SOLUTION FOR HOMEWORK 5, STAT 4351

Welcome to your 5th homework, which finishes our study of chapter 3. Here we are exploring basics of multivariate random variables (rv).

Now let us look at your problems.
1. Problem 3.43(a). By definition,
\[ F(1.2, .9) = P(X \leq 1.2, Y \leq .9) = P(X = 0, Y = 0) + P(X = 1, Y = 0) = 1/12 + 1/6. \]

2. Problem 3.44. We know that the joint probability mass function should be summable to 1, and this will allow us to find \( c \). Write,
\[
1 = \sum_{x,y} f(x, y) = \sum_{x \in \{-1,0,1,3\}} \sum_{y \in \{-1,2,3\}} c(x^2 + y^2)
\]
\[
= c \sum_{x \in \{-1,0,1,3\}} [3x^2 + (1 + 4 + 9)] = c[(3 + 0 + 3 + 27) + (4)(14)] = c[33 + 56] = 89c.
\]
Answer: \( c = 1/89 \).

3. Problem 3.50. Here it is very important to write the density mathematically and then draw the support of the density together with the area of integration. We have:
\[
f(x, y) = 24xyI(0 < x < 1)I(0 < y < 1)I(x + y < 1).
\]
Then
\[
P(X + Y < 1/2) = \int_{x+y<1/2} f(x, y)dx dy = \int_0^{1/2} \left[ \int_0^{1/2-x} f(x, y)dy \right] dx
\]
\[
= \int_0^{1/2} \left[ \int_0^{1/2-x} 24xydy \right] dx = \int_0^{1/2} 24x\left[ (1/2)y^2 \right]_0^{1/2-x} dx = \int_0^{1/2} 24x(1/2)(1/2-x)^2 dx
\]
\[
= 12 \int_0^{1/2} [x/4 - x^2 + x^3] dx = 12 \left[ x^2/8 - x^3/3 + x^4/4 \right]_0^{1/2}
\]
\[
= 12[1/32 - 1/24 + 1/64].
\]

4. Problem 3.51(c). Again, we write the joint density:
\[
f(x, y) = 2I(x > 0)I(y > 0)I(x + y < 1).
\]
Then we solve the problem:
\[
P(X > 2Y) = \int_{x>2y} f(x, y)dx dy = \int_0^1 \left[ \int_y^{\min(1-x,x/2)} 2dy \right] dx.
\]
Now I pause for a second. I may continue using mathematics, but it is simpler to look at the graphic and the constant density to understand that the answer is the doubled area under
the min(1 − x, x/2). Note that min(1 − x, x/2) = x/2 for x < 2/3 and min(1 − x, x/2) = 1 − x for x ≥ 2/3. This yields that (by the way, check that the set of interest is the triangle and its area is (1)(1/3)(1/2) = 1/6)


Note that a direct integration implies the same outcome!

5. Problem 3.73. We begin with (a). Write:

\[f(x, y) = (1/4)[I(−1, −1) + I(−1, 1) + I(1, −1) + I(1, 1)]\]

Well, let us make some calculations:

\[f^X(−1) = (1/4)[1 + 1] = 1/2, \quad f^X(1) = (1/4)[1 + 1] = 1/2,\]

\[f^Y(−1) = (1/4)[1 + 1] = 1/2, \quad f^Y(1) = (1/4)[1 + 1] = 1/2,\]

Then we see that \(f(x, y) = f^X(x)f^Y(y)\). Answer: Independent.

Remark: a simpler solution is just rewrite the jpmf as

\[f(x, y) = (1/4)I(x ∈ \{−1, 1\})I(y ∈ \{−1, 1\}).\]

Then the independence follows from the factorization of the joint density into the product of a function in \(x\) and a function in \(y\).

(b). Here

\[f(x, y) = (1/3)[I(0, 0) + I(0, 1) + I(1, 1)].\]

Then

\[f^X(0) = (1/3)[1 + 1] = 2/3, \quad f^X(1) = (1/3)[1] = 1/3,\]

\[f^Y(0) = (1/3)[1] = 1/3, \quad f^Y(1) = (1/3)[1 + 1] = 2/3.\]

As a result, for instance,

\[1/3 = f(0, 0) \neq (2/3)(1/3) = f^X(0)f^Y(0).\]

Thus the random variables are dependent.

6. Problem 3.74. The joint probability density is

\[f(x, y) = (1/4)(2x + y)I(0 < x < 1)I(0 < y < 2).\]

(a) Let us find the marginal density \(f^X(x)\):

\[f^X(x) = \int_{−∞}^{∞} f(x, y)dy = \int_{0}^{2} (1/4)(2x + y)dyI(o < x < 1) = (1/4)[2xy + y^2/2]_0^2 I(o < x < 1)\]

\[= (1/4)[4x + 2]I(o < x < 1).\]

Please pay attention to the fact that I write the support everywhere!
Remark: It can be a good habit to check your answer via verifying that the marginal density is integrated to 1. Let us do this:

\[ \int_{-\infty}^{\infty} f_X(x) \, dx = \int_{0}^{1} (1/4)(4x+2) \, dx \]

\[ = (1/4) \left[ 2x^2 + 2x \right]_0^1 = (1/4)[2+2] = 1. \]

(b) The conditional density of Y given X = x is

\[ f(y|X = x) = f(y|x) = \frac{f(x,y)}{f_X(x)}. \]

For the specific \( X = x = 1/4 \) we get

\[ f(y|X = 1/4) = \frac{(1/4)(1/2 + y)}{3/4} I(0 < y < 2) = (1/6 + y/3) I(0 < y < 2). \]

Attention: an answer without the support (here \( y \in (0,2) \)) is incomplete/incorrect.

