5.1. The Simple Cash Balance Problem

the Lagrange multipliers $\eta_1$ and $\eta_2$ satisfy the complementary slackness conditions

\begin{align*}
\eta_1(t) &\geq 0, \quad \eta_1(t)x(t) = 0, \quad \dot{\eta}_1(t) \leq 0, \\
\eta_2(t) &\geq 0, \quad \eta_2(t)y(t) = 0, \quad \dot{\eta}_2(t) \leq 0,
\end{align*}

(5.21)

(5.22)

and

\[ \frac{\partial L}{\partial u} = 0 \]

(5.23)

for all $t \in [0, T]$. Note that the transversality conditions in (5.19) and (5.20) on the adjoint variables are written as in Row 3 of Table 3.1. This form is easily seen to be equivalent to the transversality condition in (4.29).

In general, the solution of this problem is difficult and requires the use of a computer. However, we illustrate an easy case in which $\alpha = 0$ and $r_1$ and $r_2$ are piecewise-constant functions of time.

**Example 5.1** (We are indebted to C. Norström for this example.) Consider the model when $\alpha = 0$, $T = 10$, and $r_1$ and $r_2$ vary over time as follows:

\[ r_1(t) = \begin{cases} 
0 & \text{for } 0 \leq t < 5, \\
0.3 & \text{for } 5 \leq t \leq 10,
\end{cases} \]

(5.24)

\[ r_2(t) = 0.1 \quad \text{for } 0 \leq t \leq 10. \]

(5.25)

The initial cash and security balances are, respectively,

\[ x_0 = 0 \text{ and } y_0 = 3. \]

For convenience in exposition, we assume $d(t) = 0$, $0 \leq t \leq 10$; see Remark 5.1 at the end of this subsection. The optimal solution when there is no upper bound on the control is easy to guess. It is

\[ u^*(t) = 0 \text{ for } 0 \leq t < 5, \]

(5.26)

\[ u^*(t) = 0 \text{ for } 5 < t \leq 10. \]

(5.27)

At $t = 5$, the optimal action clearly is to sell all the securities instantaneously to take advantage of the higher interest rate on cash. We can do this because there is no upper bound on the rate of sales. Such a control is called an **impulse control**. It can be conceived to be the result of selling securities at a very large rate for a very short time so that the entire security balance is converted into cash more or less instantaneously.
To formalize the impulse control at $t = 5$, we first use (5.26) in (5.2) to obtain

$$g(5^-) = 3e^{0.5}.$$  

This in words means that the security balance just before time $t = 5$ is $3e^{0.5}$, obtained by earning interest at rate 0.1 compounded from time 0 to 5 on the initial security balance of 3.

Let us now sell securities at the rate of $3e^{0.5}/(2\delta t)$ for a short time interval of $2\delta t$ beginning at time $t = 5 - \delta t$ and ending at time $t = 5 + \delta t$, i.e.,

$$u^*(t) = \frac{3e^{0.5}}{2\delta t}, \; t \in [5 - \delta t, 5 + \delta t].$$

Using this control in (5.2), we can integrate the resulting equation from $t = 5 - \delta t$ to $t = 5 + \delta t$ by applying the formula (A.1.6). Thus, we have

$$y(5 + \delta t) = y(5 - \delta t)e^{0.2\delta t} - \int_{5 - \delta t}^{5 + \delta t} e^{0.1(5+\delta t-\tau)} \frac{3e^{0.5}}{2\delta t} d\tau$$

$$= y(5 - \delta t)e^{0.2\delta t} - \frac{3e^{0.5}}{0.2\delta t} \left[ e^{0.2\delta t} - 1 \right]$$

$$= y(5 - \delta t)e^{0.2\delta t} - \frac{3e^{0.5}}{0.2\delta t} \left[ 0.2\delta t + 0(\delta t) \right].$$

(5.28)

Taking the limit of (5.28) as $\delta t \to 0$, we obtain

$$y(5^+) = y(5^-) - 3e^{0.5} = 3e^{0.5} - 3e^{0.5} = 0.$$  

Note that as $\delta t \to 0$, we have $u^*(t)$, defined above, go to infinity. The net effect of $\delta t \to 0$ and $u^*(t) \to \infty$ is that the entire security balance is sold instantaneously at time 5, giving us $y(5^+) = 0$, i.e., a zero security balance just after $t = 5$.

