Welcome to your 7th homework. Here we finish our work on the multivariate distributions and their descriptive characteristics.

Now let us look at your problems.

1. Problem 4.46. Here the pmf is

\[ f(x) = (1 + x)I(-1 < x \leq 0) + (1 - x)I(0 < x < 1). \]

Note that this density is even (symmetric) about 0 [you may draw its graphic and see a nice triangular density]. In particular, this yields \( E(X) = 0 \) (you get it without a calculation).

(a) Set \( U = X, \ V = X^2 \). Then again due to the even nature of the density and the zero expectation

\[ Cov(U, V) = E\{(X - E(X))(X^2 - E(X^2))\} = E(X^3) = 0. \]

Of course, a direct calculation will apply the same result.

(b). You may find the marginal densities and compare their product with the joint one — they should not coincide. But here, just as an alternative way to do this, I simply suggest an example which establishes dependence. Let us consider

\[ \Pr(V \leq 1/2|U > u) = \Pr(X^2 \leq 1/4|X > u). \]

Plainly, if \( u > 1/2 \) then the conditional probability is 0, but if \( u < 1/4 \) then it is positive. This shows that \( V \) and \( U \) are dependent. You may suggest many similar counter-examples on your own.

2. Problem 4.48. I will do a general case \( Y = aX_1 + bX_2 + cX_3 \). Then

\[ E(Y) = aE(X_1) + bE(X_2) + cE(X_3) = (a)(4) + (b)(9) + (c)(3). \]

(b) Because \( X_1 \), \( X_2 \) and \( X_3 \) are independent:

\[ Var(y) = a^2Var(X_1) + b^2Var(X_2) + c^2Var(X_3) = (a)^2(3) + (b)^2(7) + (c)^2(5). \]

3. Problem 4.50. Here you have several options. One is to find marginal densities and then use them to calculate corresponding characteristics. I will take a path of answering the question directly.

Note that adding a constant does not change the variance, and thus

\[ Var(W) = Var(V), \quad V := 3X + 4Y. \]

Then

\[ E(V^2) = 9E(X^2) + 16E(Y^2) + 24E(XY). \]
Now we are considering these terms in turn.

\[ E(X^2) = \frac{1}{3} \int_0^2 \left[ \int_0^1 x^2(x + y) \, dx \right] \, dy = \frac{1}{3} \int_0^2 \left[ \frac{x^4}{4} + \frac{yx^3}{3} \right]_{x=0}^1 \, dy \]

\[ = \frac{1}{3} \int_0^2 (1/4 + y/3) \, dy = \frac{1}{3} \left[ \frac{y/4 + y^2/6}{} \right]_0^2 = \frac{1}{3}[1/2 + 4/6] = 7/18. \]

Further,

\[ E(Y^2) = \frac{1}{3} \int_0^2 \left[ \int_0^1 y^2(x + y) \, dx \right] \, dy = \frac{1}{3} \int_0^2 y^2(1/2 + y) \, dy \]

\[ = \frac{1}{3} \left[ \frac{y^3/6 + y^4/4}{} \right]_0^2 = \frac{1}{3}[8/6 + 16/4] = 16/9. \]

Further,

\[ E(XY) = \frac{1}{3} \int_0^2 y \left[ \int_0^1 x(x + y) \, dx \right] \, dy \]

\[ = \frac{1}{3} \int_0^2 (1/3 + y/2) \, dy = \frac{1}{3} \left[ \frac{y^2/6 + y^3/6}{} \right]_0^2 = \frac{1}{3}[4/6 + 8/6] = 2/3. \]

We conclude that

\[ E(V^2) = (9)(7/18) + (16)(16/9) + (24)(2/3) = 7/2 + 16^2/9 + 48/3. \]

Well, now we need to calculate the expectation.

\[ E(V) = E(3X + 4Y) = \frac{1}{3} \int_0^2 \left[ \int_0^1 (3x + 4y)(x + y) \, dx \right] \, dy \]

\[ = \frac{1}{3} \int_0^2 \left[ \int_0^1 (3x^2 + 4y^2 + 7xy) \, dy \right] \, dx = \frac{1}{3} \int_0^2 [1 + 4y^2 + (7/2)y] \, dx \]

\[ = \frac{1}{3} \left[ y + (4/3)y^3 + (7/4)y^2 \right]_0^2 = \frac{1}{3}[2 + 32/3 + 28/40]. \]

Finally,

\[ Var(W) = E(V^2) - (E(V))^2, \]

and you need to plug-in numbers. I stop here.

