ASSIGNMENT 4

Reading Assignment: Textbook, Sections 8.1 through 8.7

Problems: (to be handed in at the beginning of class)   Due Date: Monday September 25

§1. Preprocessing the output. One is given a communication channel with transition probabilities
\( p(y \mid x) \) and channel capacity \( C = \max_{p(x)} I(X; Y) \). A helpful statistician preprocesses
the output by forming \( \tilde{Y} = g(Y) \). He claims that this will strictly improve the capacity.

(a) Show that he is wrong.
(b) Under what conditions does he not strictly decrease the capacity?

§2. Maximum likelihood decoding. A source produces independent, equally probable symbols from
an alphabet \((a_1, a_2)\) at a rate of one symbol every 3 seconds. These symbols are transmitted
over a binary symmetric channel which is used once each second by encoding the source symbol
\( a_1 \) as 000 and the source symbol \( a_2 \) as 111. If in the corresponding 3 second interval of the
channel output, any of the sequences 000,001,010,100 is received, \( a_1 \) is decoded; otherwise, \( a_2 \)
is decoded. Let \( \epsilon < \frac{1}{2} \) be the channel crossover probability.

(a) For each possible received 3-bit sequence in the interval corresponding to a given source
letter, find the probability that \( a_1 \) came out of the source given that received sequence.
(b) Using part (a), show that the above decoding rule minimizes the probability of an in-
correct decision.
(c) Find the probability of an incorrect decision (using part (a) is not the easy way here).
(d) If the source is slowed down to produce one letter every \( 2n + 1 \) seconds, \( a_1 \) being encoded
by \( 2n + 1 \) 0’s and \( a_2 \) being encoded by \( 2n + 1 \) 1’s. What decision rule minimizes the
probability of error at the decoder? Find the probability of error as \( n \to \infty \).

§3. An additive noise channel. Find the channel capacity of the following discrete memoryless
channel:

\[
\begin{array}{c}
Z \\
\downarrow \\
X + \rightarrow Y
\end{array}
\]

where \( \Pr\{Z = 0\} = \Pr\{Z = a\} = \frac{1}{2} \). The alphabet for \( x \) is \( X = \{0, 1\} \). Assume that \( Z \) is
independent of \( X \).

Observe that the channel capacity depends on the value of \( a \).
§4. Channels with memory have higher capacity. Consider a binary symmetric channel with
\( Y_i = X_i \oplus Z_i \), where \( \oplus \) is mod 2 addition, and \( X_i, Y_i \in \{0, 1\} \).

Suppose that \( \{Z_i\} \) has constant marginal probabilities \( \Pr\{Z_i = 1\} = p = 1 - \Pr\{Z_i = 0\} \),
but that \( Z_1, Z_2, \ldots, Z_n \) are not necessarily independent. Assume that \( Z^n \) is independent of
the input \( X^n \). Let \( C = 1 - H(p, 1 - p) \). Show that
\[
\max_{p(x_1, x_2, \ldots, x_n)} I(X_1, X_2, \ldots, X_n; Y_1, Y_2, \ldots, Y_n) \geq nC.
\]

§5. Channel capacity. Consider the discrete memoryless channel \( Y = X + Z \) (mod 11), where
\[
Z = \begin{pmatrix}
1, & 2, & 3 \\
1/3, & 1/3, & 1/3
\end{pmatrix}
\]
and \( X \in \{0, 1, \ldots, 10\} \). Assume that \( Z \) is independent of \( X \).

(a) Find the capacity.
(b) What is the maximizing \( p^n(x) \)?