Auctioning Keywords in Online Search

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Web Appendix

Derivation of the Expected Revenue. Denote $V(v)$ as the equilibrium payoff of an advertiser of type $v$, and $P_j(v)$ as the equilibrium probability for him to win the $j$th share. So

\[(W1) \quad V(v) = v \sum_{j=1}^{n} Q(s_j)P_j(v) - E[\text{equilibrium payment of type } v]\]

First notice that

\[(W2) \quad V(v) \geq V(\bar{v}) + (v - \bar{v}) \sum_{j=1}^{n} Q(s_j)P_j(\bar{v}),\]

where the right-hand side is the payoff of a type-$v$ bidder if he follows instead an equilibrium strategy of a type-$\bar{v}$ bidder.

Therefore we have

\[(W3) \quad V(v) \geq V(v + dv) + (-dv) \sum_{j=1}^{n} Q(s_j)P_j(v + dv)\]

\[(W4) \quad V(v + dv) \geq V(v) + dv \sum_{j=1}^{n} Q(s_j)P_j(v)\]

Reorganizing the above two equations yields

\[(W5) \quad \sum_{j=1}^{n} Q(s_j)P_j(v + dv) \geq \frac{V(v + dv) - V(v)}{dv} \geq \sum_{j=1}^{n} Q(s_j)P_j(v)\]
Taking the limit of $dv$ to zero, we obtain
\[ \frac{dV(v)}{dv} = \sum_{j=1}^{n} Q(s_j) P_j(v). \]
Moving $dv$ to the right hand side, integrating both sides from $v$ to $v$, and assuming $V(v) = 0$ (the lowest type gets zero payoff), we get

\[ V(v) = \sum_{j=1}^{n} Q(s_j) \int_{v}^{v} P_j(x) dx, \text{ for } v \in [v, \bar{v}]. \]

Notice the expected payment from an advertiser of type $v$ is $v \sum_{j=1}^{n} Q(s_j) P_j(v) - V(v)$ by (W1). So the expected payment from one bidder is

\[ E \left[ v \sum_{j=1}^{n} Q(s_j) P_j(v) - V(v) \right] = \int_{v}^{\bar{v}} \left[ v \sum_{j=1}^{n} Q(s_j) P_j(v) - \sum_{j=1}^{n} Q(s_j) \int_{v}^{v} P_j(t) dt \right] f(v) dv \]
\[ = \int_{v}^{\bar{v}} \left[ v \sum_{j=1}^{n} Q(s_j) P_j(v) f(v) - (1 - F(v)) \sum_{j=1}^{n} Q(s_j) P_j(v) \right] dv \]

\[ = \sum_{j=1}^{n} Q(s_j) \int_{v}^{\bar{v}} P_j(v) \left( v - \frac{1 - F(v)}{f(v)} \right) f(v) dv \]

The total expected revenue from all advertisers is $n$ times the above. With strictly increasing bidding functions, the equilibrium probability for an advertiser of type $v$ to win the $j$th share is the likelihood that $j - 1$ of his competitors have higher valuation than his and the rest of the competitors have lower valuation, and thus $P_j(v)$ can be specified as in (7). □

**Proof of Lemma 1.** (a) For $j = 1, 2, ..., n - 1$,

\[ \alpha_j = n \int_{v}^{\bar{v}} P_j(v) \left[ vf(v) - (1 - F(v)) \right] dv = n \int_{v}^{\bar{v}} P_j(v) d \left[ -v (1 - F(v)) \right] \]
\[ = n \int_{v}^{\bar{v}} v (1 - F(v)) dP_j(v) \]
\[ = n \int_{v}^{\bar{v}} \binom{n-1}{j-1} F(v)^{n-j-1} (1 - F(v))^{j-1} \left[ (n - j) - (n - 1) F(v) \right] v f(v) dv \]

where the third step is due to integration by parts. We can easily verify $\alpha_1 > 0$. 2
For $j = 2, 3, ..., n - 1$, 

$$\alpha_1 - \alpha_j = n \int_{\underline{v}}^{\bar{v}} \{ (n - 1) F(v)^{n-2} (1 - F(v)) - \frac{(n-1)}{(n-j)} (1 - F(v))^{j-2} [(n - j) - (n - 1) F(v)] \} v f(v) \, dv$$

(W9)

