(\frac{\varepsilon}{\rho}). \quad (1.23)$$

According to lemma 2.3 (in appendix section 2), we have

$$\angle(T_p, \hat{T}_p) \leq O(\frac{r}{\rho}). \quad (1.24)$$

By combining (1.23), (1.24) with (1.22), we have

$$\begin{aligned} \sin\theta &\leq O(\frac{r}{\rho}) + O(\frac{\varepsilon}{\rho}), && (1.25)\\ \tan\theta &= \frac{\sin\theta}{\cos\theta} && (1.26)\\ &\leq O\left(\frac{(\varepsilon+r)/\rho}{\sqrt{1-(\varepsilon+r)^2/\rho^2}}\right) && (1.27)\\ &\leq O\left(\frac{\varepsilon+r}{\sqrt{\rho^2-(\varepsilon+r)^2}}\right) && (1.28)\\ &\leq O\left(\frac{\varepsilon+r}{\rho-\varepsilon-r}\right). && (1.29) \end{aligned}$$

We consider the points on the Voronoi cell boundaries: $m \in \partial Vr_{\mathcal{M}}(p)$, as shown in Fig. 3. According to Lemma 2.2, we have $\sin(\angle(pm, T_p)) \leq O(\varepsilon/\rho)$, since $\|p-m\| \leq \varepsilon$ from Lemma 1.1. When all angles are small, we have

$$\begin{aligned} &\sin(\angle(pm, \hat{T}_p)) && (1.30)\\ \leq& \sin(\angle(pm, T_p) + \angle(T_p, \hat{T}_p)) && (1.31)\\ \leq& \sin(\angle(pm, T_p)) + \sin(\angle(T_p, \hat{T}_p)) && (1.32)\\ \leq& \sin(\angle(pm, T_p)) + \angle(T_p, \hat{T}_p) && (1.33)\\ \leq& \frac{\varepsilon}{2\rho} + \frac{r}{\rho} && (1.34)\\ \leq& O(\frac{\varepsilon}{\rho}) && (1.35) \end{aligned}$$

holds since we have $r = 10\varepsilon$. Then we have the bound of the distance from m to $\hat{T}_p$:

$$d(m, \hat{T}_p) \leq \varepsilon \cdot \sin(\angle(pm, \hat{T}_p)) \leq O(\varepsilon^2/\rho). \quad (1.36)$$

Suppose $q \in P_\delta$, $q \neq p$ is the influencing point for $m \in \partial Vr_{\mathcal{M}}(p)$, i.e., $\|p-m\| = \|q-m\|$. By combining (1.29) with (1.36), we have the bound of the distance from m to the plane $L_{pq'}$:

$$\begin{aligned} d(m, L_{pq'}) &= \tan\theta \cdot d(m, \hat{T}_p) && (1.37)\\ &\leq O\left(\frac{\varepsilon+r}{\rho-\varepsilon-r}\right) \cdot O(\frac{\varepsilon^2}{\rho}) && (1.38)\\ &\leq O\left(\frac{\varepsilon^2(\varepsilon+r)}{\rho(\rho-\varepsilon-r)}\right). && (1.39) \end{aligned}$$

That is, $\exists c \in \mathbb{R}$, $c > 0$, $d(m, L_{pq'}) \leq c \cdot \frac{\varepsilon^2(\varepsilon+r)}{\rho(\rho-\varepsilon-r)}$.

Next we build 2 planes for each $q \in P_\delta$, $q \neq p$: L_{pq+} and L_{pq-} that satisfy $L_{pq+} \parallel L_{p\hat{\Pi}(q)} \parallel L_{pq-}$, and

$$d(L_{pq+}, L_{pq'}) = d(L_{pq-}, L_{pq'}) \quad (1.40)$$

$$= c \cdot \frac{\varepsilon^2(\varepsilon + r)}{\rho(\rho - \varepsilon - r)}, \quad (1.41)$$

$$d(p, L_{pq-}) \leq d(p, L_{pq'}) \leq d(p, L_{pq+}). \quad (1.42)$$

Since $L_{p\hat{\Pi}(q)} \perp \hat{T}_p$, we know that $L_{pq+} \perp \hat{T}_p$, $L_{pq-} \perp \hat{T}_p$. As shown in Fig. 3, we also have

$$d(p, L_{pq'}) = \frac{\|p - q\|}{2\cos\theta} \quad (1.43)$$

As shown in Fig. 3, we build the points q', q^+, q^- according to $L_{pq'}$, L_{pq+} and L_{pq-}, so that these planes are the bisecting planes between the point-pairs $\{p, q'\}$, $\{p, q^+\}$ and $\{p, q^-\}$, respectively.