Note that $x(5^-) = 0$. Following the impulse sale of securities at $t = 5$, $x(5^+) = 3e^{0.5}$. Further discussion and applications of impulse controls will be given in Chapters 7, 10, and 12.

We must now find the adjoint variables $\lambda_1$ and $\lambda_2$ and Lagrange multipliers $\eta_1$ and $\eta_2$ so that the maximum principle holds. From (5.23) with $\alpha = 0$, we get $\lambda_1 - \lambda_2 + \eta_1 - \eta_2 = 0$ or

$$\lambda_1 + \eta_1 = \lambda_2 + \eta_2.$$  

(5.29)

We now solve for these quantities in three time intervals $0 \leq t < 5$, $t = 5$, and $5 < t \leq 10$. We will start with $t$ in the last interval:
Given these definitions, the current earnings rate is \( x = ry \). The rate of change in the current earnings rate is given by

\[
\dot{x} = ry = r(\text{cu} + v)x, \quad x(0) = x_0.
\]  

(5.30)

Furthermore, the upper bound on the rate of growth of the assets implies the following constraint on the control variables:

\[
\dot{y}/y = (\text{cu} + v)x/(x/r) = r(\text{cu} + v) \leq g.
\]  

(5.31)

Finally, the objective of the firm is to maximize its value, which is taken to be the present value of the future dividend stream accruing to the shares outstanding at time zero. To derive this expression, note that

\[
\int_0^T (1 - v)xe^{-\rho t} dt
\]

represents the present value of total dividends issued by the firm. A portion of these dividends go to the new equity, which under the assumption of an efficient market will get a rate of return exactly equal to the discount rate \( \rho \). This should therefore be equal to the present value

\[
\int_0^T uxe^{-\rho t} dt
\]

of the external equity raised over time.

Thus, the net present value of the total future dividends that accrue to the initial shares is the difference of the above two expressions, i.e.,

\[
J = \int_0^T e^{-\rho t}(1 - v - u)x dt;
\]  

(5.32)

see Miller and Modigliani (1961), Sethi, Derzko, and Lehoczky (1982), and Sethi (1996) for further discussion. Note that in the case of a finite horizon, a more realistic objective function would include a salvage value or bequest term \( S[x(T)] \). This is not very difficult to incorporate. See Exercise 5.9 where the bequest function is linear. We will also solve the infinite horizon problem (i.e., \( T = \infty \)) after we have solved the finite horizon problem.

The optimal control problem is to choose \( u \) and \( v \) over time so as to maximize \( J \) in (5.32) subject to (5.30), the constraints (5.31), \( u \geq 0, \)

and \( v \geq 0 \).
and $0 \leq v \leq 1$. For convenience, we restate this problem as

$$
\begin{align*}
\max_{u,v} \left\{ J = \int_0^T e^{-\rho t} (1 - v - u)x dt \right\} \\
\text{subject to} \\
\dot{x} = r(cu + v)x, \quad x(0) = x_0, \\
\text{and the control constraints} \\
cu + v \leq g/r, \quad u \geq 0, \quad 0 \leq v \leq 1.
\end{align*}
$$

### 5.2.2 Application of the Maximum Principle

This is a bilinear problem with two control variables which is a special case of Row $(f)$ in Table 3.3. The current-value Hamiltonian is

