NP Completeness

Classes $P \cap NP$

Decision problems only
(Yes/No)

$NP$
Problem, Input: $W \in \Sigma^*$

Yes/No? If an additional "guess" $g$ is supplied, then in polynomial time, we can verify that $W$ is a

Yes instance of problem.

$\begin{array}{c|c}
\text{Yes} & \text{No} \\
\hline
\exists \text{ guess } g \text{ such that verification algorithm outputs "Yes" in detem poly time} & \not\exists \text{ g, verification alg always outputs "No"}
\end{array}$
Suppose \( G \)

\[
G \rightarrow G^{+}
\]

\[
\text{for } a \in V \text{ do } k \rightarrow k + 1 \text{ while } (G', k) \text{ is an instance of clique do } k + 1
\]

Then we can find a max clique in poly time.

\( G \) is a clique (decider) has a polynomial-time algorithm, \((E, P)\).

If clique (decider) has a polynomial-time algorithm, \((E, P)\), does \( G \) have a clique of cardinality \( k \)?

Decision (clique): Given \((G, k)\), integer \( k \),

- the induced subgraph \( G[S] \) is a complete graph
- \((G, k)\) is a clique if \( G[S] \) is a clique
- \((G, k)\) is a clique if \( G[S] \) is a clique

1. Clique problem: Given \((G, k)\), find a clique

(wrt respect to polynomial time)

Equivalence between decision optimization problems
Output: \( t \uparrow X \uparrow t \)

Then \( \text{Pt? X? back into X.} \)

If \( X \) is not a No Instance of SubsetSum

Remove \( X \) from \( X \)

For \( i = 0 \) to \( n \) do

\[ \{ x_1, \ldots, x_n \} \rightarrow X \]

While \( i \leq n \), \( t \) is a No Instance of SubsetSum

\[ \text{If } \sum_{j=1}^{i} x_j < x_{i+1} + \sum_{j=1}^{i} x_j \text{, then } t \rightarrow x_1 + x_2 + \ldots + x_n \]

Decision: Given \( x_1, \ldots, x_n \), \( t \), find a subset of \( x_1, \ldots, x_n \) whose sum is as close as possible to \( t \), without going over it.

Find a subset of \( x_1, \ldots, x_n \) whose sum is as close as possible to \( t \).

\[ \left\lfloor \frac{x \text{?} 0 \text{? } t}{x \text{?} 0 \text{? } t} \right\rfloor \]

2. SubsetSum problem (Karpred):
Some NP Completeness proofs

Known NP-complete problems: Satisfiability, Clique, Subset Sum

New problem: Subgraph isomorphism

\[ \langle G_1, G_2 \rangle : \text{Is there a subgraph of } G_2 \text{ that is isomorphic to } G_1? \]

Claim: Subgraph isomorphism is NP-complete

Proof:
1. \( SI \in NP \). Given a guess: mapping from \( G_1 \to G_2 \),
   we can verify in poly time that this mapping is correct
   \[ \text{Verify that } m \text{ is 1 to 1 and also for each } (u,v) \in E_1, \quad (m(u), m(v)) \in E_2 \]
2. All problems in \( NP \) can be reduced to \( SI \).
   Known NP-complete problem
   \( \text{Clique} \leq_p SI \).
Consider an input \( (G', k) \) to the clique problem:

\[ G = (V, E) \] is \( k \)-complete graph on \( k \) nodes.

Construct an instance of SI:

\[ C_1 = \overline{k}=K \]

\[ C_2 = G \]

\[ \rightarrow \] If \( C_2 \) does not have a clique of size \( k \) (NO Instance)

\[ \Rightarrow \] is isomorphic to \( C_1 \rightarrow \text{yes instance of SI} \]

A yes instance of SI. A K-clique in \( G = G' \) is

\[ \Rightarrow \] If \( C \) has a clique of size \( k \), then \( (G, k) \) is

\[ \rightarrow \] (K-clique)

\[ \Rightarrow \] Consider an instance of SI:
\[
\begin{align*}
&\text{Let } x = \frac{2}{x} \text{ and } x = 2 - x.
\end{align*}
\]