An Analysis of the Methods Used to Predict the Flight Path of Malaysia Airlines Flight 370

John Zweck

Department of Mathematical Sciences
The University of Texas at Dallas

http://www.utdallas.edu/~zweck/
zweck@utdallas.edu

TAMC, Nov 3, 2014
Malaysia Airlines flight 370 disappeared on March 8 en route to Beijing from Kuala Lumpur.

To date there is no confirmed debris from the aircraft and no survivors have been found.

An international search effort is trying to provide some resolution for the family members of the 239 passengers, crew, and pilots of MH370.
Search efforts have been terminated in the Gulf of Thailand and the South China Sea, near where the plane first disappeared from civilian radar.
Last known possible position of MH370 based on satellite data (somewhere on red lines)
Where did MH370 go?

- The Malaysian Govt set up a **Working Group** of experts to determine the flight path of MH370.
- Mar 17: Search refocused to a region in the Indian Ocean, 3000 km S.W. of Perth.
- Mar 28: Search area shifted to a region 1500 km west of Perth.
- Apr 7: Possible black box signals detected 2300 km northwest of Perth. Area searched without success.
- June 26: Search area 1500 km west of Perth refined.
- Oct 8: Search area moved 800 km further SW.

**Why was the search area changed five times?**
A wide range of proposed search areas!
Goal: Explain the math!
Overview of Radar and Satellite Data

- Radar data suggest that the flight path of MH370 was very erratic during the first 2-3 hours with
  - sharp turns,
  - large changes in speed and elevation.

- After last radar contact, the Inmarsat 3-F1 satellite “pinged” the aircraft 7 times over a 6 hour period.

- At each ping time, the engineers could determine
  - an arc of possible locations of the aircraft,
  - the **burst frequency offset [BFO]** which is a quantity related to the aircraft-satellite Doppler shift.
Ping Arcs

1Derived from May 27 data release and June 26 ATSB report.
The time that it takes the signal to be sent and received, via the satellite, to the ground station can be used to establish the distance of the aircraft from the satellite.

All points on the red circle are at the same distance from the satellite as is the aircraft.
MH370 measured data against predicted tracks

MH370: Burst Frequency Offset Analysis (450 knots)

- Possible turn
- Take-off
The Math Problem

- Determine flight path from
  - Ping times and ping arc angles
  - BFO/Doppler data
  - Position on first arc

- Requires a **flight path model:**
  1. Known-speed, concatenation-of-geodesics model
  2. Unknown-speed, concatenation-of-geodesics model
  3. Concatenation of small circles, BFO model
Why was the search area changed?

1. Use of BFO in addition to ping arc data
2. More sophisticated engineering analysis of data
3. Improvements to math models and algorithms
Modeling Assumptions

1. The earth is a perfect sphere.
2. Aircraft has constant elevation and ground speed.
3. Input parameters:
   - Aircraft initial position,
   - Aircraft speed.
4. Flight path is a concat\(^n\) of great circle segments.
5. The satellite is in a geostationary orbit.
6. Doppler shift data was not used.
Rotate great circle path about initial position to reach next ping arc at next ping time.
1. **Spherical satellite coordinates**, $(\Theta, \Phi)$, of $y$:

$$y = y(\Theta, \Phi) = \cos \Phi i_S + \cos \Theta \sin \Phi j_S + \sin \Theta \sin \Phi k_S.$$

2. Ping arcs are circles of latitude, $\Phi = \Phi_n$ (known).

3. Initial Position on 1st ping arc, $\Theta = \Theta_1$ (known).

4. Solve for $\Theta_2 \cdots \Theta_N$ in succession.
Great circle flight path with position, $\mathbf{r}_n$, and velocity, $\mathbf{v}_n$, at the time, $t_n$, is

$$\mathbf{r}_A(t) = \cos \left[ \frac{\mathbf{v}_n(t - t_n)}{R_A} \right] \mathbf{r}_n + \frac{R_A}{v_n} \sin \left[ \frac{\mathbf{v}_n(t - t_n)}{R_A} \right] \mathbf{v}_n.$$

Unknown **heading direction**, $\beta_n$, of the aircraft:

$$\mathbf{v}_n = v_n \cos \beta_n \mathbf{y}_\Theta + v_n \sin \beta_n \mathbf{y}_\Phi.$$
The Ping Arc Equation

1. Condition that aircraft crosses the \((n + 1)\)-st ping arc at time \(t_{n+1}\):

\[
r_A(t_{n+1}) = y(\Theta_{n+1}, \Phi_{n+1}).
\]

