Notes on 1st and 2nd order Bode Plots

\[ H(j\omega) = \frac{1}{1+j\omega} \]

*Magnitude: \[ |H(j\omega)|^2 = |H(j\omega)|^2 + \text{Re}[H(j\omega)]^2 + \text{Im}[H(j\omega)]^2 \]

\[ |H(j\omega)| = \left| \frac{-j\omega}{1-j\omega} \right| \ast \left| \frac{-j\omega}{1+\omega^2} \right| = \frac{1}{1+\omega^2}, \text{ in normalized form } = \frac{1}{1+(\omega/\omega_0)^2} \]

\[ |H(j\omega)|_{\text{dB}} = 20 \log_{10} |H(j\omega)| = -10 \log_{10} (1+\omega^2) \]

in normalized form \[ -10 \log_{10} \left[ 1+(\omega/\omega_0)^2 \right] \]

*Phase: \[ \text{Phase}(H(j\omega)) = \tan^{-1} \left( \left| \frac{\text{Im}[H(j\omega)]}{\text{Re}[H(j\omega)]} \right| \frac{1}{1+(\omega/\omega_0)^2} \right) \]

\[ \text{Phase}(H(j\omega)) = \angle H(j\omega) = \tan^{-1} (-\omega) = -\tan^{-1} (\omega) \]

in normalized form \[ \angle H(j\omega) = \tan^{-1} (\omega/\omega_0) \]

**Diagram:**
- **Magnitude:**
  - 0 dB
  - -3 dB cut-off
  - -20 dB/dec
  - **Phase:**
    - -90°
  - Asymptote

\[ H(j\omega) = \frac{1}{(j\omega)^2 + 2\zeta j\omega + 1} = \frac{1}{(1 - \omega^2) + j2\zeta \omega} \]

Magnitude \[ |H(j\omega)| = |H(j\omega)| = \sqrt{Re^2[H(j\omega)] + Im^2[H(j\omega)]} \]

\[ |H(j\omega)| = \frac{1}{\sqrt{(1 - \omega^2)^2 + (2\zeta \omega)^2}} = \frac{1}{\sqrt{1 + 2(\zeta^2 - 1)(\frac{\omega}{\omega_0})^2 + (\frac{\omega}{\omega_0})^4}} \]

in normalized form

Phase:
\[ \angle H(j\omega) = -\tan^{-1}\left[\frac{2\zeta \omega}{1 - \omega^2}\right] \]

in normalized form:
\[ \angle H(j\frac{\omega}{\omega_0}) = -\tan^{-1}\left[\frac{2\zeta (\frac{\omega}{\omega_0})}{1 - (\frac{\omega}{\omega_0})^2}\right] \]

**NOTE:** for \( \zeta = \frac{\sqrt{2}}{2} + \frac{\omega}{\omega_0} = 1 \)

\[ |H(j\frac{\omega}{\omega_0})| \approx -3 \text{ dB and} \]

\[ \angle H[j(\frac{\omega}{\omega_0})]_{\omega = \omega_0} = -90^\circ \]

also:
\[ \angle H[j(\frac{\omega}{\omega_0})]_{\omega = \omega_0} = -2\tan^{-1}(\frac{\omega}{\omega_0}) \]

\[ 1/(2\zeta) \]

\[ \zeta = 0.1 \]
\[ \zeta = 0.2 \]
\[ \zeta = 0.3 \]
\[ \zeta = 0.5 \]
\[ \zeta = 0.7 \]
\[ \zeta = 1.0 \]

-40 dB → dec
Bode Plot of a Constant Gain

The constant $K_B$ has a magnitude $|K_B|$, a phase angle of $0^\circ$ if $K_B$ is positive, and $-180^\circ$ if $K_B$ is negative. Therefore the Bode plots for $K_B$ are simply horizontal straight lines as shown in Figs. 15-1 and 15-2.

![Bode Plot of a Constant Gain](image)

Fig. 15-1

Fig. 15-2
Bode Plot of a Pole of Order $l$

The frequency response function (or sinusoidal transfer function) for a pole of order $l$ at the origin is

$$\frac{1}{(j\omega)^l}$$

(15.4)

The bode plots for this function are straight lines, as shown in Figs. 15-3 and 15-4.