Denoting $A(v) \equiv (n - 1) F(v)^{j-1} - \frac{(n-1)}{(n-j)} (1 - F(v))^{j-2} [(n - j) - (n - 1) F(v)]$, we can rewrite (W9) as $\alpha_1 - \alpha_j = n \int_{\underline{v}}^{\bar{v}} [1 - F(v)] F(v)^{n-j-1} A(v) v f(v) \, dv$. We argue that $A(v)$ single-crosses zero from below on $[\underline{v}, \bar{v}]$. To see, let $v^0$ be the solution to $(n - j) - (n - 1) F(v) = 0$. We can verify that $A(v) < 0$, $A(v)$ increases in $v$ for $v \leq v^0$, and $A(v)$ is positive for all $v > v^0$. Thus $A(v)$ crosses zero only once from below, implying $[1 - F(v)] F(v)^{n-j-1} A(v) f(v)$ also single-crosses zero from below on $(\underline{v}, \bar{v})$. Denoting the crossing point of the latter as $v^c$, we have 

$$\alpha_1 - \alpha_j = n \int_{\underline{v}}^{\bar{v}} (1 - F(v)) F(v)^{n-j-1} A(v) f(v) v dv$$

$$> n v^c \int_{\underline{v}}^{\bar{v}} (1 - F(v)) F(v)^{n-j-1} A(v) f(v) dv$$

(W10) 

$$= n v^c \int_{\underline{v}}^{\bar{v}} [P_2(v) - j P_{j+1}(v) + (j - 1) P_j(v)] f(v) dv$$

where the last equality results from substituting the definition of $A(v)$ and rearranging terms. The right side of (W10) is zero because for $j = 1, ..., n$,

$$\int_{\underline{v}}^{\bar{v}} P_j(v) f(v) \, dv = \binom{n-1}{n-j} \int_{\underline{v}}^{\bar{v}} F(v)^{n-j} [1 - F(v)]^{j-1} dF(v)$$

$$= \binom{n-1}{n-j} \int_{0}^{1} x^{n-j} (1 - x)^{j-1} dx$$

(W11) 

$$= \binom{n-1}{n-j} \binom{n-1}{n-j}^{-1} \frac{1}{n} = \frac{1}{n}$$

where the second step is due to integration by substitution and the third step is due to repeated integration by parts. Therefore, $\alpha_1 - \alpha_j > 0$ for $j = 2, 3, ..., n - 1$.  

3
We next show that \( \alpha_1 - \alpha_n > 0 \).

\[
\alpha_1 - \alpha_n = n \int_\mathbb{V} \{ F(v)^{n-1} - [1 - F(v)]^{n-1} \} \, d[-v(1 - F(v))]
\]
\[
= -n\bar{v} + n \int_\mathbb{V} v [1 - F(v)] (n - 1) [F(v)^{n-2} + (1 - F(v))^{n-2}] \, f(v) \, dv
\]
\[
> -n\bar{v} + n\bar{v} \int_\mathbb{V} [1 - F(v)] (n - 1) [F(v)^{n-2} + (1 - F(v))^{n-2}] \, f(v) \, dv
\]
\[
= -n\bar{v} + n\bar{v} \left( \frac{1}{n} + \frac{n-1}{n} \right) = 0
\]

where the second step is due to integration by parts and the fourth step is due to (W11).

(b) Denote \( h_j(x) \equiv nP_j(x)f(x) \). By (W11), \( \int_\mathbb{V} h_j(x) \, dx = 1 \). Thus we can regard \( h_j(x) \) as a probability density function. We next show that for \( j = 1, 2, ..., n - 1 \), \( h_j(x) \) first-order stochastically dominates \( h_{j+1}(x) \).

\[
h_j(x) - h_{j+1}(x)
\]
\[
= n f(x) \left\{ (\binom{n-1}{n-j} F(x)^{n-j} [1 - F(x)]^{j-1} - (\binom{n-1}{n-j-1} F(x)^{n-j-1} [1 - F(x)]^j) \right\}
\]
\[
= \binom{n}{j} f(x) F(x)^{n-j-1} [1 - F(x)]^{j-1} [nF(x) - (n - j)]
\]

