SOLUTION FOR HOMEWORK 5, ACTS 4306

Welcome to your 5th homework.
In what follows, when a definite integral is taken, I use the notation
\[ g(x)|_a^b := g(x)|_{x=a}^{b} := g(b) - g(a). \]

1. Find: \( P(Y \in (1, 3) | X = 2) \).

Solution: Write
\[ P(Y \in (1, 3) | X = 2) = \int_1^3 f_{Y|X}(y|2) dy, \tag{1} \]
where
\[ f_{Y|X}(y|x) := \frac{f_{XY}(x, y)}{f_X(x)} \tag{2} \]
and \( f_X \) is the marginal pdf,
\[ f_X(x) := \int_{-\infty}^\infty f_{XY}(x, y) dy \]
(I plug in the given joint density — note that here density is the shorthand for the pdf)
\[
= \int_{-\infty}^{\infty} \frac{2}{x^2(x - 1)} y^{-(2x - 1)/(x - 1)} I(x > 1) I(y > 1) dy = \int_{1}^{\infty} \frac{2}{x^2(x - 1)} y^{-(2x - 1)/(x - 1)} I(x > 1) dy
\]
\[
= \frac{2I(x > 1)}{x^2(x - 1)[(\frac{2x - 1}{x - 1} + 1)]y^{-(\frac{2x - 1}{x - 1} + 1)}y^{\infty}}_{y=1}
\]
\[
= \frac{(-1)(2)I(x > 1)}{x^2(x - 1)[(\frac{2x - 1}{x - 1} + 1)]} = 2x^{-3}I(x > 1).
\]

Because the calculation was long, let us check the obtained result. We know that the pdf must be nonnegative and integrated to 1, that is
\[ 2x^{-3}I(x > 1) \geq 0; \quad \int_{-\infty}^{\infty} f_X(x) dx = \int_{1}^{\infty} 2x^{-3} dx = -x^{-2}|_{x=1}^{\infty} = 1. \]
Thus the calculated density is bona fide (it is OK).

Plug the calculated marginal pdf in (2) and the result plug in (1) and get
\[ P(Y \in (1, 3) | X = 2) = \int_1^3 \frac{2}{2^3(2-1)} y^{-(2)(2-1)/(2-1)} \frac{y^{-2}dy}{2/3} \]
\[ = 2 \int_1^3 y^{-3}dy = (2)(-1/2)y^{-2/1}|_1^3 = 1 - 1/9 = 8/9. \]

Remark: I solved the problem for a rather general setting. To solve the problem faster, you may plug in \( X = 2 \) just in the beginning of the solution.
Answer: E

2. Given: \( f_X(x) = I(x \in (0, 1)), f_{Y|X}(y|x) = I(y \in (x, x + 1)) \).
Find: \( P(Y > .5) \).

Solution: Write,

\[
P(Y > .5) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(y > .5) f_{XY}(x,y) dy \, dx = \int_{-\infty}^{\infty} f_X(x) \left[ \int_{.5}^{\infty} f_{Y|X}(y|x) dy \right] dx
\]

(plug in the given joint pdf and, in what follows, graphic of the area of integration may help you to understand my calculus)

\[
= \int_{0}^{1} \left[ \int_{.5}^{\infty} I(y \in (x, x + 1)) dy \right] dx = \int_{0}^{1} \left[ \int_{\max(0,5)}^{x+1} dy \right] dx
\]

\[
= \int_{0}^{5} (x + .5) dx + \int_{5}^{1} dx = [(1/2)x^2 + (.5)x]_{x=5}^{x=0} + [x]_{x=5}^{x=0} = [1/8 + 1/4] + [1/2] = 7/8.
\]

Answer: D

3. Find: \( P((X > 20) \cap (Y > 20)) \).

Solution: Write,

\[
P((X > 20) \cap (Y > 20)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x > 20) I(y > 20) f_{XY}(x,y) dx \, dy
\]

(plug in the given joint pdf)

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x > 20) I(y > 20) I(0 < x < 50 - y < 50) [6/125,000] (50 - x - y) dx \, dy.
\]