2. Two independent equations for \(\beta_n\) and \(\Theta_{n+1}\).

3. Eliminate \(\Theta_{n+1}\) to get

\[
\sin \beta_n = \frac{\cos \varphi_n \cos \left( \frac{\Delta t_n v_n}{R_A} \right) - \cos \varphi_{n+1}}{\sin \varphi_n \sin \left( \frac{\Delta t_n v_n}{R_A} \right)}.
\]
Constant Speed Flight Paths [MH370]

- 75° E
- 90° E
- 105° E
- 120° E
- 30° S
- 15° S
- 0°
- 15° N
- 325 knots
- 400 knots
- 450 knots

J. Zweck (UTD)
MH370 Math
TAMC 20 / 48
The search area for MH370 was moved 1000 km N.E. because radar data showed the plane had initially travelled faster and further west.
Positive and Negative Flight Paths

- If $\beta_n \neq \pm \pi/2$ there are two solutions, $\beta_n^\pm$:
  1. $\cos \beta_n^+ > 0$,
  2. $\cos \beta_n^- < 0$.

- For $n > 1$ you can often choose between $\beta_n^\pm$.

- Upshot: There are two plausible flight paths:
  1. **Positive**: $\beta_1 = \beta_1^+$,
  2. **Negative**: $\beta_1 = \beta_1^-$.
The problem is **symmetric** about the **great circle** containing the satellite and starting location.
Types of Satellite Orbits

1. **Geostationary orbit:** Satellite is located at a fixed point in the sky.
   - Doppler shifts for $+$ and $-$ paths are identical.

2. **Geosynchronous orbit:** Satellite returns to the same point in the sky at the same time each day.
   - The slight motion of INMARSAT 3-F1 satellite in the sky breaks the symmetry between $+$ and $-$ paths.
Geosynchronous Satellite Orbit

Data (blue) from aqqa.org

Model (red) used below [Formulae not shown].

J. Zweck (UTD)
Geosynchronous Satellite Orbit

Position error < 6 km. Velocity error < 0.5 km/hr.
(II) Concat$^n$-of-geodesics [unknown v]

Modeling Assumptions

1. Input parameter: Aircraft initial position.
2. Flight path is a concat$^n$ of great circle segments.
3. The satellite is in a geosynchronous orbit.
4. Doppler shift data is used.

Different great circle segments can have different a priori unknown speeds.
The Main Ideas in the Model

- What model to get from one ping arc to the next?
- Simplest choice is a constant speed great circle:

\[ r(t) = \cos \left( \frac{v_n(t - t_n)}{R_A} \right) r_n + \frac{R_A}{v_n} \sin \left( \frac{v_n(t - t_n)}{R_A} \right) v_n \]

- Given current location, \( r_n \) on \( n \)-th ping arc, determine the velocity vector, \( v_n \), that gets us to the next ping arc at the right time.
We need three constraints to determine $v_n \in \mathbb{R}^3$

1. Constant elevation constraint: $v_n \in T_{r_n}S^2$.
2. Ping arc traversal constraint: $F(v_n) = 0$.
3. The Doppler shift data gives a third constraint.
The Doppler Effect

- Suppose an electromagnetic signal that is sent by the aircraft to the satellite.

- If the aircraft is moving **towards** the satellite the light received will be **more blue** than the light sent.

- If the aircraft is **moving away**, the received light will be **more red**.

\[ \frac{\Delta f}{f} = \frac{-1}{c} \hat{u} \cdot \mathbf{v} \]

**Doppler Shift** \(\propto\) **Speed along line of sight**
Doppler Shift (Geostationary Case)

Doppler shift provides partial velocity information

\[ \frac{\Delta f}{f} = -\frac{R_S}{cD_{SA}} v_\Phi \sin \Phi \]

- \( R_S \) is distance from satellite to center of earth.
- \( \Phi \) is ping arc angle.
- \( D_{SA} = D_{SA}(\Phi) \) is distance from satellite to aircraft.
- \( v_\Phi \) is component of aircraft velocity \( \perp \) to ping arc.

\( \Delta f \) cannot distinguish between \( \pm \) tracks:

\( v_\Phi \) is same for both.
Doppler Shift (Geosynchronous Case)

\[
\frac{\Delta f}{f} = -\frac{R_s}{cD_{SA}} [v_\phi \sin \phi - \hat{u}(\phi, \Theta) \cdot v_S]
\]

- \(\hat{u}\) is unit vector in direction of aircraft from satellite
- \(v_S\) is satellite velocity w.r.t. frame rotating with earth