It is obvious that q', q^+, q^- reside on the same line of $p\hat{\Pi}(q)$. Then we can build the Voronoi diagrams over $\hat{T}_p$ with points $\{p\} \cup \{q^+\}$, $\{p\} \cup \{q'\}$ and $\{p\} \cup \{q^-\}$. Denote the Voronoi cell of p of these diagrams as $Vr^+(p)$, $Vr'(p)$ and $Vr^-(p)$. According to Lemma 1.2 and Lemma 1.4, we can ignore other points $q \notin P_\delta$ without affecting these Voronoi cells for p. Thus we will only use points $q \in P_\delta$, $q \neq p$.

Since the point cloud P is an $(\varepsilon, s\varepsilon)$-sample of $\mathcal{M}$, we have $s\varepsilon \leq \|p - q\| \leq 10\varepsilon$. When ε is small enough, we always have $c \cdot \frac{\varepsilon^2(\varepsilon+r)}{\rho(\rho-\varepsilon-r)} < O(\varepsilon) \leq \frac{\|p-q\|}{2\cos\theta}$. Thus p will not stay in between L_{pq-} and L_{pq+}. Then we can have:

$$\|p - \hat{\Pi}(q)\| = \|p - q\| \cdot \cos\theta, \quad (1.44)$$

$$\|p - q'\| = 2 \cdot d(p, L_{pq'}) = \|p - q\|/\cos\theta, \quad (1.45)$$

$$\|p - q^+\| = \|p - q\|/\cos\theta + 2 \cdot c \cdot \frac{\varepsilon^2(\varepsilon + r)}{\rho(\rho - \varepsilon - r)}, \quad (1.46)$$

$$\|p - q^-\| = \|p - q\|/\cos\theta - 2 \cdot c \cdot \frac{\varepsilon^2(\varepsilon + r)}{\rho(\rho - \varepsilon - r)}. \quad (1.47)$$

For $\forall m \in \partial Vr_{\mathcal{M}}(p) \cap L_{pq}$ and its corresponding influencing point q, we have

$$d(m, L_{pq'}) \leq d(L_{pq-}, L_{pq'}), \quad (1.48)$$

$$d(m, L_{pq'}) \leq d(L_{pq+}, L_{pq'}), \quad (1.49)$$

$$\|m - p\| = \|m - q\|. \quad (1.50)$$

So we know that m stays in between L_{pq-} and L_{pq+} of point q. This can lead to:

$$\|m - p\| \leq \|m - q^+\|, \quad (1.51)$$

$$\|m - p\| \geq \|m - q^-\|. \quad (1.52)$$

Recall that here q is the influencing point of $m \in \partial Vr_{\mathcal{M}}(p)$. Since $\hat{\Pi}$ is bijective projection, it is obvious that $\hat{\Pi}(\partial Vr_{\mathcal{M}}(p)) = \partial\hat{\Pi}(Vr_{\mathcal{M}}(p))$. So we have $\forall \hat{m} \in \partial\hat{\Pi}(Vr_{\mathcal{M}}(p))$, $\exists q \in P_\delta$, such that $\|\hat{m} - p\| \geq \|\hat{m} - q^-\|$. This means that $\partial\hat{\Pi}(Vr_{\mathcal{M}}(p)) \cap (Vr^-(p) - \partial Vr^-(p)) = \emptyset$.

$$\because p \in Vr^-(p), p \in \hat{\Pi}(Vr_{\mathcal{M}}(p)),$$

$$\therefore Vr^-(p) \subseteq \hat{\Pi}(Vr_{\mathcal{M}}(p)). \quad (1.53)$$

Let us assume that $\hat{\Pi}(Vr_{\mathcal{M}}(p)) \subseteq Vr^+(p)$ does not hold, then $\exists m \in (Vr_{\mathcal{M}}(p))$, such that $\|\hat{\Pi}(m) - p\| > \|\hat{\Pi}(m) - q^+\|$ for some $q \in P_\delta$. As shown in Fig. 3, m resides in the p-side of plane L_{pq}. Since m also resides on the q^+-side of plane L_{pq+}, we have $\min d(m, \hat{T}_p) > d(L_{pq'}, L_{pq+}) \cdot \cot\theta$. Recall that we construct L_{pq+} so that $d(L_{pq'}, L_{pq+}) \geq \max d(m', \hat{T}_p) \cdot \tan\theta$ for $\forall m' \in Vr_{\mathcal{M}}(p)$. Thus we have $\min d(m, \hat{T}_p) > \max d(m', \hat{T}_p)$ for $\forall m' \in Vr_{\mathcal{M}}(p)$. Thus we have $m \notin Vr_{\mathcal{M}}(p)$, which is contradictory to the assumption. So we have $\hat{\Pi}(Vr_{\mathcal{M}}(p)) \subseteq Vr^+(p)$ holds.