Now let us find the region of integration. The following graphic helps us to visualize it:
We conclude, using the graphic, that

\[ P((X > 20) \cap (Y > 20)) = \int_{20}^{30} \left[ \int_{20}^{50-x} (6/125,000)(50 - x - y)dy \right] dx. \]

Answer: B

4. Solution: First of all, please note that \( 0 \leq y \leq 1 \) on the support \( \{(x, y) : x^2 \leq y \leq x\} \) because \( x^2 < x \) implies \( 0 \leq x \leq 1 \). Then we calculate the marginal density,

\[ g(y) = f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx = \int_{-\infty}^{\infty} 15yI(x^2 \leq y \leq x) dx \]

\[ = 15yI(0 \leq y \leq 1) \int_y^{y^{1/2}} dx = 15y(y^{1/2} - y)I(0 \leq y \leq 1) = 15y^{3/2}(1 - y^{1/2})I(0 \leq y \leq 1). \]

Answer: E

5. Find: \( P(X + Y > 1) \).

Solution:

\[ P(X + Y > 1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x + y > 1) f_{X,Y}(x, y) dx dy \]

\[ = \int_0^1 \int_0^{x^2} I(y > 1 - x) \frac{2x + 2 - y}{4} dy dx \]

(look at the region of integration and using it we continue)

\[
\begin{align*}
&= \int_0^1 \left[ \int_{1-x}^{x^2} \frac{2x + 2 - y}{4} dy \right] dx \\
&= \int_0^1 \left[ \frac{2x + 2 - y^2}{8} \right]_{y=1-x}^{y=x^2} dx \\
&= \int_0^1 [x + 1/2 + (1/2)(x^2 - 1) + (1/8) - (x/4) + (x^2/8)] dx
\end{align*}
\]
\begin{align*}
&\int_0^1 [(1/8) + (3x/4) + (5x^2/8)]dx = \left[(x/8) + (3x^2/8) + (5x^3/24)\right]_{x=0}^1 \\
&= (1/8) + (3/8) + (5/24) = \frac{3 + 9 + 5}{24} = \frac{17}{24} = .71.
\end{align*}

Answer: D

6. Solution: Write,

\[ P(Y < X|X = 1/3) = \int_{-\infty}^{\infty} I(y \leq 1/3)f_{Y|X=y}(1/3)dy. \]

Let us find the conditional density. Write

\[ f_{Y|X}(y|1/3) = \frac{f_{XY}(1/3, y)}{f_X(1/3)} = \frac{(24)(1/3)yI(0 < y < 2/3)}{\int_{-\infty}^{\infty}(24)(1/3)yI(0 < y < 2/3)dy} \]

\[ = \frac{yI(0 < y < 2/3)}{(1/2)y^{2/3}} = \frac{yI(0 < y < 2/3)}{(1/2)y^{2/3}} = (9/2)yI(0 < y < 2/3). \]

Then you may check that this is indeed a bona fide density (nonnegative and integrated to 1 function).

Remark. You can find the conditional density faster using the following approach. First you note that

\[ f_{Y|X}(y|1/3) = kyI(0 < y < 2/3) \]

where \( k \) is (an unknown) constant. Then, because the density must be integrated to 1, you can find the constant \( k \) from the equation

\[ 1 = \int_0^{2/3} kyI(0 < y < 2/3) \]

which yields \( k = 9/2 \).

Now we can finish the problem:

\[ P(Y < 1/3|x = 1/3) = \int_0^{1/3} (9/2)ydy = (9/2)(1/2)(1/3)^2 = 1/4. \]

Answer: C

7. Notations: \( S \) and \( T \) are lifetimes of the two devices.

Given: \( f_{ST}(s, t) = f(s, t)I(0 < s < 1)I(0 < t < 1). \)
Find: \( P(\min(S, T) < .5). \)

Solution: Write

\[ P(\min(S, T) < .5) = 1 - P(\min(S, T) \geq .5) \]
1 - P(\(S \geq .5\) \cap \(T \geq .5\)) = 1 - \int_0^1 \int_0^1 f(s, t)dsdt

(we could finish here but the answer is formulated in a different format so we continue using the fact that the joint density is integrated to 1)

= \int_0^1 \int_0^1 f(s, t)dsdt - \int_{.5}^1 \int_{.5}^1 f(s, t)dsdt

Now is the time to look at the region of integration:

Using the graphic we conclude that

= \int_0^1 \left[ \int_0^5 f(s, t)ds \right]dt + \int_0^5 \int_0^1 f(s, t)dsdt.

