AC RL and RC Circuits

• When a sinusoidal AC voltage is applied to an RL or RC circuit, the relationship between voltage and current is altered.

• The voltage and current still have the same frequency and cosine-wave shape, but voltage and current no longer rise and fall together.

• To solve for currents in AC RL/RC circuits, we need some additional mathematical tools:
  – Using the complex plane in problem solutions.
  – Using transforms to solve for AC sinusoidal currents.
Imaginary Numbers

- Solutions to science and engineering problems often involve $\sqrt{-1}$.
- Scientists define $i = -\sqrt{-1}$.
- As we EE’s use $i$ for AC current, we define $j = +\sqrt{-1}$.
- Thus technically, $j = -i$, but that does not affect the math.
- Solutions that involve $j$ are said to use “imaginary numbers.”
- Imaginary numbers can be envisioned as existing with real numbers in a two-dimensional plane called the “Complex Plane.”

The Complex Plane
The Complex Plane

- In the complex plane, imaginary numbers lie on the y-axis, real numbers on the x-axis, and complex numbers (mixed real and imaginary) lie off-axis.
- For example, 4 is on the +x axis, −8 is on the −x axis, j6 is on the + y axis, and −j14 is on the −y axis.
- Complex numbers like 6+j4, or −12 −j3 lie off-axis, the first in the first quadrant, and the second in the third quadrant.
Why Transforms?

- Transforms move a problem from the real-world domain, where it is hard to solve, to an alternate domain where the solution is easier.
- Sinusoidal AC problems involving $R-L-C$ circuits are hard to solve in the “real” time domain but easier to solve in the $\omega$-domain.
In the time domain, RLC circuit problems must be solved using calculus.

However, by transforming them to the $\omega$ domain (a radian frequency domain, $\omega = 2\pi f$), the problems become algebra problems.

A catch: We need transforms to get the problem to the $\omega$ domain, and inverse transforms to get the solutions back to the time domain!
A Review of Euler’s Formula

- You should remember Euler’s formula from trigonometry (if not, get out your old trig textbook and review): \( e^{\pm jx} = \cos x \pm j \sin x \).
- The alternate expression for \( e^{\pm jx} \) is a complex number. The real part is \( \cos x \) and the imaginary part is \( \pm j \sin x \).
- We can say that \( \cos x = \text{Re}\{e^{\pm jx}\} \) and \( \pm j \sin x = \text{Im}\{e^{\pm jx}\} \), where \( \text{Re} = \) “real part” and \( \text{Im} = \) “imaginary part.”
- We usually express AC voltage as a cosine function. That is, an AC voltage \( v(t) \) and be expressed as \( v(t) = V_p \cos \omega t \), where \( V_p \) is the peak AC voltage.
- Therefore we can say that \( v(t) = V_p \cos \omega t = V_p \text{Re}\{e^{\pm j\omega t}\} \). This relation is important in developing inverse transforms.
Transforms into the $\omega$ Domain

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<td>$C$</td>
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- The time-domain, sinusoidal AC voltage is normally represented as a cosine function, as shown above.
- $R$, $L$ and $C$ are in Ohms, Henrys and Farads.
- Skipping some long derivations (which you will get in EE 3301), transforms for the $\omega$ domain are shown above.
- Notice that the AC voltage $\omega$-transform has no frequency information. However, frequency information is carried in the $L$ and $C$ transforms.
Comments on $\omega$ Transforms

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- Because we are studying constant-frequency sinusoidal AC circuits, the $\omega$-domain transforms are constants.
- This is a considerable advantage over the time-domain situation, where $t$ varies constantly (which is why solving for sinusoidal currents in the time domain is a calculus problem).
- Two other items:
  - In the $\omega$-domain, the units of $R$, $j\omega L$, and $1/j\omega C$ are Ohms.
  - In the $\omega$-domain, Ohm’s Law and Kirchhoff’s voltage and current laws still hold.
Solving problems in the frequency domain:
- Given a circuit with the AC voltage shown, and only a resistor in the circuit, then the transform of the voltage is 10. $R$ transforms directly as 100.
- Solving for the circuit current, $I = V/R$, or $I = 10/100 = 0.1$ A.
- This current is the $\omega$-domain answer. It must be inverse-transformed to the time domain to obtain a usable answer.