Denote \( v^c_j \) as the solution to \( nF(x) - (n - j) = 0 \). Because \( h_j(x) < h_{j+1}(x) \) for any \( x \in (\bar{v}, v^c_j) \), \( \int_\mathbb{V} h_j(x) \, dx < \int_\mathbb{V} h_{j+1}(x) \, dx \) for \( v \in (\bar{v}, v^c_j) \). Because \( h_j(x) > h_{j+1}(x) \) for any \( x \in (v^c_j, \bar{v}) \), \( \int_\mathbb{V} h_j(x) \, dx > \int_\mathbb{V} h_{j+1}(x) \, dx \) for \( v \in (v^c_j, \bar{v}) \), which implies \( \int_\mathbb{V} h_j(x) \, dx < \int_\mathbb{V} h_{j+1}(x) \, dx \) for \( v \in (v^c_j, \bar{v}) \) (note that \( \int_\mathbb{V} h_j(x) \, dx = 1 - \int_\mathbb{V} h_j(x) \, dx \)). In all, we have \( \int_\mathbb{V} h_j(x) \, dx < \int_\mathbb{V} h_{j+1}(x) \, dx \) for any \( v \in (\bar{v}, \bar{v}) \), implying that \( h_j(x) \) first-order stochastically dominates \( h_{j+1}(x) \). According to the property of first-order stochastic dominance (e.g., Proposition 6.D.1 at page 195 of Mas-Colell, Whinston, and Green (1995)), if \( J(x) \) is an increasing function of \( x \), \( \int_\mathbb{V} h_j(x) J(x) \, dx > \int_\mathbb{V} h_{j+1}(x) J(x) \, dx \). Therefore \( \alpha_j > \alpha_{j+1} \).
Proof of Lemma 2. Assume the optimal share structure is \((s^*_1, s^*_2, ..., s^*_n)\). Denote \(\sum_{j=j+1}^{j+1} s^*_j \equiv \sigma\) and notice that \(s^*_j \geq \frac{1}{j-k+1} \sigma \geq s^*_j+1 \geq 0\) because of the size-order constraint. 

\((s^*_j, s^*_j+2, ..., s^*_j+1)\) must be the solution to the following maximization problem: 

(W12) \(\max \sum_{j=j+1}^{j+1} \alpha_j Q(s_j)\), subject to: \(s_{j+1} \geq \ldots \geq s_{j+1}\) and \(\sum_{j=j+1}^{j+1} s_j \leq \sigma\)

(W13) \(s^*_j \geq s_{j+1}\) and \(s_{j+1} \geq s^*_j+1\)

We will work on the related maximization problem without constraint (W13) and check (W13) later. The Lagrangian function then can be written as

\[ L = \sum_{j=j+1}^{j+1} \alpha_j Q(s_j) + \mu \left( \sigma - \sum_{j=j+1}^{j+1} s_j \right) + \sum_{j=j+1}^{j+1} \gamma_j (s_j - s_{j+1}) \]

where \(\mu\) and \(\gamma_j\) are Lagrange multipliers. Hence, the Kuhn-Tucher conditions are (let \(\gamma_{j+1} \equiv 0\), \(\gamma_{j+1} \equiv 0\))

(W14) \(\alpha_j Q'(s_j) - \mu + \gamma_j - \gamma_{j-1} = 0\), for \(j = j+1, \ldots, j+1\)

Averaging (W14) for the first \(l\) shares and the remaining shares, respectively, we have

(W15) \(\frac{1}{l} \left( \sum_{j=j+1}^{j+l} \alpha_j Q'(s_j) + \gamma_{j+l} \right) = \frac{1}{j+1 - j + l} \left( \sum_{j=j+1}^{j+l} \alpha_j Q'(s_j) - \gamma_{j+l} \right)\).

By definition, \(j+1\) is the maximizer for the average return factor starting from \(j+1\), so

(W16) \(\frac{1}{l} \sum_{j=j+1}^{j+l} \alpha_j \leq \frac{1}{j+1 - j + l} \sum_{j=j+1}^{j+l} \alpha_j\)

Also note that \(Q'(s_j)\) is nondecreasing in \(j\). Therefore, we have \(\frac{1}{l} \sum_{j=j+1}^{j+l} \alpha_j Q'(s_j) \leq \frac{1}{j+1 - j + l} \sum_{j=j+1}^{j+l} \alpha_j Q'(s_j)\). If \(\gamma_{j+l} = 0\), (W15) can hold only if (W16) holds in equality and \(s_{j+l} = \ldots = s_{j+l}\). In other words, if any \(\gamma_j = 0\) \((j < j+1)\), we must have \(s_{j+l} = \ldots = s_{j+l}\). Otherwise, we have \(\gamma_j > 0\) for all \(j < j < j+1\), which implies \(s_{j+1} = \ldots = s_{j+1}\)
by the Kuhn-Tucker condition. So, regardless, we have \( s_{jk+1} = \ldots = s_{j+1} = \frac{1}{j_{k+1} - j_k} \sigma \), which naturally satisfies constraint (W13). ■

**Proof of Lemma 3.** It is sufficient to show that if \( \hat{s}_j \geq s_j \) then \( \hat{s}_{j+1} \geq s_{j+1} \), for any \( j \).