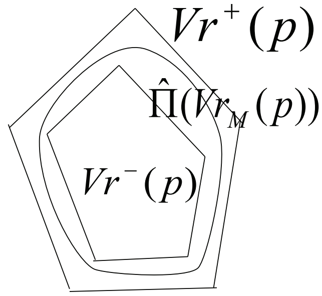

Fig. 4. The nestling of $Vr^-(p)$, $Vr^+(p)$, and $\hat{\Pi}(Vr_{\mathcal{M}}(p))$.

As shown in Fig. 4, we have

$$Vr^-(p) \subseteq \hat{\Pi}(Vr_{\mathcal{M}}(p)) \subseteq Vr^+(p), \quad (1.54)$$

which means that:

$$\mathrm{vol}(Vr^-(p)) \leq \mathrm{vol}(\hat{\Pi}(Vr_{\mathcal{M}}(p))) \leq \mathrm{vol}(Vr^+(p)). \quad (1.55)$$

Since we have $s\varepsilon \leq \|p - q\| \leq r = 10\varepsilon$, by combining equations (1.44), (1.45), (1.46), and (1.47), we can have

$$\frac{\|p - q^+\|}{\|p - q'\|} \leq 1 + O(\varepsilon^2/\rho^2), \quad (1.56)$$

$$\frac{\|p - q^-\|}{\|p - q'\|} \geq 1 - O(\varepsilon^2/\rho^2), \quad (1.57)$$

$$\frac{\|p - q'\|}{\|p - \hat{\Pi}(q)\|} = \frac{1}{\cos^2\theta} \leq 1 + O(\varepsilon^2/\rho^2). \quad (1.58)$$

We get the above last equation from Lemma 1.2 and similar calculation as in equation (1.35). According to Lemma 3.3 (in appendix section 3), we have

$$\left\|\frac{\mathrm{vol}(Vr^+(p))}{\mathrm{vol}(Vr'(p))} - 1\right\| \leq O(\varepsilon^2/\rho^2), \quad (1.59)$$

$$\left\|\frac{\mathrm{vol}(Vr^-(p))}{\mathrm{vol}(Vr'(p))} - 1\right\| \leq O(\varepsilon^2/\rho^2), \quad (1.60)$$

$$\left\|\frac{\mathrm{vol}(Vr'(p))}{\mathrm{vol}(Vr_{\hat{T}}(p))} - 1\right\| \leq O(\varepsilon^2/\rho^2). \quad (1.61)$$

Finally we have

$$\left\| \frac{\mathrm{vol}(\hat{\Pi}(Vr_{\mathcal{M}}(p)))}{\mathrm{vol}(Vr_{\hat{T}}(p))} - 1 \right\| \leq O(\varepsilon^2/\rho^2). \quad (1.62)$$

By combining equation (1.62) with Lemma 1.5, we have

$$\left\| \frac{\mathrm{vol}(Vr_{\mathcal{M}}(p))}{\mathrm{vol}(Vr_{\hat{T}}(p))} - 1 \right\| \leq O(\varepsilon^2/\rho^2). \quad (1.63)$$

□

1.2 Proof of Theorem 4.2: Convergence of Integration Approximation

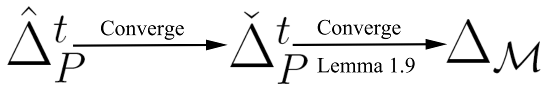

Fig. 5. Proof structure of Theorem 4.2.

As shown in figure 5, the proof of Theorem 4.2 is organized as follows: An intermediate discrete LBO $\check{\Delta}_P^t$ is defined first in equation (1.64). $\check{\Delta}_P^t$ is the approximation result by computing the integration directly over the manifold. In Lemma 1.9 we show that $\check{\Delta}_P^t$ is converging to $\Delta_{\mathcal{M}}$. With the Voronoi cell area convergence result in Theorem 4.1, we then show that our discrete LBO $\hat{\Delta}_P^t$ (with Voronoi cell area estimated on the tangent plane) is converging to $\check{\Delta}_P^t$, which means that $\hat{\Delta}_P^t$ is converging to $\Delta_{\mathcal{M}}$ as well.

