HOMEWORK 10, STAT 6332

1. Consider the cosine basis on $[0, 1]$ discussed in class. Suppose that a function $f$ is differentiable on the unit intervals. Prove that the Fourier coefficient satisfies

$$|\theta_j| \leq 2^{1/2}(\pi j)^{-1} \int_0^1 |f^{(1)}(x)| \, dx, \ j \geq 1.$$  

As you see, Fourier coefficients are decreasing at the rate $j^{-1}$.

Hint: Think about integration by parts.

2. Establish a similar inequality for the case when a function has two derivatives. Make a conclusion.

3. Establish a similar inequality for the case when a function has three derivatives. Make a conclusion. (Hint: This is not a simple case — do your calculations accurately.)

4. Let $f_S(x) := \sum_{j=0}^S \theta_j \varphi_j(x)$. Find

$$\int_0^1 (f_{J+L}(x) - f_J(x))^2 \, dx.$$  

5. Let $[0, 1]$ be the support of a density. Do you need to estimate $\theta_0$?

6. Suppose that Fourier coefficients of the density $f(x)$ supported on $[0, 1]$ are known. For the sample mean estimator

$$\hat{\theta}_j = n^{-1} \sum_{i=1}^n \varphi_j(X_i)$$

find the expression for

$$E\{ (\hat{\theta}_j - \theta_j)^2 \}.$$  

7. Consider a linear density estimator

$$\tilde{f}(x) := 1 + \sum_{j=1}^{\infty} \lambda_j \hat{\theta}_j \varphi_j(x).$$

Here $\lambda_j \in [0, 1]$ are smoothing weights. Assume that the density $f$ is given (note that this implies that its Fourier coefficients are also given). Then find the optimal smoothing weights $\lambda_j^*$ which minimize the MISE of the linear density estimator, that is, they minimize

$$E\{ \int_0^1 (\tilde{f}(x) - f(x))^2 \, dx \}.$$  

What you get is called the Wiener’s smoothing coefficient (filter). Hint: Begin with Parseval’s identity and go from there.