If \( j \) and \( j + 1 \) are located in the same plateau, \( \hat{s}_j = \hat{s}_{j+1} \) and \( s_j = s_{j+1} \), and \( \hat{s}_{j+1} \geq s_{j+1} \) holds trivially. So we assume \( j \) and \( j + 1 \) are located in plateau \( k \) and \( k + 1 \), respectively.

If \( s_{j+1} = 0 \), the result holds trivially. Suppose \( s_{j+1} > 0 \) (so that \( \hat{s}_j \geq s_j > s_{j+1} > 0 \)). By Proposition 2, we have \( \hat{\alpha}_k Q' (\hat{s}_j) \geq \hat{\alpha}_{k+1} Q' (\hat{s}_{j+1}) \) and \( \hat{\alpha}_k Q' (s_j) = \hat{\alpha}_{k+1} Q' (s_{j+1}) \). Together with condition (17), we have

\[
(W17) \quad \frac{Q' (\hat{s}_j)}{Q' (\hat{s}_{j+1})} \geq \frac{\hat{\alpha}_{k+1}}{\hat{\alpha}_k} \geq \frac{\hat{\alpha}_{k+1}}{\hat{\alpha}_k} = \frac{Q' (s_j)}{Q' (s_{j+1})}
\]

which implies \( \frac{Q' (\hat{s}_j)}{Q' (s_j)} \geq \frac{Q' (\hat{s}_{j+1})}{Q' (s_{j+1})} \). Note that \( \frac{Q' (\hat{s}_j)}{Q' (s_j)} \leq 1 \) by concavity of \( Q(\cdot) \) and \( \hat{s}_j \geq s_j \). So we have \( \frac{Q' (\hat{s}_{j+1})}{Q' (s_{j+1})} \leq 1 \), which implies \( \hat{s}_{j+1} \geq s_{j+1} \). ■

**Lemma W1 (Ranking of \( \alpha_j (v_0) \))** Under the MHR condition, for any marginal type \( v_0 \in (v, \bar{v}) \), if \( \alpha_j (v_0) > 0 \), \( \alpha_j (v_0) > \alpha_{j+1} (v_0) \); if \( \alpha_j (v_0) \leq 0 \), \( \alpha_{j+1} (v_0) < 0 \).

**Proof.** For the case \( v_0 \in (v, \bar{v}) \), define \( h_j (x|x \geq v_0) = \frac{h_j (x)}{\int_{v_0}^{\bar{v}} h_j (t) dt} \). Following steps in the proof of Lemma 1(b), we can similarly show that \( h_i (x|x \geq v_0) \) first-order stochastically dominates \( h_{i+1} (x|x \geq v_0) \). Thus,

\[
(W18) \quad \int_{v_0}^{\bar{v}} h_j (x|x \geq v_0) J (x) dx \geq \int_{v_0}^{\bar{v}} h_{j+1} (x|x \geq v_0) J (x) dx, \text{ for } j = 1, 2, \ldots, n - 1.
\]

Substituting \( h_j (x|x \geq v_0) \) with \( \frac{h_j (x)}{\int_{v_0}^{\bar{v}} h_j (t) dt} \) and rearranging, we have

\[
(W19) \quad \alpha_j (v_0) = \int_{v_0}^{\bar{v}} h_j (x) J (x) dx > \frac{\int_{v_0}^{\bar{v}} h_j (t) dt}{\int_{v_0}^{\bar{v}} h_j (t) dt \int_{v_0}^{\bar{v}} h_{j+1} (x) J (x) dx} \int_{v_0}^{\bar{v}} h_{j+1} (t) dt \alpha_{j+1} (v_0).
\]
Suppose $\alpha_j(v_0) > 0$. If $\alpha_{j+1}(v_0) \geq 0$, from $\int_{v_0}^{\bar{v}} h_j(t) \, dt > \int_{v_0}^{\bar{v}} h_{j+1}(t) \, dt > 0$ (because $h_j(t)$ first-order stochastically dominates $h_{j+1}(t)$), we have $\alpha_j(v_0) > \alpha_{j+1}(v_0)$. If $\alpha_{j+1}(v_0) < 0$, it is easy to see $\alpha_j(v_0) > \alpha_{j+1}(v_0)$.

If $\alpha_j(v_0) \leq 0$, (W19) implies $\alpha_{j+1}(v_0) < 0$. ■

References