**Fig. 15-3** Frequency $\omega$, rad/sec

**Fig. 15-4** Frequency $\omega$, rad/sec
Bode Plot of a Zero of Order \( l \) at the Origin

For a zero of order \( l \) at the origin,

\[
(j\omega)^l.
\]  \hspace{1cm} (15.5)

the Bode plots are the reflections about the 0-db and 0° lines of Figs. 15-3 and 15-4, as shown in Figs. 15-5 and 15-6.
Bode Plot of a Single Pole

Consider the single-pole transfer function \( p/(s + p), \ p > 0 \). The Bode plots for its frequency response function

\[
\frac{1}{1 + j\omega/p}
\]

(15.6)

are given in Figs. 15-7 and 15-8. Note that the logarithmic frequency scale is normalized in terms of \( p \).
Bode Plot of a Single Zero

The Bode plots and their asymptotic approximations for the single-zero frequency response function

\[
1 + \frac{j\omega}{z_1}
\]

(15.7)

are shown in Figs. 15-9 and 15-10.
Bode Magnitude & Phase Plots of 2nd Order Systems

The Bode plots and their asymptotic approximations for the second-order frequency response function with complex poles,

\[
\frac{1}{1 + j2\xi \omega / \omega_n - (\omega / \omega_n)^2}
\]

\[0 \leq \xi \leq 1\]  \hspace{1cm} (15.8)

are shown in Figs. 15-11 and 15-12. Note that the damping ratio \( \xi \) is a parameter on these graphs.

The magnitude asymptote shown in Fig. 15-11 has a corner frequency at \( \omega = \omega_n \) and a high-frequency slope twice that of the asymptote for the single-pole case of Fig. 15-7. The phase angle asymptote is similar to that of Fig. 15-8 except that the high-frequency portion is at \(-180^\circ\) instead of \(-90^\circ\) and the point of tangency, or inflection, is at \(-90^\circ\).

The Bode plots for a pair of complex zeros are the reflections about the 0 db and 0° lines of those for the complex poles.
Why Bode?
The great popularity of Bode magnitude plots stems from the following useful properties of logarithms:

If \( H(s) = \frac{(s + a)^n (s + b)^m}{(s + c)^l (s + d)^k} \) then

\[
\log_{10}[H(s)] = n \log_{10}(s + a) + m \log_{10}(s + b) - l \log_{10}(s + c) - k \log_{10}(s + d)
\]

Thus the magnitude functions are asymptotic to straight lines on a log-log plot.

dB or not dB? That is the question

Bode Plots are Magnitude and Phase versus frequency graphs. There are two log-log conventions for plotting magnitude versus frequency: log magnitude and decibels (dB).

Decibels (dB) to Magnitude Conversion

Magnitude dB = 20\log_{10}(\text{Magnitude})

Magnitude Conversion to Decibels (dB)

Magnitude = 10^{(\text{Magnitude dB])/20} Here is a conversion table:

<table>
<thead>
<tr>
<th>Decibel Examples</th>
<th>dB</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,000,000,000</td>
<td>+180</td>
</tr>
<tr>
<td>100,000,000</td>
<td>+160</td>
</tr>
<tr>
<td>10,000,000</td>
<td>+140</td>
</tr>
<tr>
<td>1,000,000</td>
<td>+120</td>
</tr>
<tr>
<td>100,000</td>
<td>+100</td>
</tr>
<tr>
<td>10,000</td>
<td>+80</td>
</tr>
<tr>
<td>1,000</td>
<td>+60</td>
</tr>
<tr>
<td>100</td>
<td>+40</td>
</tr>
<tr>
<td>10</td>
<td>+20</td>
</tr>
<tr>
<td>4</td>
<td>+12</td>
</tr>
<tr>
<td>2</td>
<td>+6</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1/2</td>
<td>-6</td>
</tr>
<tr>
<td>1/4</td>
<td>-12</td>
</tr>
<tr>
<td>0.1</td>
<td>-20</td>
</tr>
<tr>
<td>0.01</td>
<td>-40</td>
</tr>
<tr>
<td>0.001</td>
<td>-60</td>
</tr>
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</tr>
<tr>
<td>0.00001</td>
<td>-100</td>
</tr>
<tr>
<td>0.000001</td>
<td>-120</td>
</tr>
<tr>
<td>0.0000001</td>
<td>-140</td>
</tr>
<tr>
<td>0.00000001</td>
<td>-160</td>
</tr>
<tr>
<td>0.000000001</td>
<td>-180</td>
</tr>
</tbody>
</table>
Bode Plot Slopes for Poles and Zeros at the Origin