$$
H = (1 - v - u)x + \lambda r(cu + v)x,
$$

where the current-value adjoint variable $\lambda$ satisfies

$$
\dot{\lambda} = \rho \lambda - (1 - v - u) - \lambda r(cu + v)
$$

with the transversality condition

$$
\lambda(T) = 0.
$$

The first term in the Hamiltonian in (5.34) is the dividend payout rate to stockholders of record at time $t$. According to Section 2.2.1, $\lambda$ is the marginal value (in time $t$ dollars) of a unit change in the earnings rate at time $t$. Thus, $\lambda r(cu + v)x$ is the value at time $t$ of the incremental earnings rate due to the investment of retained earnings $vx$ and the net amount of external financing $cux$. This explains why we should maximize $H$ with respect to $u$ and $v$ at each $t$. To interpret (5.35) as in Section 2.2.4, consider an earnings rate of one dollar at time $t$. It is worth $\lambda$, on which the stockholders expect a return of $\rho \lambda dt$ at time $dt$. In equilibrium this must be equal to the “capital gain” $d\lambda$, plus the immediate dividend $(1 - v)dt$ less $udt$, the “claims” of the new stockholders, plus the value $\lambda r(cu + v)dt$ of the incremental earnings rate $r(cu + v)dt$ at time $t + dt$.

To specify the form of optimal policy, we rewrite the Hamiltonian as

$$
H = [W_1u + W_2v + 1]x,
$$

(5.37)
defined for finite horizon problems. However, the infinite horizon solution can usually be inferred from the finite horizon solution if sufficient care is exercised. This will be done in Section 5.2.4.

Our analysis of the finite horizon problem (5.33) proceeds with the assumption that the terminal time $T$ is assumed to be sufficiently large. We shall make this assumption precise during our analysis. Moreover, we shall discuss the solution when $T$ is not sufficiently large in Remarks 5.2 and 5.4.

Define the reverse-time variable $\tau$ as

$$\tau = T - t,$$

so that

$$\dot{y} = \frac{dy}{d\tau} = \frac{dy}{dt} \frac{dt}{d\tau} = -\dot{y}.$$
This is the starting point for our switching point synthesis. First, we consider Case A.

**Case A:** $g \leq r$.

Note that the constraint $v \leq 1$ is superfluous in this case and the only feasible subcases are A1, A3, and A6. Since $\lambda(0) = 0$, we have $W_1(0) = W_2(0) = -1$, and Subcase A1 obtains.

**Subcase A1:** $W_1 = cr\lambda - 1 < 0$ and $W_2 = r\lambda - 1 < 0$.

From Row (1) of Table 5.1 we have $u^* = v^* = 0$, which gives the state equation (5.43) and the adjoint equation (5.44) as

$$\dot{x} = 0 \quad \text{and} \quad \dot{\lambda} = 1 - \rho \lambda. \quad (5.45)$$

With the initial conditions given in (5.41), the solutions for $x$ and $\lambda$ are

$$x(\tau) = \alpha_A \quad \text{and} \quad \lambda(\tau) = (1/\rho)[1 - e^{-\rho\tau}]. \quad (5.46)$$

It is easy to see that because of the assumption $0 \leq c < 1$, it follows that if $W_2 = r\lambda - 1 < 0$, then $W_1 = cr\lambda - 1 < 0$. Therefore, to remain in this subcase as $\tau$ increases, $W_2(\tau)$ must remain negative for some time as $\tau$ increases. From (5.46), however, $\lambda(\tau)$ is increasing asymptotically toward the value $1/\rho$ and therefore, $W_2(\tau)$ is increasing asymptotically toward the value $r/\rho - 1$. Since, we have assumed $r > \rho$, there exists a $\tau_1$ such that $W_2(\tau_1) = (1 - e^{-\rho\tau_1})r/\rho - 1 = 0$. It is easy to compute

$$\tau_1 = (1/\rho) \ln[r/(r - \rho)]. \quad (5.47)$$

From this expression, it is clear that the firm leaves Subcase A1 provided $\tau_1 < T$. Moreover, this observation also makes precise the notion of a sufficiently large $T$ in Case A by having $T > \tau_1$.

**Remark 5.2** When $T$ is not sufficiently large, i.e., when $T \leq \tau_1$ in Case A, the firm stays in Subcase A1. The optimal solution in this case is $u^* = 0$ and $v^* = 0$, i.e., a policy of no investment.