$\omega$-domain voltage = $V_p = 10$

$\omega$-domain current = $V_p / R = 10/100 = 0.1$ ampere
An $\omega$ Domain Solution for an $L$ Circuit

- The $\omega$-domain voltage is still 10.
- The $\omega$-domain transform of $L = j\omega L = j(1000)10(10)^{-3} = j10$.
- The units of the $L$ transform is in Ohms ($\Omega$), i.e., the $\omega$-domain transform of $L$ is $j10 \Omega$.
- The value $\omega L$ is called **inductive reactance** ($X$). The quantity $j\omega L$ is called **impedance** ($Z$).
- Time-domain answer in a few slides!

\[\omega\text{-domain voltage} = V_p = 10\]
\[\omega\text{-domain current} = \frac{V_p}{j\omega L} = \frac{10}{j10} = -j1 = -j \text{ ampere}\]
An \( \omega \) Domain Solution for a \( C \) Circuit

- The \( \omega \)-domain voltage still = 10.
- The \( \omega \)-domain transform of \( C = 1/j\omega C \) 
  \( = 1/j(1000)100(10)^{-6} = 1/j0.1 = -j10 \).
- The units of the \( C \) transform is in Ohms (\( \Omega \)), i.e., the \( \omega \)-domain transform of \( C \) is \(-j10 \, \Omega \).
- The value \( 1/\omega C \) is called **capacitive reactance**, and \( 1/j\omega C \) is also called **impedance** (here, capacitive impedance).
- Finding the current: \( I = V/Z = 10/-j10 = 1/-j = j1 \) (rationalizing) = \( j \).
- Time-domain answer coming up!

\[ \text{AC Voltage} = 10 \cos (1000t) \]

\[ \omega \)-domain voltage = \( V_p = 10 \]

\[ \omega \)-domain current = \( V_p /(1/j\omega C) \]

\[ = 10/-j10 = j1 = j \text{ amperes} \]
An $RL \omega$-Domain Solution

- The $\omega$-domain voltage still = 10.
- The $\omega$-domain impedance is $10+j10$.
- Resistance is still called resistance in the $\omega$-domain. The $R$ and $L$ transforms are called impedance, and a combination of resistance and imaginary impedances is also called impedance.
- Note: all series impedances add directly in the $\omega$-domain.
- Finding the current: $I = \frac{V}{Z} = \frac{10}{10+j10} = \text{(rationalizing)} (\frac{100-j100}{200} = 0.5-j0.5$.
- Time-domain answers next!

$\omega$-domain voltage $= V_p = 10$

$\omega$-domain current $= \frac{V_p}{(R+j\omega L)} = \frac{10}{10+j10} = 0.5-j0.5$ ampere
Inverse Transforms

- Our $\omega$-domain solutions do us no good, since we are inhabitants of the time domain.
- We required a methodology for inverse transforms, mathematical expressions that can convert the frequency domain currents we have produced into their time-domain counterparts.
- It turns out that there is a fairly straightforward inverse transform methodology which we can employ.
- First, some preliminary considerations.
Cartesian-to-Polar Transformations

- Our $\omega$-domain answers are complex numbers – currents expressed in the $X$-$Y$ coordinates of the complex plane.
- Coordinates in a two-dimensional plane may also be expressed in $R$-$\theta$ coordinates: a radius length $R$ plus a counterclockwise angle $\theta$ from the positive $X$-axis (at right).
- That is, there is a coordinate $R, \theta$ that can express an equivalent position to an $X,Y$ coordinate.
Cartesian-to-Polar Transformations (2)

• The $R, \theta$ coordinate is equivalent to the $X,Y$ coordinate if $\theta = \arctan(Y/X)$ and $R = \sqrt{X^2 + Y^2}$.

