Details and Rules of the Test

Introduction to Digital Signal Processing
EE 4361
Midterm Exam
Monday-June 21, 2004
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Please Print Clearly.

Last Name___________________     First Name _________________   Student ID______________

Instructions:
WRITE CLEARLY AND NEATLY

1. Exam Duration 1 hour and 15 minutes (Test will end at (9:50)
2. One 8.5" x 11" SINGLE-sided Crib sheet
3. One calculator is allowed
4. Read the test first and note that each problem is worth a different point value
5. A Z-Transform Table is provided on the last page of the test
6. Answer in the space/sheets provided
7. Where possible, show all your work; answers without any justification will not be credited
8. ANY copying or cheating will result in appropriate action as per university regulations
Complex Numbers in Communications Engineering

\[ j = i = \sqrt{-1} \]

**Euler’s Formula:**

\[ e^{\pm j\theta} = \cos \theta \pm j \sin \theta \]

**Useful Related Expressions:**

\[ \cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \]

\[ \sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \]
Principal \textit{Nth} root of a complex number 
(\(N=16\) for this example)

\[
z = r(\cos \theta + j \sin \theta)
\]

\[
z^{1/N} = [r(\cos \theta + j \sin \theta)]^{1/N} =
\]

\[
r^{1/N} \left\{ \cos \left( \theta + \frac{2k\pi}{N} \right) + j \sin \left( \theta + \frac{2k\pi}{N} \right) \right\}
\]

for \(k = 0, 1, 2, \ldots N-1\)
Phase and Arctangent

\[ \theta = \text{Arctan}_2 \text{ARG}(x) \]

Two Argument Arctangent function

\[ \theta = \tan^{-1}_{1\text{ARG}} \left( \frac{\text{Imag}(z)}{\text{Real}(z)} \right) \]

\[ \tan^{-1}_{2\text{ARG}} \left( \frac{\text{Imag}(z)}{\text{Real}(z)} \right) \]

Atan_{1\text{arg}}[\sin(45^\circ)/\cos(45^\circ)] = Atan_{2\text{arg}}[\cos(45^\circ),\sin(45^\circ)] = 45^\circ

Atan_{1\text{arg}}[\sin(-45^\circ)/\cos(-45^\circ)] = Atan_{2\text{arg}}[\cos(-45^\circ),\sin(-45^\circ)] = -45^\circ

Atan_{2\text{arg}}[\cos(135^\circ),\sin(135^\circ)] = Atan_{1\text{arg}}[\sin(135^\circ)/\cos(135^\circ)] + 180^\circ = 135^\circ

Atan_{2\text{arg}}[\cos(-135^\circ),\sin(-135^\circ)] = Atan_{1\text{arg}}[\sin(-135^\circ)/\cos(-135^\circ)] - 180^\circ = -135^\circ

\[ \text{for } z \in \text{Quadrant I or IV} \]

\[ \text{for } z \in \text{Quadrant II} \]

\[ \text{for } z \in \text{Quadrant III} \]
More on Phase and Complex Numbers

• Phase of a real number: $z = re^{j\theta} = r[\cos(\theta) + j\cdot0]$
  
  Phase[$z$] = Phase[$re^{j\theta}$]; $z$ is real => $\sin(\theta) = 0$. Two cases:
  1) $\theta = 0$ for $z$ positive, since $\cos(0) = 1$
  2) $\theta = \pi$ for $z$ negative, since $\cos(\pi) = -1$

• Phase of a purely complex number: $z = re^{j\theta} = r[0 + j\sin(\theta)]$
  
  Phase[$z$] = Phase[$re^{j\theta}$]; $z$ is purely complex => $\cos(\theta) = 0$. Two cases:
  1) $\theta = \pi/2$ for $z/j$ positive, since $\sin(\pi/2) = 1$
  2) $\theta = -\pi/2$ for $z/j$ negative, since $\sin(-\pi/2) = -1$

• Phase of the product of two complex numbers is the sum of the phases of the individual numbers:
  
  $\text{Phase}[z_1z_2] = \text{Phase}[r_1e^{j\theta_1}r_2e^{j\theta_2}] = \text{Phase}[r_1r_2e^{j(\theta_1+\theta_2)}] = \theta_1 + \theta_2$

• Phase of the quotient of two complex numbers is the difference between the phase of the numerator and the phase of the denominator:
  
  $\text{Phase}[z_1/z_2] = \text{Phase}[r_1e^{j\theta_1}/r_2e^{j\theta_2}] = \text{Phase}[r_1/r_2e^{j(\theta_1-\theta_2)}] = \theta_1 - \theta_2$
Chapter 1

**Sampling Theorem**

If the highest frequency contained in an analog signal is $F_{\text{max}} = B$ and the signal is sampled at a rate $F_s > 2F_{\text{max}} = 2B$, then the analog signal can be exactly recovered from its sample values using the interpolation function:

$$h(t) = \frac{\sin 2\pi B t}{2\pi B t}$$

- This is like putting a tomato in a veggomatic and then putting it back together again.
- Note that $h(t)$ can be considered a continuous time lowpass antialiasing filter.
- We can use the sampling theorem to resample the signal at any arbitrary resampling frequency without having to go back to analog domain.