**Remark 5.3** Note that if we had assumed $r < \rho$, the firm never exits from Subcase A1 regardless of the value of $T$. Obviously, there is no use investing if the rate of return is less than the discount rate.
5.2. Optimal Financing Model

At reverse time $\tau_1$, we have $W_2 = 0$ and $W_1 < 0$ and the firm, therefore, is in Subcase A6.

**Subcase A6:** $W_1 = cr\lambda - 1 < 0$ and $W_2 = r\lambda - 1 = 0$.

In this subcase, the optimal controls

$$u^* = 0, \quad 0 \leq v^* \leq g/r$$

from Row (3) of Table 5.1 are singular with respect to $v$. This case is termed singular because the Hamiltonian maximizing condition does not yield a unique value for the control $v$. In such cases, the optimal controls are obtained by conditions required to sustain $W_2 = 0$ for a finite time interval. This means we must have $W = 0$, which in turn implies $\lambda = 0$.

To compute $\lambda$, we substitute (5.48) into (5.44) and obtain

$$\lambda^* = (1 - v^*) - \lambda[\rho - rv^*].$$

(5.49)

Substituting $\lambda = 1/r$ (since $W_2 = 0$) in (5.49) and equating the right-hand side to zero we obtain

$$r = \rho$$

(5.50)

as a necessary condition required to maintain singularity over a finite time interval following $\tau_1$. Condition (5.50) is fortuitous and will not generally hold. In fact we have assumed $r > \rho$. Thus, the firm will not stay in Subcase A6 for a finite time interval. Furthermore, since $r > \rho$, we have $\lambda(\tau_1) = (r - \rho/r) > 0$. Therefore, $W_2$ is increasing from zero and becomes positive after $\tau_1$. Thus, at $\tau_1^+$ the firm switches to Subcase A3.

**Subcase A3:** $W_2 = r\lambda - 1 > 0$.

The optimal controls in this subcase from Row (2) of Table (5.1) are

$$u^* = 0, \quad v^* = g/r.$$  

(5.51)

The state and the adjoint equations are

$$\dot{x} = -gx, \quad x(\tau_1) = \alpha_A,$$

(5.52)

$$\dot{\lambda} = (1 - g/r) - \lambda(\rho - g), \quad \lambda(\tau_1) = 1/r,$$

(5.53)

with values at $\tau = \tau_1$ deduced from (5.46) and (5.47).
EXERCISES FOR CHAPTER 5

5.1 Find the optimal policies for the simple cash balance model (Sections 5.1.1 and 5.1.2) with \( x_0 = 2, \ y_0 = 2, \ U_1 = U_2 = 5, \ T = 1, \ \alpha = 0.01, \) and the following specifications for the interest rates:

(a) \( r_1(t) = \frac{1}{2}, \ r_2(t) = \frac{1}{3}. \)
(b) \( r_1(t) = \frac{t}{2}, \ r_2(t) = \frac{1}{3}. \)
(c) Sketch the optimal policy in (b) in the \((t, \lambda_2/\lambda_1)\) space, like in Figure 5.2.

5.2 Formulate the extension of the model in Section 5.1.3 with finite positive bounds \( U_1 \) and \( U_2 \) on the control variables for

(a) \( \alpha = 0. \)
(b) \( \alpha > 0. \)

[Hint: Adjoin the control constraints to the Hamiltonian in forming the Lagrangian. For (b), write \( u = u_1 - u_2 \) as in (5.10).]

5.3 It is also possible to guess the solution for Example 5.1 when \( \alpha > 0. \)
Show that the optimum policy remains unchanged if \( \alpha < 1 - 1/e. \)
[Hint: Use an elementary compound interest argument.]

5.4 Discuss the optimal equity financing model of Section 5.2.1 when \( c = 1. \) Show that only one control variable is needed. Then solve the problem.

5.5 What happens in the optimal equity financing model when \( r < \rho? \)
Guess the optimal solution (without actually solving it).

5.6 When \( g = r \) in Case A of the optimal equity financing model, why is the limit of the solution not the solution to the infinite horizon problem?