Recall that in our algorithm, the approximation LBO $\hat{\Delta}_P^t$ is defined as:

$$\hat{\Delta}_P^t f(p) = \frac{1}{4\pi t^2} \sum_{q \in P_\delta} (e^{-\frac{\|q-p\|^2}{4t}} (f(q) - f(p)) \, \mathrm{vol}(Vr_{\hat{T}}(q))),$$

where $Vr_{\hat{T}}(q)$ is the Voronoi cell of the point q over $\hat{T}_q$. In order to prove that $\hat{\Delta}_P^t$ is converging to the LBO $\Delta_{\mathcal{M}}$, we introduce the following intermediate LBO:

$$\check{\Delta}_P^t f(p) = \frac{1}{4\pi t^2} \sum_{q \in P_\delta} (e^{-\frac{\|q-p\|^2}{4t}} (f(q) - f(p)) \, \mathrm{vol}(Vr_{\mathcal{M}}(q))). \quad (1.64)$$

It is obvious that the only difference between $\hat{\Delta}_P^t$ and $\check{\Delta}_P^t$ is that we use $Vr_{\hat{T}}(q)$ instead of $Vr_{\mathcal{M}}(q)$ as in (1.64), because it is impossible to get $\mathrm{vol}(Vr_{\mathcal{M}}(q))$ in most real applications.

***Definition** 1.6:* For $\forall p \in P$, recall we have $P_\delta = P \cap B(p, \delta)$. Define

$$N_p = \mathcal{M} \cap B(p, \delta) \quad (1.65)$$
$$N_{Vp} = \cup_{q \in P_\delta} Vr_{\mathcal{M}}(q) \quad (1.66)$$
$$N_p^+ = \mathcal{M} \cap B(p, 1.1\delta) \quad (1.67)$$
$$N_p^- = \mathcal{M} \cap B(p, 0.9\delta) \quad (1.68)$$

given $\delta \geq 10\varepsilon$.

According to Lemma 2.5, the continuous LBO $\Delta_{\mathcal{M}}$ can be computed as the integration over the whole manifold $\mathcal{M}$. Our essential idea is to approximate such integration locally over N_{Vp} instead. And the following Lemma 1.8 shows that such local approximation is reasonable, which can lead to the convergence of $\check{\Delta}_P^t$ to $\Delta_{\mathcal{M}}$, as shown in Lemma 1.9. In order to prove Lemma 1.8, we need to show that N_{Vp} is bounded in between N_p^- and N_p^+ which are independent of the sampling size ε, which is addressed in the following Lemma 1.7.

***Lemma** 1.7 (Bound of N_{Vp}):*

$$N_p^- \subseteq N_{Vp} \subseteq N_p^+. \quad (1.69)$$

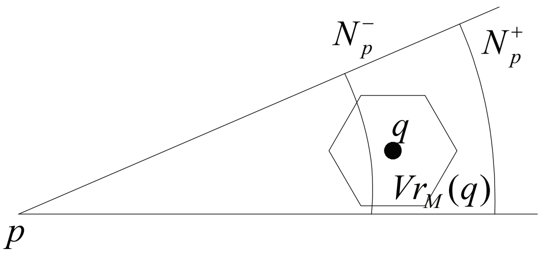

Fig. 6. The nestling of N_p^-, N_p^+, and N_{Vp}.

Proof: This nestling relationship is shown in Figure 6.

First we prove $N_p^- \subseteq N_{Vp}$: For $\forall m \in N_p^-$, there exists $\exists q \in P$, such that $m \in Vr_{\mathcal{M}}(q)$. According to Lemma 1.1, we have $\|q - m\| \leq \varepsilon$.

$\because \|p - q\| \leq \|p - m\| + \|q - m\| \leq 0.9\delta + \varepsilon \leq \delta$,

$\therefore q \in P_\delta$, $Vr_{\mathcal{M}}(q) \subseteq N_{Vp}$, $m \in N_{Vp}$.

Next we prove $N_{Vp} \subseteq N_p^+$: For $\forall m \in N_{Vp}$, there exists $\exists q \in P_\delta$, $m \in Vr_{\mathcal{M}}(q)$. Recall that $\|q - m\| \leq \varepsilon$ and $\|p - q\| \leq \delta$, so we have $\|p - m\| \leq \|p - q\| + \|q - m\| \leq \delta + \varepsilon \leq 1.1\delta$. Thus $m \in N_p^+$. □

***Lemma** 1.8 (Approximation using N_{Vp}):*

$$\int_{N_{Vp}} e^{-\frac{\|p-y\|}{4t}} f(y) d\mu_y - \int_{\mathcal{M}} e^{-\frac{\|p-y\|}{4t}} f(y) d\mu_y = o(t^l), \quad (1.70)$$

for any positive natural number l.

