SOLUTION FOR HOMEWORK 5, STAT 4352

Welcome to your last homework before Exam 2. We finish the interval estimation; your
Exam 2 is next week and it will be close to HW4-HW5.

Try to find mistakes (and get extra points) in my solutions. Typically they are silly
arithmetic mistakes (not methodological ones). They allow me to check that you did your
HW on your own. Please do not e-mail me about your findings — just mention them on the
first page of your solution and count extra points.

Now let us look at your problems.

1. Problem 11.17. It is given that \( X \sim Bin(\theta, n) \), and \( \theta \) is small. Then
\[
P_\theta(0 < \theta < \frac{1}{2n-1} \chi^2_{\alpha,2(n+1)}) = 1 - \alpha. \tag{1}
\]
Please note that \( X \) is the RV in (1). Given:
\( n = 200, \ X = 3, \ 1 - \alpha = .99 \). Then
\[
P_\theta(\theta < \frac{\chi^2_{.01,2(3+1)}/[(2)(200)]}{\chi^2_{.01,8}/400}) = 1 - \alpha.
\]
As a result, using the Table we get the upper confidence bound
\[U_{.01} = 20.09/400.\]

2. Problem 11.18. Let \( S^2_1 \) and \( S^2_2 \) be two sample variances from two independent nor-
mal populations based on samples of sizes \( n_1 \) and \( n_2 \), respectively. Then \((n_1 - 1)S^2_1/\sigma^2_1 \sim Chisq(n_1 - 1)\) and \((n_2 - 1)S^2_2/\sigma^2_2 \sim Chisq(n_2 - 1)\). Because we are interested in the ratio \( \sigma^2_1/\sigma^2_2 \), it is natural to consider the pivot
\[
\frac{[(n_1 - 1)S^2_1/\sigma^2_1]/(n_1 - 1)}{[(n_2 - 1)S^2_2/\sigma^2_2]/(n_2 - 1)} = \frac{\chi^2_{n_1-1}/(n_1 - 1)}{\chi^2_{n_2-1}/(n_2 - 1)} = F_{n_1-1,n_2-1}.
\]
Here \( F_{\nu,\mu} \) denotes a RV with \( F(\mu, \nu) \) distribution. Then
\[
P_{\sigma_1/\sigma_2}\left(f_{1-\alpha/2,n_1-1,n_2-1} \leq \frac{S^2_1}{\sigma^2_1} \leq f_{\alpha/2,n_1-1,n_2-1}\right) = 1 - \alpha. \tag{2}
\]

Via a simple algebra (2) is equivalent to
\[
P_{\sigma_1/\sigma_2}\left(f_{1-\alpha/2,n_1-1,n_2-1}^{-1} \leq \frac{S^2_1}{\sigma^2_1} \leq f_{\alpha/2,n_1-1,n_2-1}^{-1}\right) = 1 - \alpha.
\]

Now let us understand a relationship between \( f_{1-\beta,k,m} \) and \( f_{\beta,r,s} \). We have:
\[
\beta = P\left(\frac{\chi^2_k}{\chi^2_{n}/m} < f_{1-\beta,k,m}\right) = P\left(\frac{\chi^2_m}{\chi^2_{k}/k} > 1/f_{1-\beta,k,m}\right). \tag{3}
\]
At the same time, we also have
\[
P\left(\frac{\chi^2_m}{\chi^2_{k}/k} > f_{\beta,m,k}\right) = \beta. \tag{4}
\]
Then (3) and (4) imply the identity

\[ f_{1-\beta,k,m} = \frac{1}{f_{\beta,m,k}}. \]  

(5)

We can conclude that

\[ P_{\sigma_1/\sigma_2} \left( \frac{S_1^2/[S_2^2 f_{\alpha/2,n_1-1,n_2-1}]}{\sigma_1^2/\sigma_2^2} \leq f_{\alpha/2,n_2-1,n_1-1} S_1^2/S_2^2 \right) = 1 - \alpha. \]

3. Problem 11.19. Suppose that the sample standard deviation \( S \) has Normal \((\sigma, \sigma^2/2n)\) distribution. Then the pivot is

\[ Z = \frac{S - \sigma}{\sigma/(2n)^{1/2}}. \]

This yields

\[ P_\sigma(-z_{\alpha/2} < \frac{S - \sigma}{\sigma/(2n)^{1/2}} < z_{\alpha/2}) = 1 - \alpha. \]

From here we get two inequalities

\[-z_{\alpha/2}\sigma/(2n)^{1/2} < S - \sigma, \quad S - \sigma < z_{\alpha/2}\sigma/(2n)^{1/2}. \]

They are equivalent to

\[ \sigma(1 - z_{\alpha/2}/(2n)^{1/2}) < S, \quad S < \sigma(1 + z_{\alpha/2}/(2n)^{1/2}). \]

Thus, we get that

\[ P_\sigma \left( \frac{S}{1 + z_{\alpha/2}/(2n)^{1/2}} < \sigma < \frac{S}{1 - z_{\alpha/2}/(2n)^{1/2}} \right) = 1 - \alpha. \]

4. Problem 11.28. Here \( 1 - \alpha = .95, E = 2.5, \sigma = 12.2 \). Note that the units (here sec) are the same for all statistics. If they are different, you must choose a particular unit and recalculate the data.

