SOLUTIONS FOR HOMEWORK 5, STAT 4382

1. Exerc. 3.1.1. Well, the probability in question is \( \text{Pr}(X_0 = 0, X_1 = 1, X_2 = 2) \). This joint probability can be written via conditional ones as follows,
\[
\text{Pr}(X_0 = 0, X_1 = 1, X_2 = 2) = \text{Pr}(X_0 = 0)\text{Pr}(X_1 = 1|X_0 = 0)\text{Pr}(X_2 = 2|X_1 = 1, X_0 = 0).
\]
Then, using Markov property of the chain we can simplify the right side,
\[
\text{Pr}(X_0 = 0, X_1 = 1, X_2 = 2) = \text{Pr}(X_0 = 0)\text{Pr}(X_1 = 1|X_0 = 0)\text{Pr}(X_2 = 2|X_1 = 1).
\]
(1)
Now we can plug in numbers,
\[
\text{Pr}(X_0 = 0, X_1 = 1, X_2 = 2) = (.3)(.2)(0) = 0.
\]

2. Exerc. 3.1.2. Well, here we have a similar situation. Write
\[
\text{Pr}(X_2 = 1, X_3 = 1|X_1 = 0) = \text{Pr}(X_3 = 1|X_2 = 1, X_1 = 0)\text{Pr}(X_2 = 1|X_1 = 0)
\]
[and continue using Markov property]
\[
= \text{Pr}(X_3 = 1|X_2 = 1)\text{Pr}(X_2 = 1|X_1 = 0).
\]
Now plug in the given values from the transition probability matrix,
\[
\text{Pr}(X_2 = 1, X_3 = 1|X_1 = 0) = (.6)(.2) = .12.
\]
For the second probability in question the formula is the same,
\[
\text{Pr}(X_1 = 1, X_2 = 1|X_0 = 0) = \text{Pr}(X_2 = 1|X_1 = 1)\text{Pr}(X_1 = 1|X_0 = 0) = .12.
\]
Remark: Note that the probabilities are the same. It should be clear why: there is just a unit shift in index (time) and, because the Markov chain is stationary — we have this equality.

3. Exerc. 3.1.3. Note that the probability in question is not as it is written in the text but
\[
\text{Pr}(X_0 = 1, X_1 = 0, X_2 = 2|X_0 = 1)
\]
because it is given that \( X_0 = 1 \). In the strict sense, the problem is not written correctly in the text. But in any case, now we can continue, using the Markov property of the chain,
\[
\text{Pr}(X_0 = 1, X_1 = 0, X_2 = 2|X_0 = 1) = \text{Pr}(X_2 = 1|X_1 = 1)\text{Pr}(X_1 = 1|X_0 = 0)\text{Pr}(X_0 = 0|X_0 = 0)
\]
[and then we plug in numbers and continue]
\[
= (.3)(.1) = .03.
\]
4. Exerc. 3.1.4. Note that here again the two probabilities are just shifts in time, and thus due to stationarity must be the same. Write,
\[\Pr(X_1 = 1, X_2 = 1|X_0 = 0) = \Pr(X_2 = 1|X_1 = 1)\Pr(X_1 = 1|X_0 = 0) = (.2)(.1) = .02.\]
The second probability is the same.

5. Exerc. 3.1.5. Write,
\[\Pr(X_0 = 1, X_1 = 1, X_2 = 0) = \Pr(X_0 = 1)\Pr(X_1 = 1|X_0 = 1)\Pr(X_2 = 0|X_1 = 1, X_0 = 1).\]
Using the Markov property of the chain we can simplify the right side,
\[\Pr(X_0 = 1, X_1 = 1, X_2 = 0) = \Pr(X_0 = 1)\Pr(X_1 = 1|X_0 = 1)\Pr(X_2 = 0|X_1 = 1)\]
\[= (.5)(.1)(.5) = .025.\] (2)
The second probability is more involved because we do not have \(\Pr(X_1 = i)\) and must calculate it. In any case, similarly to (2) we write
\[\Pr(X_1 = 1, X_2 = 1, X_3 = 0) = \Pr(X_1 = 1)\Pr(X_2 = 1|X_1 = 1)\Pr(X_3 = 0|X_2 = 1)\]
\[= \Pr(X_1 = 1)(.1)(.5).\] (3)
Now we need to calculate \(\Pr(X_1 = 1)\). Using Total Probability Theorem we get
\[P(X_1 = 1) = P(X_1 = 1, X_0 = 0) + P(X_1 = 1, X_0 = 1) + P(X_1 = 1, X_0 = 2)\]
\[= P(X_0 = 0)P(X_1 = 1|X_0 = 0) + P(X_0 = 1)P(X_1 = 1|X_0 = 1) + P(X_0 = 2)P(X_1 = 1|X_0 = 2)\]
\[= (.5)(.2) + (.5)(.1) + (0)(.2) = .1 + .05 = .15.\]
Now we can finish. Plug in the result in (3) and get
\[\Pr(X_1 = 1, X_2 = 1, X_3 = 0) = (.15)(.1)(.5) = .0075.\]