Proof: Similar to the proof of Lemma 2.4(in this material), which is Lemma 1 in [4]:

$$\left| \int_{N_{Vp}} e^{-\frac{\|p-y\|}{4t}} f(y) d\mu_y - \int_{\mathcal{M}} e^{-\frac{\|p-y\|}{4t}} f(y) d\mu_y \right|$$
$$= \left| \int_{\mathcal{M} - N_{Vp}} e^{-\frac{\|p-y\|}{4t}} f(y) d\mu_y \right|$$
$$\leq \mathrm{vol}(\mathcal{M}) \sup_{x \in \mathcal{M}, x \notin N_{Vp}} (|f(x)|) e^{-\frac{d_1^2}{4t}} \quad (1.71)$$
$$\leq \mathrm{vol}(\mathcal{M}) \sup_{x \in \mathcal{M}, x \notin N_p^-} (|f(x)|) e^{-\frac{d_2^2}{4t}} = o(t^l), \quad (1.72)$$

where $d_1 = \inf_{x \notin N_{Vp}} \|p - x\|$, and $d_2 = \inf_{x \notin N_p^-} \|p - x\|$. □

Lemma *1.9 (Convergence of $\check{\Delta}_P^t$ to $\Delta_{\mathcal{M}}$):*

$$\lim_{\varepsilon\to 0}\|\check{\Delta}_P^t f-\Delta_{\mathcal{M}}f\|_\infty=0, \tag{1.73}$$

where $t(\varepsilon)=\varepsilon^{\frac{1}{2+\xi}}$, and $\xi>0$ is any positive fixed number.

Proof: Note that $\varepsilon=t^{2+\xi}$. According to Lemma 1.1, we know $\forall y\in Vr_{\mathcal{M}}(q)$, $\|y-q\|\le\varepsilon$ or $\|y-q\|\le O(\varepsilon)$. According to Lemma 2.5, we can approximate $\Delta_{\mathcal{M}}$ using integration over $\mathcal{M}$. Thus we have:

$$\lim_{t\to 0}\left|\check{\Delta}_P^t f(p)-\int_{N_{V_p}}e^{-\frac{\|p-y\|^2}{4t}}(f(y)-f(p))d\mu_y\right| \tag{1.74}$$

$$=\lim_{t\to 0}|\sum_{q\in P_\delta}\int_{Vr_{\mathcal{M}}(q)}\frac{1}{4\pi t^2}[e^{-\frac{\|q-p\|^2}{4t}}(f(q)-f(p)) - e^{-\frac{\|y-p\|^2}{4t}}(f(y)-f(p))]d\mu_y| \tag{1.75}$$

$$=\lim_{t\to 0}|\sum_{q\in P_\delta}\int_{Vr_{\mathcal{M}}(q)}\frac{1}{4\pi t^2}[e^{-\frac{\|q-p\|^2}{4t}}(f(q)-f(p))-e^{-\frac{\|q-p\|^2}{4t}}(f(y)-f(p)) + e^{-\frac{\|q-p\|^2}{4t}}(f(y)-f(p))-e^{-\frac{\|y-p\|^2}{4t}}(f(y)-f(p))]d\mu_y| \tag{1.76}$$

$$=\lim_{t\to 0}|\sum_{q\in P_\delta}\int_{Vr_{\mathcal{M}}(q)}\frac{1}{4\pi t^2}e^{-\frac{\|y-p\|^2}{4t}}[e^{\frac{\|y-p\|^2-\|q-p\|^2}{4t}}(f(q)-f(y)) + (e^{\frac{\|y-p\|^2-\|q-p\|^2}{4t}}-1)(f(y)-f(p))]d\mu_y| \tag{1.77}$$

$$\le\lim_{t\to 0}\sum_{q\in P_\delta}\int_{Vr_{\mathcal{M}}(q)}\frac{1}{4\pi t^2}e^{-\frac{\|y-p\|^2}{4t}}[|f(y)-f(p)|\cdot \left|e^{-\frac{O(\varepsilon)}{4t}}-1\right|+O(\varepsilon)]d\mu_y \tag{1.78}$$

$$\le\lim_{t\to 0}\frac{(f_{\max,\mathcal{M}}-f_{\min,\mathcal{M}})\cdot\left|e^{-\frac{O(\varepsilon)}{4t}}-1\right|+O(\varepsilon)}{t} \cdot\int_{N_{V_p}}\frac{1}{4\pi t}e^{-\frac{\|y-p\|^2}{4t}}d\mu_y \tag{1.79}$$

$$\le\lim_{t\to 0}\frac{(f_{\max,\mathcal{M}}-f_{\min,\mathcal{M}})\left|e^{-\frac{O(\varepsilon)}{4t}}-1\right|+O(\varepsilon)}{t} \cdot\text{Constant} \tag{1.80}$$