• In our $X$-$Y$ plane, the $X$ axis is the real axis, and the $Y$ axis is the imaginary axis. Thus the coordinates of a point in the complex plane with (for example) $X$ coordinate $A$ and $Y$ coordinate $+B$ is $A+jB$.

• Now, remember Euler’s formula:
  $$e^{\pm jx} = \cos x \pm j \sin x$$
Cartesian-to-Polar Transformations (3)

- If $e^{\pm j\omega} = \cos \omega \pm j \sin \omega$, then $Re^{\pm j\omega} = R \cos \omega \pm Rj \sin \omega$.
- But in our figure, $R \cos \theta = X$, and $R \sin \theta = Y$.
- Or, $Re^{\pm j\theta} = X \pm jY$!
- What this says is that when we convert our $\omega$-domain AC current answers into polar coordinates, we can express the values in $Re^{\pm j\theta}$ format as well as $R, \theta$ format.
- The $Re^{\pm j\theta}$ is very important in the inverse transforms.
Inverse Transform Methodology

- We seek a time-domain current solution of the form \( i(t) = I_p \cos(\omega t) \). where \( I_p \) is some peak current.
- This is difficult to do with the \( \omega \)-domain answer in Cartesian (\( A \pm jB \)) form.
- So, we convert the \( \omega \)-domain current solution to \( R, \theta \) format, then convert that form to the \( Re^{\pm j\theta} \) form, where we know that \( \theta = \arctan \left( \frac{Y}{X} \right) \), and \( R = \sqrt{X^2 + Y^2} \).
- Once the \( \omega \)-domain current is in \( Re^{\pm j\theta} \) form (and skipping a lot of derivation), we can get the time-domain current as follows:
Inverse Transform Methodology (2)

• Given the $Re^{\pm j\theta}$ expression of the $\omega$-domain current, we have only to do two things:
  – Multiply the $Re^{\pm j\theta}$ expression by $e^{j\omega t}$.
  – Take the real part.

• This may seem a little magical at this point, but remember, $Re (e^{j\omega t})$ is $\cos \omega t$, and we are looking for a current that is a cosine function of time.

• We can see examples of this methodology by converting our four $\omega$-domain current solutions to real time-domain answers.
In the resistor case, our $\omega$-domain current is a real number, 0.1 A. Then $X=0.1$, $Y=0$.

Then $R = \sqrt{X^2 + Y^2} = \sqrt{(0.1)^2} = 0.1$, and $\theta = \arctan(Y/X) = \arctan 0 = 0$.

Thus current = $\Re\{0.1 e^{j\omega t} e^{j\theta}\} = 0.1 \Re\{0.1 e^{j\omega t}\} = 0.1 \cos 1000\pi t$ A.

Physically, this means that the AC current is cosinusoidal, like the voltage. It rises and falls in lock step with the voltages, and has a maximum value of 0.1 A (figure at right).
• For the inductor circuit, \( I = -j1 = -j \).

• Converting to polar: \( R = \sqrt{X^2 + Y^2} = \sqrt{(1)^2} = 1 \)

• \( \theta = \arctan \frac{Y}{X} = \arctan \frac{-1}{0} = \arctan -\infty = -90^\circ \).

• \( I_\omega = 1, -90^\circ = 1e^{-j90^\circ} = e^{-j90^\circ} \).

• Multiplying by \( e^{j\omega t} \) and taking the real part: \( i(t) = \text{Re}\{e^{j\omega t} \cdot e^{-j90^\circ}\} = \text{Re}\{e^{j(\omega t - 90^\circ)}\} = (1)\cos(\omega t - 90^\circ) = \cos(\omega t - 90^\circ) \text{ A} \).

• Physical interpretation: \( i(t) \) is a maximum of 1 A, is cosinusoidal like the voltage, but lags the voltage by exactly 90\(^{\circ}\) (plot at right).