Because of \(\Theta\)-dependence

\(\Delta f\) can distinguish between \(\pm\) tracks!
The Moving Satellite Frame

1. Choose orthonormal basis (frame)

\[ \{ \hat{i}_n, \hat{j}_n, \hat{k}_n \} = \{ \mathbf{i}_S(t_n), \mathbf{j}_S(t_n), \mathbf{k}_S(t_n) \} \]

so that satellite position is

\[ \mathbf{r}_S(t_n) = R_S \mathbf{i}_S(t_n). \]

2. **Spherical satellite coordinates** of aircraft:

\[ \mathbf{r}_A(t_n) = R_A \mathbf{y}_n \]

\[ \mathbf{y}_n = \cos \phi_n \hat{i}_n + \cos \Theta_n \sin \phi_n \hat{j}_n + \sin \Theta_n \sin \phi_n \hat{k}_n. \]

3. Aircraft velocity in 2-D tangent space to sphere:

\[ \mathbf{v}_n = \mathbf{v}_A(t_n) = v_{\Theta,n} \mathbf{y}_\Theta + v_{\phi,n} \mathbf{y}_\phi. \]
Ping Arc Equation

1. Satellite frame: \( F_n = [\hat{i}_n \ \hat{j}_n \ \hat{k}_n] \in SO(3) \)

2. \( Y_n = F_n^T [y_n \ y_{\phi,n} \ y_{\theta,n}] \in SO(3) \)

3. Position of aircraft at \( t_{n+1} \) on great circle:

\[
\mathbf{r}_{n+1} = R_A F_n Y_n \mathbf{x}_n, \quad (1)
\]

where \( \mathbf{x}_n \) is the unit column vector

\[
\mathbf{x}_n = \frac{1}{v_n} \begin{bmatrix} v_n \cos \alpha_n & v_{\phi,n} \sin \alpha_n & v_{\theta,n} \sin \alpha_n \end{bmatrix}^T, \quad (2)
\]

with \( \alpha_n = v_n \Delta t_n / R_A \).
1. For aircraft to cross the next ping arc at $t_{n+1}$ need

$$r_{n+1} = RA_{n+1}(Y_{n+1}) \ast 1.$$  

2. So, the ping arc equation is

$$x_n = A_n(Y_{n+1}) \ast 1, \quad \text{where } A_n = Y_n^T F_n^T F_{n+1} \in SO(3).$$

3. Two independent equations in $v_{\Phi,n}$, $v_{\Theta,n}$, and $\Theta_{n+1}$.

4. Eliminate $\Theta_{n+1}$ to obtain

$$f_{Ping}(v_{\Phi,n}, v_{\Theta,n}) = 0.$$
Summary for Model II

For exact data

1. Given $r_n$ on current ping arc find $(v_\phi,n, v_\Theta,n)$ so that great circle path ends at $r_{n+1}$ on next ping arc.
2. Solve Doppler Shift Equation for $v_\phi,n$.
3. Solve Ping Arc Equation for $v_\Theta,n$.

For uncertain data

Numerically minimize

$$F(v_\phi,n, v_\Theta,n) = f_{DS,1}^2(v_\phi,n) + f_{Ping}^2(v_\phi,n, v_\Theta,n) + f_{DS,2}^2(v_\phi,n, v_\Theta,n)$$

using unconstrained, gradient-based optimization.
Great circle scenario:
Constant aircraft heading relative to the air.

BUT

Due to wind, path of aircraft over earth may not have exactly been a great circle.

Modeling Assumptions

1. Flight path is a concat^n of small circle segments.²
2. BFO data is used.

²A small circle is any circle on a sphere.
Burst Frequency Offset

\[ \text{BFO} = f_{\text{Received}} - f_{\text{Design}} \]

\[ \text{BFO} = \Delta f_{\text{AS}} - \Delta f_{\text{AS-Comp}} + \Delta f_{\text{SG}} + \delta f_{\text{AI}} + \delta f_{\text{Bias}}. \]

- \( \Delta f_{\text{AS}} \) = Aircraft-Satellite Doppler shift (DS)
- \( \Delta f_{\text{AS-Comp}} \) = Aircraft-Geostationary Satellite DS
- \( \Delta f_{\text{SG}} \) = Satellite-Ground Station DS
- \( \delta f_{\text{AI}} = \delta f_{\text{AI}}(t) \) = Aircraft-Independent frequency shift
- \( \delta f_{\text{Bias}} = 150 \pm 5 \text{ Hz} \) = Aircraft Dependent bias

\(^3\)Made on aircraft to partially compensate for \( \Delta f_{\text{AS}} \)
The Reasons Search is so Challenging

1. Uncertainty in $\delta f_{\text{Al}}$ and $\delta f_{\text{Bias}}$
2. Small BFO error of these three flights
3. Uncertainty in initial position on 2nd ping arc
Small Circles

- The **geodesic curvature** of a small circle is

\[ K_g = \pm \sqrt{\frac{1}{R_C^2} - \frac{1}{R_A^2}}. \]

- \( K_g \) quantifies how much \( C \) is turning in \( S \).