Bode plots of $(j\omega)^{\pm\rho}$
Bode Plot with Magnitude on a dB Scale in MATLAB

```matlab
% Magnitude of a Transfer Function on a dB Plot
% Save output figures in bitmap mode for best quality
s = tf('s');
H = 0.010*(s + 20)/((s + 1)*(s + 7000));
[mag phase w] = bode(H);

% Magnitude in dB not on log scale
mag2 = 20*log10(mag);

figure;
semilogx(w, reshape(mag2, 1, length(mag2)), 'LineWidth', 2);
grid minor; % finer grid
xlabel('\omega (rad/s)');
ylabel('Magnitude in dB');

figure;
semilogx(w, reshape(phase, 1, length(phase)), 'LineWidth', 2);
grid minor; % finer grid
xlabel('\omega (rad/s)');
ylabel('Phase (degrees)');
```

Bode Plot with Magnitude on Log Scale in MATLAB

```matlab
% Log Magnitude Plot
% Save output figures in bitmap mode for best quality
s = tf('s');
H = (s + 50)/((s + 10)*(s + 60000));
[mag phase w] = bode(H);

figure;
loglog(w, reshape(mag, 1, length(mag)), 'LineWidth', 2);
grid on;
xlabel('\omega (rad/s)');
ylabel('Magnitude');

figure;
semilogx(w, reshape(phase, 1, length(phase)), 'LineWidth', 2);
grid on;
xlabel('\omega (rad/s)');
ylabel('Phase (degrees)');
```
Stability from Bode Plots

Closed Loop System is stable provided the Gain of $L(j\omega)$ is less than 1 AND the phase of $L(j\omega)$ is less than $180^\circ$ for all $\omega$

Let $\omega_\pi$ be the phase cross over frequency where the phase of the open loop transfer function crosses $180^\circ$

Let $\omega_g$ be the gain cross over frequency where the open loop gain crosses 1.

Gain Margin = $1/|L(j\omega_\pi)|$ and Phase Margin = $\arg L(j\omega_g) + \pi$

System is marginally stable when
Gain Margin = 0 dB AND Phase Margin = $0^\circ$ (i.e., $\omega_g = \omega_\pi$)
Gain & Phase Margin Defined

\[ |L(j\omega)| \]
\[ \frac{1}{\text{Gain Margin}} = |L(j\omega_\pi)| \]

\[ \text{Phase Margin} = \arg L(j\omega_g) + \pi \]
16.2 GAIN FACTOR COMPENSATION

It is possible in some cases to satisfy all system specifications by simply adjusting the open-loop gain factor \( K \). Adjustment of the gain factor \( K \) does not affect the phase angle plot. It only shifts the magnitude plot up or down to correspond to the increase or decrease in \( K \). The simplest procedure is to alter the db scale of the magnitude plot in accordance with the change in \( K \) instead of reploting the curve. For example, if \( K \) is doubled, the db scale should be shifted down by \( 20 \log_{10} 2 = 6.02 \) db.

EXAMPLE 16.1. The Bode plots for

\[
GH(j\omega) = \frac{K_B}{j\omega(1+j\omega/2)}
\]

are shown in Fig. 16-1 for \( K_B = 1 \).