6. Problem 3.1.1. Note that we are asked only about the transition probability matrix. Let \(X_n\) be the number of diseased at the end of \(n\)th iteration. Then it is clear that \(P_{0,0} = 1\), \(P_{5,5} = 1\) and \(P_{k,k-1} = 0\) for any \(k = 1, 2, 3, 4, 5\). As a result, we need to find only two transition probabilities: \(P_{k,k+1}\) and \(P_{k,k}\) for \(k = 1, 2, 3, 4\). Note that \(P_{k,k+1} = 1 - P_{k,k}\) so it is enough to find only one of them. For instance,
\[P_{k,k+1} = P(\text{choose, among } k \text{ diseased and } (N-k) \text{ healthy, 1 diseased and 1 healthy})\]
\[\times P(\text{disease is transferred})\]
The first probability is hypergeometric, the second is equal to \(\alpha\). This yields
\[P_{k,k+1} = \frac{{C_k^1}{N-k \choose 1}}{C_N^2} \alpha,\]
where $C^m_k = m!/[k!(m-k)!]$.

We defined all transition probabilities.

7. Problem 3.1.4. Let us solve it in turn “row-by-row”. We have

$$P_{0,0} = P(X_1 \leq X_0) = P(X_1 = 0) = .1.$$ 

Further, for $k = 1, 2, 3$ we can write

$$P_{0,k} = P(X_1 = k) = P(\xi = k).$$

The last probability is from the table, and thus we have numbers for the first row of the transition matrix.

Now let us find values for the second row. Plainly $P_{1,0} = 0$ because $X_n$ cannot decrease according to its definition. Further,

$$P_{1,1} = P(X_n = 1|X_{n-1} = 1) = P(\xi \leq 1) = .1 + .3 = .4.$$ 

Further,

$$P_{1,2} = P(X_n = 2|X_{n-1} = 1) = P(\xi = 2) = .2.$$ 

Further,

$$P_{1,3} = P(X_n = 3|X_{n-1} = 1) = P(\xi = 3) = .4.$$ 

Now let us consider elements for the third row. We have $P_{3,0} = P_{3,2} = 0$ because the Markov chain does not decrease. At the same time

$$P_{2,2} = P(X_n = 2|X_{n-1} = 2) = P(\xi \leq 2) = .6.$$ 

Because the total in each row must be 1, we get $P_{2,3} = .4$ (or you can calculate it directly).

Finally, for the fourth row we get $P_{3,k} = 0$ for $k = 0, 1, 2$ and $P_{3,3} = 1$.

Combining the obtained results we get the transition matrix

$$P = \begin{pmatrix}
.1 & .3 & .2 & .4 \\
0 & .4 & .2 & .4 \\
0 & 0 & .6 & .4 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}$$

8. Exerc. 3.2.1. The two-step transition matrix allows us to calculate probabilities for $X_{n+2}$ to be in a particular state if the state of $X_n$ is given. The formula is very simple,

$$P^{(2)} = P \times P = P^2$$

and using matrix multiplication we get

$$P^{(2)} = \begin{pmatrix}
.47 & .13 & .4 \\
.42 & .14 & .44 \\
.26 & .17 & .57 \\
\end{pmatrix}$$
9. Exerc. 3.2.3. Well, we need to calculate two multistep-transition matrices. For this particular example it is OK to use software to multiply matrices.

\[
P^{(3)} = P^3 = \begin{pmatrix}
.478 & .264 & .258 \\
.360 & .256 & .384 \\
.570 & .180 & .250
\end{pmatrix}
\]

\[
P^{(4)} = P^4 = \begin{pmatrix}
.4636 & .254 & .2824 \\
.444 & .2256 & .3304 \\
.5240 & .2222 & .2540
\end{pmatrix}
\]

Remark: *Always check that the sum in rows is 1!!!*

Using these transition matrices we get that \( P(X_3 = 1|X_0 = 0) = .264 \) and \( P(X_4 = 1|X_0 = 0) = .254 \).

10. Exerc. 3.2.5 Well, here again we need to multiply matrices, but it is simpler — we need just multiply two matrices. Write,

\[
P^{(2)} = P^2 = \begin{pmatrix}
.27 & .27 & .46 \\
.24 & .24 & .52 \\
.21 & .21 & .58
\end{pmatrix}
\]

Using this result we conclude that

\[
P(X_3 = 1|X_1 = 0) = P^{(2)}_{0,1} = .27.
\]

The same two-step transition matrix can be used for answering the second question,

\[
P(X_2 = 1|X_0 = 0) = P^{(2)}_{0,1} = .27.
\]

Note that the two probabilities in question must be the same due to stationarity of a Markov chain, that is, a shift in time does not change the transition probabilities.