$$\le\lim_{t\to 0}\frac{O(\varepsilon/t)+O(\varepsilon)}{t} \tag{1.81}$$

$$=\lim_{t\to 0}\frac{O(t^{1+\xi})+O(t^{2+\xi})}{t}=0 \tag{1.82}$$

where $f_{\max,\mathcal{M}}$ and $f_{\min,\mathcal{M}}$ stands for the maximum and minimum of function f on manifold $\mathcal{M}$. In the above derivation, we applied $\|q-p\|-\|y-q\|\le\|y-p\|\le\|q-p\|+\|y-q\|$, $\|y-q\|\le\varepsilon$, $|f(y)-f(q)|=O(|y-q|)$ (since $f\in C^2$) and following inequality on equation (1.77) to get equation (1.78).

$$\left|\|y-p\|^2-\|q-p\|^2\right| \tag{1.83}$$
$$=|(\|y-p\|+\|q-p\|)\cdot(\|y-p\|-\|q-p\|)| \tag{1.84}$$
$$\le(\|y-p\|+\|q-p\|)\cdot\varepsilon \tag{1.85}$$
$$\le(2\|q-p\|+\|y-q\|)\cdot\varepsilon \tag{1.86}$$
$$\le(2\|q-p\|+\varepsilon)\cdot\varepsilon \tag{1.87}$$
$$\le 2\delta\cdot\varepsilon+\varepsilon^2\le O(\varepsilon). \tag{1.88}$$

By applying the following inequality to equation (1.79), we get equation (1.80).

$$\int_{N_{V_p}}\frac{1}{4\pi t}e^{-\frac{\|y-p\|^2}{4t}}d\mu_y \tag{1.89}$$
$$=\int_{\hat{\Pi}(N_{V_p})}\frac{1}{4\pi t}e^{-\frac{\|\hat{\Phi}(y)-p\|^2}{4t}}|J(\hat{\Phi})|_y d\mu_y \tag{1.90}$$
$$\le\int_{\hat{\Pi}(N_{V_p})}\frac{1}{4\pi t}e^{-\frac{\|y-p\|^2}{4t}}|J(\hat{\Phi})|_y d\mu_y \tag{1.91}$$
$$\le\max_{y\in\hat{\Pi}(N_p^+)}(|J(\hat{\Phi})|_y)\int_{\hat{T}_p}\frac{1}{4\pi t}e^{-\frac{\|y-p\|^2}{4t}}d\mu_y \tag{1.92}$$
$$\le\max_{y\in\hat{\Pi}(N_p^+)}(|J(\hat{\Phi})|_y)=\text{Constant}. \tag{1.93}$$

Here J stands for the Jacobian Matrix.

By combining Lemma 2.5 and Lemma 1.8 (in this material) with equation (1.82), we have $\lim_{\varepsilon\to 0}\|\check{\Delta}_P^t f-\Delta_{\mathcal{M}}f\|_\infty=0$ proved. □

With Lemma 1.9, we can prove Theorem 4.2 as follows:

Proof:

According to Theorem 4.1, we have

$$\left\|\frac{\mathrm{vol}(Vr_{\mathcal{M}}(p))}{\mathrm{vol}(Vr_{\hat{T}}(p))}-1\right\|=O(\frac{\varepsilon^2}{\rho^2}). \tag{1.94}$$

Thus:

$$\left\|\frac{\check{\Delta}_P^t f(p)}{\hat{\Delta}_P^t f(p)}-1\right\|=O(\frac{\varepsilon^2}{\rho^2}), \tag{1.95}$$

which means:

$$\lim_{t\to 0}\|\check{\Delta}_P^t f(p)-\hat{\Delta}_P^t f(p)\|_\infty=0. \tag{1.96}$$

By combining (1.96) with Lemma 1.9, we have this theorem proved. □

2 Referred Lemmas

This appendix section shows the Lemmas that we referred from other papers. These Lemmas are used in our proof of convergence in appendix section 1.

Lemma *2.1 (Lemma 3.1 in [5]):* Given two points $p,q\in\mathcal{M}$ with $\|p-q\|\le\rho/3$, the angle between their normals n_p and n_q satisfies $\angle(n_p,n_q)<\frac{\|p-q\|}{\rho-\|p-q\|}$.

Lemma *2.2 (Lemma 6 in [6]):* For any point $p,q\in\mathcal{M}$ with $\|p-q\|<\rho$, we have that $\sin\angle(pq,T_p)\le\frac{\|q-p\|}{2\rho}$, and the distance from q to T_p is bounded by $\frac{\|q-p\|^2}{2\rho}$, where ρ is the local feature size of p, T_p is the tangent plane at p.