We know that

\[ E = z_{\alpha/2}\sigma/n^{1/2}. \]

Then

\[ n = \frac{z_{\alpha/2}\sigma^2}{E^2} = \frac{(1.96)^2(12.2)^2}{(2.5)^2}. \]

5. Problem 11.30. Here \( n = 10, \bar{X} = 5.68 \) and \( S = .29, X \sim N(\mu, \sigma^2) \), \( 1 - \alpha = .95 \). Then the confidence interval for \( \mu \) is

\[ [\bar{X} - t_{\alpha/2,n-1}S/n^{1/2}, \bar{X} + t_{\alpha/2,n-1}S/n^{1/2}]. \]

From the Table \( t_{0.025,9} = 2.262 \) so

\[ P_{\mu,\sigma}(5.68 - (2.262)(.29)/(10)^{1/2} < \mu < 5.68 + (2.262)(.29)/(10)^{1/2}) = .95. \]
6. Problem 11.42. Here \( n = 100, X = 18, X \sim Binom(\theta, n), 1 - \alpha = .99 \). Then
\[
\Pr(\bar{X} - z_{\alpha/2}[\bar{X}(1 - \bar{X})]^{1/2}/n^{1/2} < \theta < \bar{X} + z_{\alpha/2}[\bar{X}(1 - \bar{X})]^{1/2}/n^{1/2}) = 1 - \alpha.
\]
We plug-in numbers and get a confidence interval
\[
[.18 - 2.575[.18)(.82)]^{1/2}/10, .18 + 2.575[.18)(.82)]^{1/2}/10].
\]
Remark: Because \( \theta > 0 \), if the lower bound is negative you must set it to zero.

7. Problem 11.50. Here \( n_1 = 500, X_1 = 48, n_2 = 400, X_2 = 68 \). Here we are considering the case of two independent populations with \( \bar{X}_1 \sim Binom(\theta_1, 500) \) and \( X_2 \sim Bin(\theta_2, 400) \). Then
\[
\bar{X}_1 - \bar{X}_2 \sim N(\mu := \theta_1 - \theta_2, \sigma^2 := \bar{X}_1(1 - \bar{X}_1)n_1^{-1} + \bar{X}_2(1 - \bar{X}_2)n_2^{-1}).
\]
As a result,
\[
P_{\theta_1, \theta_2}\left((\bar{X}_1 - \bar{X}_2) - \sigma z_{\alpha/2} < \theta_1 - \theta_2 < (\bar{X}_1 - \bar{X}_2) + \sigma z_{\alpha/2}\right) = 1 - \alpha.
\]
For our numbers we get the following confidence interval:
\[
\left(\frac{48}{500} - \frac{68}{400}\right) - \left[\left(\frac{(48/500)(452/500)}{500}\right)^{1/2} + \left(\frac{(68/400)(332/400)}{400}\right)^{1/2}\right] (1/96)
< \theta_1 - \theta_2 < \left(\frac{48}{500} - \frac{68}{400}\right) + \left[\left(\frac{(48/500)(452/500)}{500}\right)^{1/2} + \left(\frac{(68/400)(332/400)}{400}\right)^{1/2}\right] (1/96)
\]

8. Problem 11.53. Here \( 1 - \alpha = .95, n = 10, S = .20 \). Then we know that
\[
P_\sigma(\chi_{1-\alpha/2,n-1}^2 < \frac{(n - 1)S^2}{\sigma^2} < \chi_{\alpha/2,n-1}^2) = 1 - \alpha.
\]
Thus the confidence interval is
\[
[(9)(.29)^2/\chi_{1-\alpha/2,n-1}^2, (9)(.29)^2/\chi_{\alpha/2,n-1}^2].
\]
The Table gives us \( \chi_{.025,9} = 19.023, \chi_{.975,9} = 2.7 \). We conclude that
\[
P_\sigma(\frac{(9)(.29)^2}{19.023} < \sigma < \frac{(9)(.29)^2}{2.7}) = .95.
\]

9. Problem 11.57. Here \( \alpha = .98, S_1 = 19.4, n_1 = 61, S_2 = 18.8 \) and \( n_2 = 61 \). Then
\[
P_{\sigma_1/\sigma_2}\left(f_{1-\alpha/2,n_1-1,n_2-1} < \frac{S_2^2/\sigma_2^2}{S_2^2/\sigma_2^2} < f_{\alpha/2,n_1-1,n_2-1}\right) = 1 - \alpha.
\]
From the Table \( f_{.01,60,60} = 1.84 \), and we get:
\[
P_{\sigma_1/\sigma_2}\left(\frac{(19.4)^2}{(18.8)^2} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{(1.84)(19.4)^2}{(18.8)^2}\right).
\]

10. Problem 11.58. Here \( 1 - \alpha = .98, S_1 = 1.2, n_1 = 12, S_2 = 1.5, n_2 = 15 \). From the Table \( f_{.01,11,14} = 3.87 \) (I used a linear interpolation). Also, from the Table \( f_{.01,14,11} = 4.3 \) (again I used interpolation). We get
\[
P_{\sigma_1/\sigma_2}\left(\frac{(1.2)^2}{(3.87)(1.5)^2} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{(4.3)(1.2)^2}{(1.5)^2}\right).
\]