The maximum amount \( K_B \) may be increased to improve the system steady state performance without decreasing the phase margin below 45° is determined as follows. In Fig. 16-1, the phase margin is 45° if the gain crossover frequency \( \omega_c \) is 2 rad/sec and the magnitude plot can be raised by as much as 9 db before \( \omega_c \) becomes 2 rad/sec. Thus \( K_B \) can be increased by 9 db without decreasing the phase margin below 45°.
**FINDING CLOSED LOOP STABILITY MARGINS FROM THE OPEN LOOP GAIN**

**Problem:** For the Open Loop Magnitude and Phase Plots shown below, find 1) Gain Crossover Frequency, $\omega_g$ 2) the Phase Crossover Frequency, $\omega_\pi$ 3) Gain Margin (linear and dB), and 4) the Phase margin (in degrees and radians)

Gain Margin:

$$K = \frac{1}{|L(j\omega_g)|} = \frac{1}{0.22} \approx 4.5 = 13 \text{ dB}$$
FINDING THE CLOSED LOOP PHASE MARGIN

Phase Margin = +\arg[L(j\omega)] = +47^\circ = +0.82 \text{ rad}
EXAMPLE: FINDING K FOR A GIVEN PHASE MARGIN REQUIREMENT

Problem: Find the value of $K$ such that the open loop system with the frequency response shown has a closed loop PM of $+20^\circ$ (and thus the closed loop system is stable). (When PM=$0^\circ$, $K$ is Gain Margin)

$$K = \frac{1}{|L(j\omega_d)|} = \frac{1}{0.45} = 2.22$$
EXAMPLE: FINDING K FOR A GIVEN PHASE MARGIN REQUIREMENT

\[ \text{PM} = +20^\circ \]

\[ \omega_d = 0.7 \text{rad/ sec} \]
CHAPTER 6 FREQUENCY RESPONSE

MAGNITUDE PLOT FOR $K=2.22$

$|KL(j\omega)| = \omega$

$\omega_d = 0.7rad / sec.$
CHAPTER 6 FREQUENCY RESPONSE

PHASE PLOT FOR $K=2.22$

$\arg[KL(j\omega)]$

$\omega$ (rad/s)

PM = + 20°
What is the open loop transfer function $KL(j\omega)$ for the system whose magnitude and phase plots are shown on the previous few slides?

Inspection of magnitude and phase plot indicates that $KL(j\omega)$ is of the form:

$$KL(j\omega) = \frac{K_1}{j\omega\left(1 + \frac{j\omega}{\omega_0}\right)^2}$$

where $|KL(j\omega)|_{\omega=0.10} = 10 \approx |K_1/(j\omega)|_{\omega=0.10} = K_1/\omega_{0.10} = 10K_1 \Rightarrow K_1 = 1$

$\phi(\omega=\omega_0) = \phi[1/(j\omega_0)] + \phi\{1/[1+j(\omega_0/\omega_0)]^2\} = -90^\circ - 90^\circ = -180^\circ \Rightarrow \omega_0 = 1$

Hence the transfer function is:

$$KL(j\omega) = \frac{1}{j\omega(1 + j\omega)^2}$$
\[ L(\omega) = \frac{1}{\omega (\omega + 1)(\omega + 5)} = \frac{1}{\omega^3 + 6\omega^2 + 5\omega} \]

\[ L(j\omega) = \frac{0.2}{j\omega(j\omega + 1)(j\omega/5 + 1)} \]

**Magnitude (dB)**

-6 dB/oct
-14 dB
-12 dB/oct
-40 dB/dec
-18 dB/oct
-60 dB/dec

**Phase (Degrees)**

-90°
-135°
-180°
-225°
-270°

**Frequency (rad/sec)**

0.25
0.5
1
2
4
8
16

-90° - \tan^{-1}(\omega) - \tan^{-1}(\omega/5)
MATLAB CODE:

$$\text{sys=tf([1],[1 6 5 0])}$$

Transfer function:

$$\frac{1}{s^3 + 6s^2 + 5s}$$

$$\text{>> margin(sys)}$$

The system is neutrally stable for a gain of $K=30$ and the crossover frequencies are at $\omega=\pm \sqrt{5} = 2.236$. This agrees with the Bode plot, since $K=10^\left(Gm dB/20\right) = 29.8$ and the phase crossover frequency is $\omega=2.24$ rad/sec.