- **Extra parameter:** \( K_g \).

- **Extra constraint:** \( \Delta f \) at the next ping arc.

---

**If you know \( K_g \) of \( C \) on \( S^2 \), then you know \( C \).\(^4\)**

---

\(^4\)By Darboux Frame Formulae
Objective Function to Minimize

\[
f_{BFO,1}(v\Phi, n, v\Theta, n) = 0, \\
f_{Ping}(v\Phi, n, v\Theta, n, Kg, n) = 0, \\
f_{BFO,2}(v\Phi, n, v\Theta, n, Kg, n) = 0. \\
\]

\[
F = f_{Ping}^2 + f_{BFO,1}^2 + f_{BFO,2}^2 + f_{Penalty}^2. 
\]

Solve using a gradient-descent method.
Simulation Results

![Diagram showing simulation results with True, Model II, and Model III markers.]
Very Good Agreement with June 26 ATSB Report
MH370 Results [JZ, June 26 Data]
MH370 Results [ATSB, Oct 8 Data]

Figure 5: Representation of probability distribution at 0011 arc for constrained autopilot dynamics (red) and data error optimization (green). Red flight paths are most probable paths from the two types of analysis. Area of interest on 0019 arc covers 80% of probable paths from the two analyses at 0011 and consideration of the MRC approximate southern boundary.

Source: Google Earth/ Flight path reconstruction group

This analysis indicates that the total probability areas overlap between approximately 35°S and 39°S at 0011. The 6th arc between latitudes 32.5°S and 38.1°S covers 80% of the highest probability paths for both analyses. Extrapolating paths and limiting the southern boundary by the MRC intersection with the 7th arc provides an area between approximately 33.5°S and 38.3°S at 0019 (7th arc).

Update on factors relating to likely MH370 proximity to the 7th arc

End-of-flight scenarios

To estimate and have confidence in a reasonable search area width, it is important to understand the aircraft system status at the time of the SATCOM transmission from the aircraft at 0019.29 (log-on request), and the variations in aircraft behaviour and trajectory that were possible from that time.

The log-on request recorded at the final arc occurred very near the estimated time of fuel exhaustion. The recorded BFO values indicated that the aircraft could have been descending at that time. Aircraft systems analysis, in particular the electrical system and autoflight system, has been ongoing. In support of the systems analysis, the aircraft manufacturer and the operator have observed and documented various end-of-flight scenarios in their B777 simulators.

The simulator activities involved fuel exhaustion of the right engine followed by flameout of the left engine with no control inputs. This scenario resulted in the aircraft entering a descending spiralling low bank angle left turn and the aircraft entering the water in a relatively short distance after the last engine flameout. However when consideration of the arc tolerances, log on messages and simulator activities are combined, it indicates that the aircraft may be located within relatively close proximity to the arc. Whilst the systems analysis and simulation activities are ongoing, based on the analysis to date, the search area width described in the June report remains reasonable with the underwater search to commence at the 7th arc and progress outwards both easterly and westerly.

On Oct 8, search area was shifted 800 km SW.
Explanation for Oct 8 Shift

- Shift based on two unanswered ground-to-aircraft phone calls between 1st and 2nd ping
- These suggest aircraft turned south earlier than previously thought

“The sensitivity of the reconstructed flightpath to frequency errors is such that there remains significant uncertainty in the final location”.  

“Ongoing work may result in changes to the prioritization and location of search activity within the current search area.”.

---

5Inmarsat, *The Journal of Navigation*
6ATSB, Oct 8. as reported in *Aviation Week*
Historical Context

- Engineers at JHU-APL tracked the flight path of Sputnik in 1957.
- They realized that, knowing the path of Sputnik, they could use the Doppler effect to determine their location on earth.
- In 1964, JHU-APL engineers developed TRANSIT, the first satellite navigation system.
- TRANSIT was used by the US Navy to provide accurate location information to Polaris submarines.
- TRANSIT was superceded by GPS in 1996.
- Doppler tracking is still widely used for many applications.
Acknowledgements

- Official Search Strategy Group and ATSB
- Duncan Steel and the Independent Group
- Math 2415.002, Spring 2014
- Alan Boyle (NBC News)
- Yanping Chen (UTD Math)
- Matthew Goeckner (UTD Math)
- Brian Marks (JHU-APL)
- Justin Jacobs (NSA)
- Sue Minkoff (UTD Math) and our boys.