Remark: You can also solve the problem by writing the probability in question as

P(\(\min(S, T) < .5\)) = P((S < .5) \cup (T < .5)) = \int \int_{(s, t):s<.5\&t<.5} f(t, s)dt\,ds

and then using the same graphic. Note that by De Morgan's law \(((S < .5) \cup (T < .5))' = (S > .5) \cap (T > .5)\) and this relates the two approaches.

Answer: E

8. Find: P(\(Y < .05\mid X = .1\)).

Solution: Let us find \(f_{Y\mid X}(y\mid .1)\). Write

\[f_{Y\mid X}(y\mid .1) = \frac{2(.1 + y)I(0 \leq y \leq .1)}{f_X(.1)} =: k(.1 + y)I(0 \leq y \leq .1).\]

Here \(k\) is a constant which we find via the condition that the conditional density is integrated to 1. We get the equation

\[1 = \int_0^1 f_{Y\mid X}(y\mid .1)\,dy = k[.1y + (1/2)y^2]_{y=0}^{1} = k[.01 + (1/2)(.01)]\]
whose solution yields $k = 200/3$.

Now we can solve the problem:

$$P(Y < .05| x = .1) = (200/3) \int_0^{.05} [1+y]dy = (200/3) [(1)(.05) + (1/2)(.05)^2] = .417.$$

Answer: D


Solution: We begin with a calculation of the marginal pdf:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_{-\infty}^{\infty} 6(1-x-y)I(x > 0)I(y > 0)Y(x+y < 1)dy$$

$$= 6I(0 < x \leq 1) \int_0^{1-x} (1-x-y)dy = 6I(0 < x \leq 1)[(1-x)^2 - (1/2)(1-x)^2] = 3(1-x)^2I(0 < x \leq 1).$$

Let us quickly check that this is a valid pdf:

$$3 \int_0^1 (1-x)^2dx = (3)(1/3)(-1)(1-x)^3 |_{x=0}^1 = 1.$$

So it is the pdf. Then

$$P(X < .2) = \int_{0}^{.2} f_X(x)dx = (3)(1/3)(-1)(1-x)^3 |_{x=0}^2 = 1 - (.8)^3 = .488.$$

Remark: You can solve this problem more directly as well:

$$P(X < .2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x < .2)f_{X,Y}(x,y)dxdy$$

and then look at the graphic for the region of integration:

![Graphical representation of the region of integration](image)

The result should be the same.
Answer: C

10. Given: \( f_X(x) = (1/\alpha)e^{-x/\alpha}I(x > 0) \), \( f_Y(y) = (1/\beta)e^{-y/\beta}I(y > 0) \), \( f_{XY}(x,y) = f_X(x)f_Y(y) \).
Find: \( P(X > Y) \).

Solution: Write,

\[
P(X > Y) = \int_0^\infty \int_0^\infty I(x > 0)f_{XY}(x,y)dxdy
\]

\[
= \int_0^\infty \left[ \int_y^\infty f_X(x)dx \right]f_Y(y)dy.
\]

Note that the inner integral is

\[
\int_y^\infty (1/\alpha)e^{-x/\alpha}dx = e^{-y/\alpha},
\]

and then

\[
P(X > Y) = \int_0^\infty (1/\beta)e^{-y[(1/\alpha)+(1/\beta)]}dy = \frac{1}{\beta}\left[\frac{1}{\alpha} + \frac{1}{\beta}\right] = \frac{\alpha}{\alpha + \beta}.
\]

Answer: A