• The angle \( \theta \) between voltage and current is called the phase angle. Cos \( \theta \) is called the power factor, a measure of power dissipation in an inductor or capacitor circuit.
For the capacitor circuit, \( I = -j \) A.

Converting to polar:
\[
R = \sqrt{X^2 + Y^2} = \sqrt{(-1)^2} = 1.
\]

\( \theta = \arctan \frac{Y}{X} = \arctan \frac{1}{0} = \arctan \infty = 90^\circ \), so that \( I_\omega = 1, 90^\circ = e^{j90^\circ} \).

Multiplying by \( e^{j\omega t} \) and taking the real part:
\[
i(t) = \text{Re}\{e^{j\omega t} \cdot e^{j90^\circ}\} = \text{Re}\{e^{j(\omega t + 90^\circ)}\} = \cos(\omega t + 90^\circ) \text{ A}.
\]

Physically, \( i(t) \) has a maximum amplitude of 1 A, is cosinusoidal like the voltage, but leads the voltage by exactly 90° (figure at right).
For the RL circuit, \( I = 0.5 - j0.5 \) ampere.

Converting to polar:
\[
R = \sqrt{X^2 + Y^2} = \sqrt{(0.5)^2 + (-0.5)^2} \approx 0.707.
\]

And \( \theta = \arctan \frac{Y}{X} = \arctan \frac{-0.5}{0.5} = \arctan (-1) = -45^\circ; \quad I_\omega = 0.707, \quad -90^\circ = 0.707e^{-j45^\circ} \).

Multiplying by \( e^{j\omega t} \) and taking the real part:
\[
i(t) = \text{Re}\{0.707e^{j\omega t} \cdot e^{-j45^\circ}\} = 0.707\text{Re}\{e^{j(\omega t-45^\circ)}\} = 0.707\cos(\omega t-45^\circ) = 0.707\cos(\omega t-45^\circ) \text{ A}.
\]

Note the physical interpretation: \( i(t) \) has a maximum amplitude of 0.707 A, is cosinusoidal like the voltage, and lags the voltage by 45\(^\circ\). Lagging current is an inductive characteristic, but it is less than 90\(^\circ\), due to the influence of the resistor.
Summary: Solving for Currents Using $\omega$ Transforms

- Transform values to the $\omega$-domain:

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- Solve for $I_\omega$, using Ohm’s and Kirchoff’s laws.
  - Solution will be of the form $A \pm jB$ (Cartesian complex plane).

- Use inverse transforms to obtain $i(t)$.
  - Convert the Cartesian solution ($A \pm jB$) to $R, \theta$ format and thence to $Re^{\pm j\theta}$ form.
  - Multiply by $e^{\pm j\omega t}$ and take the real part to get a cosine-expression for $i(t)$. 
Measuring AC Current Indirectly

• Because we do not have current probes for the oscilloscope, we will use an indirect measurement to find \( i(t) \) (reference Figs. 11 and 13 in Exercise 5).

• As the circuit resistance is real, it does not contribute to the phase angle of the current. Then a measure of voltage across the circuit resistance is a direct measure of the phase of \( i(t) \).

• Further, a measure of the \( \Delta t \) between the \( i,v \) peaks is a direct measure of the phase difference in seconds.

• We will use this method to determine the actual phase angle and magnitude of the current in Lab. 5.
Discovery Exercises

- Lab. 5 includes two exercises that uses inductive and capacitive impedance calculations to allow the discovery of the equivalent inductance of series inductors and the equivalent capacitance of series capacitors.
- Question 7.6 then asks you to infer the equivalent inductance of parallel inductors and the equivalent capacitance of parallel capacitors.
- Although you are really making an educated guess at that point, you can validate your guess using $\omega$-domain circuit theory, with one additional bit of knowledge not covered in the lab text:
  - In the $\omega$-domain, parallel impedances add reciprocally, just like resistances in a DC circuit.
  - (Remember that in the $\omega$-domain, series impedances add directly).