***Lemma** 2.3 (Theorem 3.2 in [1]):* Suppose P is an ε-sample of $\mathcal{M}$. For $p \in P$ with local feature size ρ and real tangent plane T_p. Compute $\hat{T}_p$ as in Algorithm PCDLaplace [1], $\angle(T_p, \hat{T}_p) = O(r/\rho)$ for $r < \rho/2$ and $r \geq 10\varepsilon$.

***Lemma** 2.4 (Lemma 1 in [4]):* Given any openset $B \subset \mathcal{M}$, $p \in B$, for any positive natural number l,

$$\int_{B \subset M} e^{-\frac{\|p-y\|}{4t}} f(y) d\mu_y - \int_M e^{-\frac{\|p-y\|}{4t}} f(y) d\mu_y = o(t^l).$$

***Lemma** 2.5 (Lemma 5 in [4]):*

$$\Delta_{\mathcal{M}} f(p) = \lim_{t \to 0} \frac{1}{4\pi t^2} \Big(\int_{\mathcal{M}} e^{-\frac{\|p-y\|^2}{4t}} f(p) d\mu_y - \int_{\mathcal{M}} e^{-\frac{\|p-y\|^2}{4t}} f(y) d\mu_y \Big).$$

3 Lemmas about Voronoi Cells

In this paper we are using the Voronoi cells $Vr_{\hat{T}}(p)$ on the estimated tangent planes $\hat{T}_p$. This appendix section shows some results that are related to Voronoi cells over 2-planes and are used for our convergence proof of Theorem 4.1.

***Lemma** 3.1:* For plane L and point set $P \subset L$, consider the Voronoi diagram of P over L. Let $p \in P$. Suppose the Voronoi cell of p is $Vr(p)$ and the cell boundary is $\partial Vr(p)$. $\forall \tilde{p} \in \partial Vr(p)$, $\|p - \tilde{p}\| \leq \varepsilon$.

If we fix point p and move all the other points $q \in P$ as: $q' = p + (q - p) \cdot t$, $t > 0$, Then the area of the new Voronoi cell $Vr'(p)$ has such property:

$$\frac{\text{vol}(Vr'(p))}{\text{vol}(Vr(p))} = t^2. \tag{3.1}$$

Proof:

Suppose the plane L is parameterized in (u, v) coordinates with p being the origin. For point $q_i \in P$ and corresponding displaced point q_i', we have their coordinate relationship: $(u_i', v_i') = t(u_i, v_i)$.

So we can build the mapping $f : L \to L$ as $m' = f(m) = f(u, v) = (tu, tv)$, where $m, m' \in L$. Thus we have

$$\forall m \in L, \|m' - p\| = t\|m - p\|,$$
$$\forall m \in L, \|m' - q_i\| = t\|m - q_i\|.$$

It's obvious that $m \in Vr(p) \leftrightarrow m' = f(m) \in Vr'(p)$. Then we have

$$\text{vol}(Vr'(p)) = \int_{Vr'(p)} du'dv' = \int_{Vr(p)} t^2 du dv = t^2 \, \text{vol}(Vr(p)).$$

□

***Lemma** 3.2:* Consider plane L and point set $P \subset L$, as defined in Lemma 3.1. If we move one point $q \in P$ as: $q' = p + (q - p) \cdot t$, $0 < t < 1$, then the area of the new Voronoi cell $Vr'(p)$ has such property:

$$\text{vol}(Vr'(p)) \leq \text{vol}(Vr(p)). \tag{3.2}$$

Proof:

Suppose l_{pq} and $l_{pq'}$ are the bisecting planes between the point-pairs $\{p, q\}$ and $\{p, q'\}$. $\forall m \in Vr'(p)$, we have $\|m - p\| < \|m - q'\|$. That is, m resides on the p-side of $l_{pq'}$. As shown in figure 7, it is obvious that $l_{pq'}$ resides on the p-side of l_{pq}. Thus we have $m \in Vr(p)$.

Thus $Vr'(p) \subseteq Vr(p)$. □

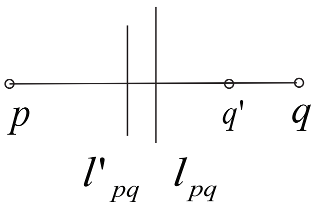

Fig. 7. Moving a point in the Voronoi diagram.

