EE6316 Fields and Waves

Homework Assignment #2 Solutions

Due on: Tue, March 7, 2006
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4. Balanis, Chapter 4, page 174, #4.3

The phasor notation of the H-field is

$$H = \left( \frac{1}{\eta_0} \right) \left( \hat{a}_x - 2\hat{a}_y \right) e^{-j\beta z} \quad (1.1)$$

And

$$H = \frac{1}{\eta_0} \mathbf{n} \times \mathbf{E} \quad (1.2)$$

, where in this particular case, the wave goes towards the z direction, so

$$\mathbf{n} = \hat{a}_z \quad (1.3)$$

(a) Complex E-field

Let (1.2) be written as

$$\mathbf{E} = \eta_0 \mathbf{H} \times \mathbf{n} \quad (1.4)$$

, into which we plug (1.1)

$$\mathbf{E} = \eta_0 \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ H_x & H_y & H_z \\ 0 & 0 & 1 \end{vmatrix} = \eta_0 \left( H_x \hat{a}_x - H_y \hat{a}_y \right)$$

$$= - \left( 2\hat{a}_x + \hat{a}_y \right) e^{-j\beta z} \quad (1.5)$$

(b) Instantaneous Poynting vector
\[
E \times H^* = 
\begin{vmatrix}
\hat{a}_x & \hat{a}_y & \hat{a}_z \\
-2e^{-j\beta z} & -e^{-j\beta z} & 0 \\
\frac{1}{\eta_0} e^{j\beta z} & \frac{1}{\eta_0} (-2)e^{j\beta z} & 0 
\end{vmatrix} = \left(\frac{5}{\eta_0}\right) \hat{a}_z 
\quad (1.6)
\]

\[
E \times H e^{j2\omega t} = e^{j2\omega t} 
\begin{vmatrix}
\hat{a}_x & \hat{a}_y & \hat{a}_z \\
-2e^{-j\beta z} & -e^{-j\beta z} & 0 \\
\frac{1}{\eta_0} e^{-j\beta z} & \frac{1}{\eta_0} (-2)e^{-j\beta z} & 0 
\end{vmatrix} = \left(\frac{5}{\eta_0}\right) \left(e^{j(2\omega t - 2\beta z)}\right) \hat{a}_z 
\quad (1.7)
\]

So, the instantaneous Poynting vector comes out to be

\[
p(x, y, z, t) = \frac{1}{2} \left[ \text{Re} \left( E \times H^* \right) + \text{Re} \left( E \times H e^{j2\omega t} \right) \right] 
= \left(\frac{5}{2\eta_0}\right) \left[ 1 + \cos(2\omega t - 2\beta z) \right] \hat{a}_z 
\quad (1.8)
\]

(c) Time-average Poynting vector

The time average of (1.8) is

\[
P_{\text{avg}} = \left(\frac{5}{2\eta_0}\right) \hat{a}_z 
\quad (1.9)
\]
5. Balanis, Chapter 4, page 175, #4.8

\[ E_0 (z = 0) = 4 \times 10^{-3} \text{ (V/m)} \]  
\[ f_0 = 300 \text{ (MHz)} \]  
\[ \omega = 2\pi f_0 = 1.88 \times 10^9 \text{ (rad/s)} \]  
\[ \beta = 2\pi f_0 / c = 6.28 \text{ (rad/m)} \]  

(a) Phasor \(E\) and \(H\)

Let the \(E\)-field in the phasor notation be

\[ \hat{E} = (\hat{a}_x) E_0 e^{j\beta z} \]  

Since

\[ \hat{H} = \frac{1}{\eta_0} \hat{n} \times \hat{E} \]  

where in this particular case, the wave goes into the \(-z\) direction, so

\[ \hat{n} = -\hat{a}_z \]  

The \(H\)-field is therefore

\[ \hat{H} = (-\hat{a}_x) \left( \frac{E_0}{\eta_0} \right) e^{j\beta z} \]  

where

\[ \frac{E_0}{\eta_0} = 1.06 \times 10^{-5} \text{ (A/m)} \]  

(b) Instantaneous \(E\) and \(H\)

\[ e(x, y, z; t) = \text{Re} \left[ E(x, y, z) e^{jot} \right] \]  

\[ = \text{Re} \left[ (\hat{a}_x) E_0 e^{j\beta z} e^{jot} \right] \]  

\[ = (\hat{a}_x) E_0 \cos(\omega t + \beta z) \]
\[ h(x, y, z; t) = \text{Re} \left[ H(x, y, z) e^{j\omega t} \right] \]
\[ = \text{Re} \left[ (-\hat{a}_t) \frac{E_0}{\eta_0} e^{j\beta z} e^{j\omega t} \right] \]
\[ = (-\hat{a}_t) \left( \frac{E_0}{\eta_0} \right) \cos(\omega t + \beta z) \]  