4. Problem 4.52. (a)

\[ Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y). \]

(b) \[ Var(X - Y) = Var(X) + Var(Y) - 2Cov(X, Y). \]

(c) Here it is convenient to note that subtracting a constant does not change the covariance and the variance; as a result we can assume without any loss of generality that \( E(X) = E(Y) = 0. \) Then

\[ Cov(X + Y, X - Y) = E\{(X + Y)(X - Y)\} \]

\[ = E(X^2 - Y^2) = E(X^2) - E(Y^2) = Var(X) - Var(Y). \]
Note that I can solve the problem for not-zero-mean RV, but then the computation is lengthier.

5. Problem 4.55. Write:

\[ E(X|Y = -1) = \sum_x x f^{X|Y}(x|Y = -1) = \frac{\sum_x f^{X,Y}(x, Y = -1)}{P(Y = -1)} \]

\[ = \frac{(-1)(1/8) + (1)(1/2)}{1/8 + 1/2} = \frac{(1/2) - (1/8)}{(1/2) + (1/8)} = \frac{4 - 1}{4 + 1} = 3/5. \]

Further,

\[ E(X^2|Y = -1) = \frac{(1)(1/8) + (1)(1/2)}{(1/8) + (1/2)} = 1. \]

Then

\[ Var(X|Y = -1) = E(X^2|Y = -1) - [E(X|Y = -1)]^2 = 1 - 9/25 = 16/25. \]

6. Problem 4.56. OK, let us do this.

\[ E(Z^2|X = 1, Y = 2) = \sum_{z=1}^2 z^2 f(Z = z|X = 1, Y = 2) = \frac{\sum_{z=1}^2 z^2 f(Z = z, X = 1, Y = 2)}{P(X = 1, Y = 2)} \]

\[ = \frac{(1)(2/108) + (2^24)/108}{(2/108) + (4/108)} = 17/6. \]

7. Problem 4.59a. This is a very nice problem which yields a powerful tool for calculation of conditional expectations.

(a) Write:

\[ F(x|a < X \leq b) = \frac{P(X \leq x, a < X \leq b)}{P(a < X \leq b)}. \]

This conditional cdf is 0 if \( x \leq a \) and 1 if \( x > 0 \) because \( \{X \leq x\} \) and \( \{a < X \leq b\} \) are disjoint in the first case and \( \{a < X \leq b\} \subseteq \{X \leq x\} \) in the latter. For \( a < x \leq b \) we have

\[ F(x|a < X \leq b) = \frac{P(a < X \leq x)}{P(a < X \leq b)} = \frac{F(x) - F(a)}{F(b) - F(a)} I(a < x \leq b). \]

In the last equality we used familiar fact that any cdf is continuous from the right.

(b) Taking derivative we get

\[ f(x|a < X \leq b) = \frac{f(x)}{F(b) - F(a)} I(a < x \leq b). \]

Then for a function \( u(x) \):

\[ E(u(X)|a < X \leq b) = \frac{\int_a^b u(x) f(x) dx}{F(b) - F(a)} = \int_a^b u(x) f(x) dx \int_a^b f(x) dx. \]
This is a very convenient formula indeed.

8. Problem 4.77. Set: inside diameter is $r$, thickness is $T$. The outside diameter is

$$R = r + 2T,$$

with

$$E(R) = E(r) + 2E(T) = 3 + (2)(.3),$$

and due to independence of $r$ and $T$

$$Var(R) = Var(r) + 4Var(T) = (.02)^2 + (4)(.005)^2.$$

Then for the standard deviation

$$Stdev(R) = \left[(.02)^2 + 4(.005)^2\right]^{1/2}.$$

9. Problem 4.78. Let $L$ denote the length, $T$ denotes thickness of the mortar, $W$ is the total length of 50 bricks. Then

$$W = (50)L + (49)T.$$

This implies that

$$E(W) = 50E(L) + 49E(T) = (50)(8) + (49)(.5),$$

and

$$Var(W) = (50)^2Var(L) + (49)^2Var(T) = (50)^2(.1) + (49)^2(.03)^2.$$

Then the square root of the variance gives the wished standard deviation.

10. Problem 4.83. Write

$$E(X | X \geq 1) = \frac{\int_1^{\infty} x f(x)dx}{\int_1^{\infty} f(x)dx} = \frac{\int_1^{2} x(x/4)dx + \int_2^{\infty} 4x^{-2}dx}{\int_1^{2} x(x/4)dx + \int_2^{\infty} 4x^{-3}dx}$$

$$= \frac{[x^3/12]^2_1 - 4[x^{-1}]^\infty_2}{[x^2/8]^2_1 - 4[x^{-2}/2]^\infty_2} = \frac{(8 - 1)/12 + 4/2}{((4 - 1)/8 + 1/2}$$

$$= \frac{(7 + 24)/12}{(3 + 4)/8} = 62/21.$$

If I made mistakes, please note them on the first page of your solution and add extra point(s).