***Lemma** 3.3:* Consider plane L and point set $P \subset L$, as defined in Lemma 3.1. If we fix point p and move all the other points $q \in P$ as: $q' = p + (q - p) \cdot t_q$, $t_q > 0$, then we have the following result about the new Voronoi cell $Vr'(p)$:

$$(\min(t_q))^2 \leq \frac{\text{vol}(Vr'(p))}{\text{vol}(Vr(p))} \leq (\max(t_q))^2. \tag{3.3}$$

Proof: From Lemma 3.2, we know that the area of $Vr'(p)$ will change monotonically with t_q. So combine it with Lemma 3.1 we can get this lemma proved. □

4 Lemmas about Small Angles

This appendix section shows the Lemmas about small angles that we used for the equation (1.22) in the proof of Theorem 4.1.

***Lemma** 4.1 (Angles of 3 Planes):* Consider 3 planes T_1, T_2 and T_3 with their corresponding unit normal vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$. Denote $\angle(\mathbf{n}_1, \mathbf{n}_2) = \alpha$, $\angle(\mathbf{n}_2, \mathbf{n}_3) = \beta$, $\angle(\mathbf{n}_1, \mathbf{n}_3) = \gamma$. Without any loss of generality, assume α, β, γ are all acute angles.

If $\alpha < \pi/4$, $\beta < \pi/4$, then we have $\gamma \leq \alpha + \beta$ holds.

Proof:

Since we are observing 3 unit vectors, it's convenient to put them on unit sphere S, as shown in figure 8 (left). Consider the geodesic distance $g(\mathbf{n}_1, \mathbf{n}_2)$, $g(\mathbf{n}_2, \mathbf{n}_3)$ and $g(\mathbf{n}_1, \mathbf{n}_3)$ on S. It's obvious that all these geodesics are part of great circles of S. From the definition of geodesic distance, we know that $g(\mathbf{n}_1, \mathbf{n}_3) \leq g(\mathbf{n}_1, \mathbf{n}_2) + g(\mathbf{n}_2, \mathbf{n}_3)$. Since S is unit sphere, we also have $g(\mathbf{n}_1, \mathbf{n}_2) = \alpha$, $g(\mathbf{n}_2, \mathbf{n}_3) = \beta$, $g(\mathbf{n}_1, \mathbf{n}_3) = \gamma$. So we have $\gamma \leq \alpha + \beta$ holds. □

***Lemma** 4.2 (Angles of 2 Planes and 1 Vector):* Consider 2 planes T_1 and T_2 with their corresponding unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$. Consider another unit vector $\mathbf{n}_3$. Denote $\angle(\mathbf{n}_1, \mathbf{n}_2) = \alpha$, $\angle(T_2, \mathbf{n}_3) = \beta$, $\angle(T_1, \mathbf{n}_3) = \gamma$.

If $\alpha < \pi/4$, $\beta < \pi/4$, then we have $\gamma \leq \alpha + \beta$ holds.

Proof:

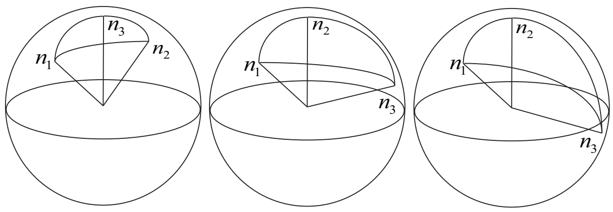

Fig. 8. Unit vectors on the unit sphere: Lemma 4.1 (left) and Lemma 4.2 (middle and right).

Similar to the proof of Lemma 4.1, we put all vectors on the unit sphere S. Select proper $\mathbf{n}_1$ and $\mathbf{n}_2$ directions to ensure that $\mathbf{n}_1 \cdot \mathbf{n}_2 \geq 0$.

In case $\mathbf{n}_2 \cdot \mathbf{n}_3 \geq 0$, as shown in figure 8 (middle), we have $\angle(\mathbf{n}_2, \mathbf{n}_3) = \pi/2 - \beta$. From Lemma 4.1, we know that:

$\angle(\mathbf{n}_1, \mathbf{n}_3) \leq \angle(\mathbf{n}_1, \mathbf{n}_2) + \angle(\mathbf{n}_2, \mathbf{n}_3) = \pi/2 + \alpha - \beta$,

$\angle(\mathbf{n}_1, \mathbf{n}_3) \geq \angle(\mathbf{n}_2, \mathbf{n}_3) - \angle(\mathbf{n}_1, \mathbf{n}_2) = \pi/2 - \alpha - \beta$.

Thus we have $\gamma = \angle(T_1, \mathbf{n}_3) \leq \alpha + \beta$ holds.

In case $\mathbf{n}_2 \cdot \mathbf{n}_3 \leq 0$, as shown in figure 8 (right), we have $\angle(\mathbf{n}_2, \mathbf{n}_3) = \pi/2 + \beta$. Similarly we also have $\gamma = \angle(T_1, \mathbf{n}_3) \leq \alpha + \beta$. □

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