(c) Poynting Vector

The instantaneous Poynting vector
\[ p(x, y, z; t) = e \times h = (-\hat{a}_z) \left( \frac{E_0^2}{2\eta_0} \right) \left[ 1 + \cos(2\omega t + 2\beta z) \right] \]  

The time average
\[ P_{\text{avg}} = (-\hat{a}_z) \left( \frac{E_0^2}{2\eta_0} \right) \]  

, where
\[ \frac{E_0^2}{2\eta_0} = 2.12 \times 10^{-8} \text{ (W/m}^2) \]

(d) Energy Densities

The instantaneous energy densities
\[ w_e = \frac{1}{2} \epsilon_0 |e|^2 = \frac{1}{4} \epsilon_0 E_0^2 \left[ 1 + \cos(2\omega t + 2\beta z) \right] \]  

\[ w_h = \frac{1}{2} \mu_0 |h|^2 = \frac{1}{4} \mu_0 \left( \frac{E_0}{\eta_0} \right)^2 \left[ 1 + \cos(2\omega t + 2\beta z) \right] \]  

The time-average energy densities
\[ W_e = \frac{1}{4} \epsilon_0 E_0^2 = 3.54 \times 10^{-17} \text{ (J/m}^3) \]  

\[ W_h = \frac{1}{4} \mu_0 \left( \frac{E_0}{\eta_0} \right)^2 = 3.53 \times 10^{-17} \text{ (J/m}^3) \]  

\[ W_e = W_h \text{ as expected.} \]
6. **Balanis, Chapter 4, page 176, #4.13**

(a) **Time-average power density**

An antenna transmits a total power of

\[ P_{\text{rad}} = 5 \times 10^{-2} \text{ (W)} \]  

(1.28)

The power density at \( r(\text{m}) \) away from the antenna is

\[ S = \frac{P_{\text{rad}}}{4\pi r^2} = 4.42 \times 10^{-10} \text{ (W/m}^2\text{)} \]  

(1.29)

(b) **RMS E- and H fields**

Let the \( E \)-field be written as

\[ E = (\hat{\mathbf{a}}_r) E_0 e^{-j\beta z} \]  

(1.30)

and the \( H \)-field as

\[ H = (-\hat{\mathbf{a}}_x) \left( \frac{E_0}{\eta_0} \right) e^{-j\beta z} \]  

(1.31)

so that the wave travels in the positive direction

\[ S = (\hat{\mathbf{a}}_r) \frac{1}{2} \text{Re}(E \times H^*) = (\hat{\mathbf{a}}_r) \left( \frac{E_0^2}{2\eta_0} \right) \]  

(1.32)

Since

\[ \frac{E_0^2}{2\eta_0} = S = 4.42 \times 10^{-10} \text{ (W/m}^2\text{)} \]  

(1.33)

The peak value of the \( E \)-field is

\[ E_0 = 5.77 \times 10^{-4} \text{ (V/m)} \]  

(1.34)

and that of the \( H \)-field is

\[ E_0/\eta_0 = 1.53 \times 10^{-6} \text{ (A/m)} \]  

(1.35)

The root-mean-square value of the \( E \)-field is

\[ E_{\text{rms}} = E_0/\sqrt{2} = 4.08 \times 10^{-4} \text{ (V/m)} \]  

(1.36)

and that of the \( H \)-field is
\[ H_{\text{rms}} = \left( \frac{E_0}{\eta_0} \right) \frac{1}{\sqrt{2}} = 1.08 \times 10^{-6} \text{ (A/m)} \quad (1.37) \]

(c) **Time average energy density**

The time-average electric energy density

\[ W_e = \frac{1}{4} \varepsilon_0 E_0^2 = 7.37 \times 10^{-19} \text{ (J/m}^3\text{)} \quad (1.38) \]

The time-average magnetic energy density

\[ W_h = \frac{1}{4} \mu_0 \left( \frac{E_0}{\eta_0} \right)^2 = 7.35 \times 10^{-19} \text{ (J/m}^3\text{)} \quad (1.39) \]

\[ W_e = W_h \text{ as expected.} \]