**A/D Quantization and Dynamic Range**

For a bandlimited signal $x(t)$ that is sufficiently oversampled, the quantization error, $e_q(t)$, is approximately piecewise linear over the time $T$ that $x(t)$ remains within a quantization level. The power of the quantization error is approximately:

- $P_0 = \frac{1}{T} \int \frac{A^2}{2} t^2 dt = \frac{A^2}{4} \left[ \frac{t^3}{3} \right]_0 = \frac{A^2}{12}$
- The power of the sinusoid is $P_s = \frac{1}{T} \int (A \cos \omega_0 t)^2 dt = \frac{A^2}{2}$
- Quantizer has $b$ bits $\Rightarrow \Delta = 2\Delta \Rightarrow P = 2^{-b}$
- $\text{SNR} \text{ (i.e. Dynamic Range)} = P_s/P = 2^b \cdot 2^{2b}$
- $\text{Signal to Quantization Noise Ratio in dB} = 1.76 + 6.02b$
Properties of Linear Shift Invariant (LSI) Systems

• Any system that is described by the following constant-coefficient difference equation is **Linear Shift Invariant (LSI)**

\[ \sum_{k=0}^{N} a_k y(n - k) = \sum_{k=0}^{M} b_k x(n - k) \]

• **Causality**: A system is causal if \( y(n), \) for \( n=n_1, \) depends on \( x(n) \) **only** for \( n < n_1, \) i.e., the impulse response \( h(n)=0 \) for \( n<n_1 \)

• **Memory** defined through examples:
  • Memoryless (Instantaneous) e.g., \( y(n) = 2x(n) \) (memory of zero length)
  • Finite Memory e.g., \( y(n) = 2x(n) + 3x(n-1) + 4x(n-2) \) (i.e., FIR with memory of length 3)
  • Infinite memory e.g. \( y(n) = ay(n-1) + x(n) \) (first order recursive lowpass filter)

• **BIBO** (bounded input bounded output) Stability
  • A system is **BIBO** stable **if and only if for every** input \( x(n) \) that is bounded on the ordinate, there is a resulting output \( y(n) \) that is also bounded on the ordinate i.e., \( |x(n)|<\infty \) and \( |y(n)|<\infty \)
  • Equivalently, a system is **BIBO** stable **if and only if** it’s impulse response series is absolutely convergent

\[ \sum_{n=-\infty}^{+\infty} |h(n)| < \infty \]
Examples BIBO Stability

• Consider the first order causal recursive lowpass filter with difference equation
  \[ y(n) = ay(n-1) + x(n) \]

• The impulse response is \( h(n) = a^n u(n), n=0,1,2,\ldots \) (why?)
  \[ \sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} |a^n| = \frac{1-|a|^{N+1}}{1-|a|} \quad \text{for } N \geq 0 \]

• This is a geometric series which converges to
  \[ \sum_{n=-\infty}^{\infty} |h(n)| < \infty \]

• The system is stable if
  \[ \sum_{n=-\infty}^{\infty} |h(n)| < \infty \]

• This means that for the system to be stable, \(|a|<1\)

  e.g., 1) \( h(n) = 2^n u(n) \) is not stable, since
  \[ \sum_{n=-\infty}^{\infty} |h(n)| = \lim_{N \to \infty} \sum_{n=-\infty}^{N} |a^n| = \lim_{N \to \infty} \frac{1-2^{N+1}}{1-2} = 2^{N+1} - 1 = \infty \]

  2) \( h(n) = (1/2)^n u(n) \) is stable,
  \[ \sum_{n=-\infty}^{\infty} |h(n)| = \lim_{N \to \infty} \sum_{n=-\infty}^{N} |a^n| = \lim_{N \to \infty} \frac{1-|1/2|^{N+1}}{1-|1/2|} = 2 < \infty \]

  3) \( h(n) = (1)^n u(n) \) is not stable. However, this system is sometimes referred to as marginally stable because it is stable for many input waveforms. However, note that if \( x(n) = u(n) \) (bounded input), then \( y(n) = r(n) \) (unbounded output) \( \Rightarrow \) unstable
System Stability iff Unit Circle is in ROC of Z-Transform

• A LSI System is stable iff the ROC of the Z-Transform of its impulse response includes the unit circle

• Convergence iff \( H(z) = \sum_n |h(n) z^{-n}| < \infty \)

• Stability iff \( H(z) \) converges for \( |z| = 1 \), i.e., the ROC includes the unit circle hence, \( \sum_n |h(n)| < \infty \)
Chapter 2

- A discrete-time (DT) signal is defined only at discrete instants of time.
- A DT signal is usually represented as a sequence of values \( x(n) \) for integer values of \( n \).
- A DT signal \( x(n) \) is periodic with period \( N \) if \( x(n + N) = x(n) \) for some integer \( N \).