7. Problem 3.76. The joint density is

\[ f_{XY}(x,y) = 24y(1-x-y)I(x > 0)I(y > 0)I(x + y < 1). \]

By the way, do you see that the random variables are dependent?

(a) The marginal density of X is

\[ f_X(x) = \int_{-\infty}^{\infty} f(x,y) \, dy = \int_{0}^{1-x} 24y(1-x-y) \, dy I(0 < x < 1) \]

\[ = 24 \left[ (1/2)y^2(1 - x) - (1/3)y^3 \right]_{y=0}^{y=1-x} I(0 < x < 1) \]

\[ = 24[(1-x)^3/2 - (1-x)^3/3] I(0 < x < 1) = 4(1-x)^3 I(0 < x < 1). \]

Please pay attention to the support (here \( 0,1 \)). Also, you may check your answer by integration of the marginal: the integral should be 1.

(b) The marginal density of Y is

\[ f_Y(y) = \int_{-\infty}^{\infty} f(x,y) \, dx = \int_{0}^{1-y} 24y(1-x-y) \, dx I(0 < y < 1) \]

\[ = 24 \left[ y(1-y)x - yx^2/2 \right]_{x=0}^{x=1-y} I(0 < y < 1) \]

\[ 24[y(1-y)^2 - y(1-y)^2/2] I(0 < y < 1) = 12y(1-y)^2 I(0 < y < 1). \]

(c) Either directly (compare the joint density with the product of marginals) or via viewing the joint density you may conclude that the random variables are dependent.
8. Problem 3.82. It is given that the two random variables are independent, so their joint is the product of individual (marginal) densities:

\[ f_{XY}(x, y) = f(x)\pi(y) = (1/6)I(0 < x < 2)I(0 < y < 3). \]

This answers (a). Please note that: (i) the joint density is uniform; (ii) only if two random variables are independent it suffices to know their marginals to find the joint density.

(b). Well, we need to take integrals:

\[
P(X^2 + Y^2 > 1) = \int_{x^2+y^2>1} f(x, y) dxdy = (1/6) \int_0^2 \left[ \int_{y^2>1-x^2} I(0 < y < 3) dy \right] dx
\]

\[
= (1/6) \int_0^1 [3 - (1 - x^2)^{1/2}] dx + (1/6) \int_1^2 3 dx = 1/2 - (1/6) \int_0^1 \sqrt{1-x^2} dx + 1/2
\]

\[
= 1 - (1/6) \int_0^1 \sqrt{1-x^2} dx.
\]

We need to calculate the integral. Let us try the substitution \( x = \sin(u), \ x \in (0, 1) \). Note that when \( x \) changes from 0 to 1, \( u \) changes from 0 to \( \pi/2 \). Also, \( dx = \cos(u) du \) and \( \sqrt{1-x^2} = \cos(u) \). Using these facts we get:

\[
\int_0^1 \sqrt{1-x^2} dx = \int_0^{\pi/2} \cos^2(u) du = \pi/4.
\]

(The last integral is elementary taken via using \( \sin^2(u) + \cos^2(u) \equiv 1 \).)

Substituting this integral we conclude that

\[
P(X^2 + Y^2 > 1) = 1 - \pi/24.
\]

Well, instead of this complicated (but straightforward) solution we can look at the geometry instead because here the density is uniform. The area of interest is \( \{(x, y) : x^2 + y^2 > 1, 0 < x < 2, 0 < y < 3\} \). Draw it and see that the area is \( 6 - \pi/4 \). Then multiply the area by 1/6 and get the answer.

9. Problem 3.91. Here

\[ f(x) = (1/5)I(227.5 < x < 232.5). \]

Note that the distribution is continuous uniform, and boundaries for probabilities are irrelevant. Then:

(a) \[ P(X \leq 228.65) = (1/5)(228.65 - 227.5). \]

(b) \[ P(229.34 \leq X \leq 231.66) = (1/5)(231.66 - 229.34). \]

(c) \[ P(X \geq 229.85) = (1/5)(232.5 - 229.85). \]
10. Problem 3.101. Here the joint probability density is

\[ f(x, y) = \left(\frac{2}{5}\right)(2x + 3y)I(0 < x < 1)I(0 < y < 1). \]

Note that the random variables are dependent.

(a) Let us calculate

\[ P(X < .4, Y < .4) = \int_0^{.4} \left[ \int_0^{.4} f(x, y)dx \right] dy = \left(\frac{2}{5}\right) \int_0^{.4} (2x + 3y)dx \int_0^{.4} dy \]

\[ = \left(\frac{2}{5}\right) \int_0^{.4} x^2 dx + 3y \int_0^{.4} dy = \left(\frac{2}{5}\right) \int_0^{.4} (.16 + 1.2y) dy \]

\[ = \left(\frac{2}{5}\right)(.4)(.16) + (.6)(.16) = \left(\frac{2}{5}\right)(.16). \]

(b) Here we are calculating

\[ P(X > .8, Y < .5) = \left(\frac{2}{5}\right) \int_0^{.8} \left[ \int_{.5}^{1} (2x + 3y)dy \right] dx \]

\[ = \left(\frac{2}{5}\right) \int_0^{.8} x^2 + 3y \int_{.5}^{1} dy = \left(\frac{2}{5}\right) \int_0^{.8} (1 - .64 + 3y(.2)) dy \]

\[ = \left(\frac{2}{5}\right) \int_0^{.8} (.36 + 3y(.2)) dy = \left(\frac{2}{5}\right) [.36y + .3y^2]_{y=.5}^{y=0} \]

\[ = \left(\frac{2}{5}\right)(.36)(.5) + (.3)(.25) = \left(\frac{2}{5}\right)(.18 + .075). \]

I hope that I made no mistakes. Agree?