Welcome to your first homework. It is devoted to some topics in calculus that you must know very well. If not — dust off books and review the stuff. I am not going to discuss it in class.

All my solutions for a homework may contain some seeded mistakes. They help me to understand if you did your HW on your own or just used my solutions. As a result, it is prudent to do all problems by yourself. If you find a mistake — do not e-mail or call me — just describe it on the top of your solution and you can take (by yourself) a partial credit for it to compensate for some unsolved problems. However, your total for all HWs cannot be more than 20 points.

Now let us look at your problems. In what follows I may write $f'(x)$ for the first derivative $d f(x)/dx$, $f''(x)$ for the second derivative $d^2 f(x)/dx^2$, etc. I also may write $[f(x)]_a^b$ for $f(b) - f(a)$, and $[f(x)]_{x=a} = [f(x)]_a = f(a)$. This simplifies notation and allows me to write shorter formulas.

Further, when you write formulas by hand, please do it as accurately as possible — for instance, a subscript should be seen as a subscript, that is $a^b$ is not the same as $ab$ or $a^b$, etc. Such small “things” are the most typical reasons for failure on exams.

1. There are at least two reasonable approaches to solve the problem:

A. Write

$$2x^2(5x^4 + 3) = 10x^6 + 6x^2$$

and then

$$d(10x^6 + 6x^2)/dx = 60x^5 + 12x.$$

B. Use the product rule $(f g)' = f'g + fg'$ and get

$$[2x^2(5x^4 + 3)]' = (4x)(5x^4 + 3) + (2x^2)(20x^3) = 60x^5 + 12x.$$

2. Write using the product rule and chain rule (which is $[f(g(x))]' = g'(x)[df(z)/dz]_{z=g(x)}$),

$$[(3x)(e^{4x^2+3})]' = 3(4x^2+3) + (3x)(8x)[e^{4x^2+3}] = 3(1 + 8x^2)e^{4x^2+3}.$$

Let me explain in more detail the used chain rule:

$$de^{4x^2+3}/dx = (4x^2 + 3)[de^z/dz]_{z=4x^2+3} = (8x)[e^z]_{z=4x^2+3} = 8xe^{4x^2+3}.$$

3. For a quotient we have the rule

$$[f(x)/g(x)]' = rac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

Using it we get

$$\left(\frac{2x^2 + 5}{\sin(x^2)}\right)' = \frac{(4x)(\sin(x^2)) - (2x^2 + 5)(2x\cos(x^2))}{\sin^2(x^2)}.$$
4. Here the denominator and the numerator vanish as \( x \to 2 \), so you use the L'Hospital rule by looking at the corresponding limit of the ratio of derivatives:

\[
\lim_{x \to 2} \frac{2^{x/2} - 2}{2^x - 4} = \lim_{x \to 2} \frac{[2^{x/2} - 2]'}{[2^x - 4]'} = \left. \frac{(1/2)2^{x/2} \ln(2)}{[2^x \ln(2)]'} \right|_{x=2} = 1/4.
\]

In the above I used formula \( [a^x]' = a^x \ln(a) \) for a positive \( a \). You can establish/verify it by writing \( a^x = e^{x \ln(a)} \) and then taking the derivative.

5. Write,

\[
\partial(x^{2y})/\partial x = 2yx^{2y-1}.
\]

Further,

\[
\partial(x^{2y})/\partial y = 2x^{2y} \ln(x),
\]

and we finish with

\[
\partial^2(x^{2y})/\partial y^2 = \partial(2x^{2y} \ln(x))/\partial y = 2 \ln(x)[2x^{2y} \ln(x)] = 4x^{2y} \ln^2(x).
\]

6. Write,

\[
\int_1^\infty x^{-3/2} dx = \left. -2x^{-1/2} \right|_1^\infty = -2[0 - 1] = 2.
\]

7. We have

\[
\int_1^5 x^{-1} dx = [\ln(x)]^5_{x=1} = \ln(5).
\]

8. Write

\[
\int_0^1 \int_1^2 x^2y^{-1} dy dx = \int_0^1 x^2 \left[ \int_1^2 y^{-1} dy \right] dx
\]

\[
= \int_0^1 x^2 \ln(2) dx = \ln(2) \left[ (1/3)x^3 \right]_0^1 = (1/3) \ln(2).
\]