The DT unit-step and impulse functions are related as

\[
\mu(n) = \sum_{k=-\infty}^{\infty} \delta(k) \\
\delta(n) = \mu(n) - \mu(n - 1)
\]

- Any DT signal \( x(n) \) can be expressed in terms of shifted impulse functions as

\[
x(n) = \sum_{k=-\infty}^{\infty} x(k) \delta(n-k)
\]

- The complex exponential \( x(n) = \exp[j\omega_n n] \) is periodic only if \( \Omega / 2\pi \) is a rational number.
- The set of harmonic signals \( x_k(n) = \exp[jk\omega_0 n] \) consists of only \( N \) distinct waveforms.
- Time scaling of DT signals may yield a signal that is completely different from the original signal.
- Concepts such as linearity, memory, time invariance, and causality in DT systems are similar to those in continuous-time (CT) systems.
- A DT LTI system is completely characterized by its impulse response.
- The output \( y(n) \) of an LTI DT system is obtained as the convolution of the input \( x(n) \) and the system impulse response \( h(n) \):

\[
y(n) = h(n) \ast x(n) = \sum_{m=-\infty}^{\infty} h(m)x(n-m)
\]
- The convolution sum gives only the forced response of the system.
- An alternative representation of a DT system is in terms of the difference equation (DE)

\[
\sum_{k=0}^{N} a_kx(n-k) = \sum_{k=0}^{N} b_kx(n-k), \quad n \geq 0
\]
- The DE can be solved either analytically or by iterating from known initial conditions. The analytical solution consists of two parts: the homogeneous (zero-input) solution and the particular (zero-state) solution. The homogeneous solution is determined by the roots of the characteristic equation. The particular solution is of the same form as the input \( x(n) \) and its delayed versions.
- The impulse response is obtained by solving the system DE with input \( x(n) = \delta(n) \) and all initial conditions zero.
- The simulation diagram for a DT system can be obtained from the DE using summers, coefficient multipliers, and delays as building blocks.
- The following conditions for the BIBO stability of a DT LTI system are equivalent:
  (a) \( \sum |h(k)| < \infty \)
  (b) The roots of the characteristic equation are inside the unit circle. These are also the poles of the system transfer function \( H(z) = Y(z)/X(z) \).

Chapter 3

- The Z-transform is the discrete-time counterpart of the Laplace transform.
- The bilateral Z-transform of the discrete-time sequence \( x(n) \) is defined as

\[
X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}
\]
- The unilateral Z-transform of a causal signal \( x(n) \) is defined as

\[
X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} \quad \text{i.e., } x(n) \text{ is a right-sided sequence}
\]
- The region of convergence (ROC) of the Z-transform consists of those values of \( z \) for which the sum converges.
- For causal sequences, the ROC in the \( z \) plane lies outside a circle containing all the poles of \( X(z) \). For anticausal signals, the ROC is inside the circle such that all poles of \( X(z) \) are external to this circle. If \( x(n) \) consists of both a causal and an anticausal part, then the ROC is an annular region, such that the poles outside this region correspond to the anticausal part of \( x(n) \), and the poles inside the annulus correspond to the causal part. For finite sequences, the ROC is \( 0 < \Re[z] < \infty \)
- The Z-transform of an anticausal sequence \( x_{-\infty}(n) \) can be determined from a table of unilateral transforms as

\[
X_{-\infty}(z) = \frac{Z[x_{-\infty}(-n)]}{z^{-1}}
\]
- Expanding \( X(z) \) in partial fractions and identifying the inverse of each term from a table of Z-transforms is the most convenient method for determining \( x(n) \). If only the first few terms of the sequence are of interest, \( x(n) \) can be obtained by expanding \( X(z) \) in a power series in \( z^{-1} \) by a process of long division.
- The properties of the Z-transform are similar to those of the Laplace transform. Among the applications of the Z-transform are the solution of difference equations and the evaluation of the convolution of two discrete sequences.
- The time-shift property of the Z-transform can be used to solve difference equations.
- If \( y(n) \) represents the convolution of two discrete sequences \( x(n) \) and \( h(n) \), then

\[
Y(z) = H(z)X(z)
\]
- The transfer function \( H(z) \) of a system with input \( x(n) \), impulse response \( h(n) \), and output \( y(n) \) is

\[
H(z) = \frac{Z[y(n)]}{Z[x(n)]} = \frac{Y(z)}{X(z)}
\]
- The relation between the Laplace transform and the Z-transform of the sampled analog signal \( x_s(t) \) is

\[
X(z) \big|_{z = e^{\omega_0 t}} = X_s(s)
\]
- The transformation \( z = \exp[Ts] \) represents a mapping from the \( s \) plane to the \( z \) plane in which the left half of the \( s \) plane is mapped inside the unit circle in the \( z \) plane, the \( j\omega \) -axis is mapped into the unit circle, and the right half of the \( s \) plane is mapped outside the unit circle. The mapping effectively divides the \( z \) plane into horizontal strips of width \( \omega_0 \), each of which is mapped into the entire \( z \) plane.