A more realistic gain would be $K=4.7$, which yields a phase margin of $45^\circ$ at a frequency of 0.75 rad/sec. The Gain Margin = 13.4 dB
\[ H(s) = \frac{s}{(s+2)(s+3)} \]

\[ |H(j\omega)| = \left| \frac{j\omega}{(3+j\omega)(2+j\omega)} \right| = \left| \frac{j\omega / c}{(1+j\omega/3)(1+j\omega/2)} \right| \]

\[ \phi = \tan^{-1} \omega - 2\omega - \tan^{-1} \left( \frac{\omega}{3} \right) - \tan^{-1} \left( \frac{\omega}{2} \right) \]

\[ \phi = 90^\circ - 2\omega - \tan^{-1} \left( \frac{\omega}{3} \right) - \tan^{-1} \left( \frac{\omega}{2} \right) = 90^\circ - 11.46 \omega - \tan^{-1} \left( \frac{\omega}{3} \right) - \tan^{-1} \left( \frac{\omega}{2} \right) \]

For \( \omega = 10.2 \), \( \phi \approx -180^\circ \)

\[ c = 2 \text{\omega rad} = -11.46 \omega \text{ degrees} \]
MATLAB CODE:  
\[
\text{>> sys =tf([1 0],[1 5 6])}
\]
Transfer function:
\[
\frac{s}{s^2 + 5s + 6}
\]
\[
\text{>> sys.outputd=0.2}
\]
Transfer function:
\[
\frac{s}{\exp(-0.2s) \cdot \frac{s}{s^2 + 5s + 6}}
\]
\[
\text{>> margin(sys)}
\]
Derive the Magnitude and Phase of the following functions of $\omega$ and plot both the Magnitude and Phase functions on the $\omega$ axis using the same logarithmic scale for $0 < \omega < \infty$. These are often referred to as Bode plots.

a) $j\omega$  
b) $(j\omega)^2$  
c) $(j\omega)^3$  
d) $1/j\omega$  
e) $1/(j\omega)^2$  
f) $1/(j\omega)^3$

c) $1+j\omega$  
d) $(1+j\omega)^2$  
e) $1/(1+j\omega)$  
f) $1/(1+j\omega)^2$

g) $(1+j3\omega)/(1+j2\omega)$  
h) $(1+j2\omega)/(1+j3\omega)$  
i) $j\omega/(1+j\omega)$

j) $j\omega/[ (1+j2\omega)(1+j3\omega) ]$  
l) $exp(j2\omega)$  
m) $exp(-j3\omega)$

n) $exp(-j4\omega)j\omega/[ (1+j2\omega)(1+j3\omega) ]$  
o) $X(j\omega) = 2\sin(\omega T)/\omega$
1st order High Pass

\[ \frac{\text{Mag}(dB)}{\log w} = \frac{20 \text{ dB}}{\text{dec}} \left| \frac{j \omega}{1+j \omega} \right| \]

\[ \phi = \tan^{-1} \omega \]

1st order Bandpass

\[ \frac{\text{Mag}(dB)}{\log w} = \frac{20 \text{ dB}}{\text{dec}} \left| \frac{j \omega}{(1+j \omega)(1+3j \omega)} \right| \]

\[ \phi = -\tan^{-1} \frac{3 \omega}{\omega} - \tan^{-1} \frac{2 \omega}{\omega} \]

0 dB \[ \left| e^{j \omega} \right| \text{ Linear } \omega \]

Linear Phase \[ \phi = 2 \omega \]

0 dB \[ \left| e^{-j \frac{3 \omega}{2}} \right| \text{ Linear } \omega \]

Linear Phase \[ \phi = -3 \omega \text{ Linear } \omega \]

same 1st order bandpass

Magnitude (as above)

Phase \[ \exp(-j\omega) \left( \frac{j \omega}{(1+2j \omega)(1+3j \omega)} \right) \]

\[ \phi = -4 \omega + 90^\circ - \tan^{-1} \frac{3 \omega}{\omega} - \tan^{-1} \frac{2 \omega}{\omega} \]
\[ X(j\omega) = \frac{2\sin(\omega T)}{\omega} \]

\[ \phi(\omega) = \angle \frac{2\sin(\omega T)}{\omega} \]

Magnitude

Phase

\[ \pi \]

\[ 0 \]

\[ -\pi / T \]

\[ \pi / T \]