5.7 Let \( g = 0.12 \) in Example 5.2. Re-solve the finite horizon problem with this new value of \( g. \) Also, for the infinite horizon problem, state a policy which yields an infinite value for the objective function.

5.8 Reformulate and solve the simple cash balance problem of Sections 5.1.1 and 5.1.2, if the earnings on bonds are paid in cash.
5.9 Add a salvage value function
\[ e^{-\rho T} B x(T), \]
where \( B \geq 0 \), to the objective function in the problem (5.33) and analyze the modified problem due to Sethi (1978b). Show how the solution changes as \( B \) varies from 0 to \( 1/rc \).

5.10* Redo Example 5.1 with the control constraints \(-1 \leq u \leq 1\).
(a) Give reasons why the solution shown in Figure 5.8 is optimal.

(b) Compute \( f(t^*) \) in terms of \( t^* \).

(c) Compute \( J \) in terms of \( t^* \). Find \( t^* \) that maximizes \( J \) by setting \( dJ/dt^* = 0 \).
   [Hint: Because this is a long and tedious calculus problem, you may wish to use Mathematica or MAPLE to solve this problem.]

![Figure 5.8: Solution for Exercise 5.10](image)

5.11 For the solution found in Exercise 5.10, show by using the maximum principle and Exercise 5.10 that the adjoint trajectories are:

\[
\lambda_1(t) = \begin{cases} 
\lambda_1(0) = e^{1.5}, & 0 \leq t \leq 5, \\
\lambda_1(5)e^{-0.3(t-5)} = e^{3-0.3t}, & 5 \leq t \leq 10,
\end{cases}
\]
Exercises for Chapter 5

and

\[ \lambda_2(t) = \begin{cases} 
\lambda_2(0)e^{-0.1t^*} = e^{1.5+0.1(t^*-t)}, & 0 \leq t \leq f(t^*) \approx 6.52, \\
\frac{2}{3} + \frac{1}{3}e^{3-0.3t}, & f(t^*) < t \leq 10,
\end{cases} \]

where \( t^* \approx 1.97 \). Sketches of these functions are shown in Figure 5.9.

Figure 5.9: Adjoint Trajectories for Exercise 5.11

5.12 Suppose we extend the model of Exercise 5.9 to include debt. For this let \( z \) denote the total debt at time \( t \) and \( w \geq 0 \) denote the amount of debt issued expressed as a proportion of current earnings. Then the state equation for \( z \) is

\[ \dot{z} = wx, \; y(0) = y_0. \]

How would you modify the objective function, the state equation for \( x \), and the growth constraint (5.31)? Assume \( i \) to be the constant interest rate on debt, and \( i < r \).
5.13 Find the form of the optimal policy for the following model due to Davis and Elzinga (1971):

\[
\max_{u,v} \left\{ J = \int_0^T e^{-\rho t} (1 - v) Er dt + P(T) e^{-\rho T} \right\}
\]

subject to

\[
\begin{align*}
\dot{P} &= k [rE(1 - v) - \rho P], \quad P(0) = P_0, \\
\dot{E} &= rE [v + u(c - E/P)], \quad E(0) = E_0,
\end{align*}
\]

and the control constraints

\[
u \geq 0, \quad v \geq 0, \quad cu + v \leq g/r.
\]

Here \(P\) denotes the price of a stock, \(E\) denotes equity per stock and \(k > 0\) is a constant. Also, assume \(r > \rho > g\) and \(1/c < r/\rho < 1/c + (ck + 1)g/(\rho ck)\). This example requires the use of the generalized Legendre-Clebsch condition (D.40) in Appendix D.

5.14 Remove the assumption of an arbitrary upper bound \(g\) on the growth rate in the financing model of Section 5.2.1 by introducing a convex cost associated with the growth rate. With \(r\) re-interpreted now as the gross rate of return, obtain the net increase in rate of earnings by the rate of increase in gross earnings less the cost associated with the growth rate. Also assume \(c = 1\) as in Exercise 5.4. Formulate the resulting model and apply the maximum principle to find the form of the optimal policy. You may assume the cost function to be quadratic in the growth rate to get an explicit form for the solution.