9. This is an example for the change of variable approach. Write,

\[
\int_0^1 x[1 - x^2]^{1/2} = \left. \text{change of variable } y = x^2 - 1 \right. \text{ and use } dy = 2xdx
\]

\[
= \int_0^1 (1/2)(-y)^{1/2} dy = \left. \text{change of variable } v = -y \right.
\]

\[
= (1/2) \int_0^1 v^{1/2} dv = (1/2) \left[ (2/3)v^{3/2} \right]_0^1 = 1/3.
\]

10. Write,

\[
\int_0^1 \int_0^{3x} y dy dx = \int_0^1 [(1/2)y^2]_0^{3x} dx = \int_0^1 (1/2)(3x)^2 dx = [(9/2)(1/3)x^3]_0^1 = 3/2.
\]

11. Here I use integration by parts which is the following method:

\[
\int_a^b f'(x)g(x) dx = [f(x)g(x)]_a^b - \int_a^b g'(x)f(x) dx.
\]
The main issue in using it is to guess about functions $f$ and $g$. In a majority of cases this is not difficult because $g(x)$ will be a polynomial (power) function, say $x^k$, $k$ is a positive integer. Then by taking the derivative the power decreases and by repeating the integration by parts we eventually get $x' = 1$ which is just a factor.

Let us see how this works out in our example. Write,

$$
\int_1^4 xe^{cx}dx = [(x)(e^{-e^{cx}})]_1^4 - \int_1^4 (dx/dx)(e^{-e^{cx}})dx = c^{-1}[4e^{4c} - e^c] - [e^{cx}]_1^4
$$

$$
= (4c^{-1} - 1)e^{4c} - (c^{-1} - 1)e^c.
$$

12. This is the sum of a geometric progression,

$$
a \sum_{k=1}^K b^k = a \frac{b - b^{K+1}}{1 - b} = ab \frac{1 - b^K}{1 - b}.
$$

To remember/check the formula, just multiply both sides of the first equality by $1 - b$.

13. Note that $f(x) = F(x^{2/3}) - F(1)$ where $F(t)$ is an antiderivative of $(1 + t^3)^{1/3}$. Then using the chain rule we get

$$
f'(x) = (2/3)x^{-1/3}[1 + (x^{2/3})^{3/3}]^{1/3} = (2/3)x^{-1/3}[1 + x^{2}]^{1/3}.
$$

Of course, you may remember a general Leibnitz’s rule for taking such derivatives which is

$$
d[\int_{a(x)}^{b(x)} f(u, x)du]/du = \int_{a(x)}^{b(x)} [\partial f(u, x)/\partial x]du + b'(x)f(b(x), x) - a'(x)f(a(x), x).
$$

14. Begin with plotting $y = x^{1/2}$ and $y = -x^{1/2}$ for $x \in [0, 1]$. Then look at the area between these curves for $x \in [0, 1]$. In the given integral you first fix $x$ from $[0, 1]$ and then take integral over the vertical line between the two curves. When you change the order of integration, you fix $y$ from $[-3^{1/2}, 3^{1/2}]$ and then take integral over the corresponding horizontal line. From the diagram you should see that the integration in $x$ is over the interval $[y^2, 3]$. As a result,

$$
\int_0^3 [\int_{x^{1/2}}^{-x^{1/2}} g(x, y)dy]dx = \int_{-3^{1/2}}^{3^{1/2}} [\int_{y^2}^3 g(x, y)dx]dy.
$$

15. OK, this is a problem from a SOA exam. I will show how it is solved “profesionally”. Denote by $X$ and $Y$ random numbers of claims during first and second weeks. Then we are interested in a new random variable $Z = X + Y$ which is the total of claims. The question is to find the probability $P(Z = 7)$ given that $X$ and $Y$ are independent and identically distributed with $P(X = k) = 2^{-(k+1)}$ for $k = 0, 1, \ldots$

Solution: See how I write it:

$$
P(Z = 7) = P(X + Y = 7) = \sum_{k=0}^7 P((X = k) \cap (Y = 7 - k)).$$

3
The last equation is valid because events in the sum are the partition of the event $Z = 7$. Because $X$ and $Y$ are independent, by definition of independent events we have

$$P((X = k) \cap (Y = 7 - k)) = P(X = k)P(Y = 7 - k).$$

We conclude that

$$P(Z = 7) = \sum_{k=0}^{7} P(X = k)P(Y = 7 - k) = \sum_{k=0}^{7} 2^{-(k+1)}2^{-(7-k+1)} = (8)2^9 = 1/64.$$