1. Exerc.9.7. (a) We begin with the LRT. The corresponding hypothesis test is $H_0 : a = a_0$ versus $H_1 : a \neq a_0$ for $X_1, \ldots, X_n$ being iid $\text{Normal}(\theta, a\theta)$ where $\theta$ is unknown but $a\theta > 0$ since it is the variance. Then

$$L(a, \theta|X) = \prod_{l=1}^{n} [2\pi a\theta]^{1/2} e^{-(X_l-\theta)^2/2a\theta}$$

and

$$\ln(L(a, \theta|X)) = -n(1/2) \ln(2\pi a\theta) - (1/2a\theta) \sum_{l=1}^{n} (X_l - \theta)^2.$$ 

To find unrestricted MLE’s we take derivatives:

$$\begin{align*}
\frac{\partial \ln(L(a, \theta|X))}{\partial a} &= -\frac{n}{2a} + \frac{1}{2a^2} \sum_{l=1}^{n} (X_l - \theta)^2 \\
\frac{\partial \ln(L(a, \theta|X))}{\partial \theta} &= -\frac{n}{2a} + \frac{1}{2a^2} \sum_{l=1}^{n} (X_l - \theta)^2 + \frac{1}{a^2} \sum_{l=1}^{n} (X_l - \theta) 
\end{align*}$$

(1)

Then we find MLE’s by equating the derivatives to zero,

$$\begin{align*}
-\frac{n}{2a} + \frac{1}{2a^2} \sum_{l=1}^{n} (X_l - \theta)^2 &= 0 \\
-\frac{n}{2a} + \frac{1}{2a^2} \sum_{l=1}^{n} (X_l - \theta)^2 + \frac{1}{a^2} \sum_{l=1}^{n} (X_l - \theta) &= 0 
\end{align*}$$

(2)

This system yields the following system

$$\begin{align*}
a &= (1/2n) \sum_{l=1}^{n} (X_l - \theta)^2 \\
-\frac{n}{2a} + \frac{n}{2a^2} \sum_{l=1}^{n} (X_l - \theta)^2 + \frac{1}{a^2} \sum_{l=1}^{n} (X_l - \theta) &= 0 
\end{align*}$$

(3)

Then I plug in the obtained $a$ into the second equation and get (note that you always go from a system of two equations to a system of two equations)

$$\begin{align*}
a &= (1/2n) \sum_{l=1}^{n} (X_l - \theta)^2 \\
-\frac{n}{2a} + \frac{n}{2a^2} \sum_{l=1}^{n} (X_l - \theta)^2 + \frac{1}{a} \sum_{l=1}^{n} (X_l - \theta) &= 0 
\end{align*}$$

(4)

And we get the solution (unrestricted MLEs)

$$\begin{align*}
\hat{a} &= \frac{1}{n^2} \sum_{l=1}^{n} (X_l - \bar{X})^2 := \hat{\sigma}_{MLE}^2 \\
\hat{\theta} &= \bar{X}.
\end{align*}$$

(5)

Note that $\hat{\sigma}_{MLE}$ is the MLEs estimator of $\sigma^2$.

To get the restricted MLE for $\theta$ given $a = a_0$, we equate (1) to zero, solve the quadratic equation and get

$$\hat{\theta}_0 = (1/2)[-a_0 + \sqrt{(a_0^2 + 4n^{-1} \sum_{l=1}^{n} X_l^2)^1/2}].$$

Then the LRT statistic is

$$\lambda(X, a_0) = \left(\frac{\hat{a}\hat{\theta}}{a_0\theta_0}\right)^{n/2} \exp\left\{ - \frac{\sum_{l=1}^{n} (X_l - \hat{\theta}_0)^2}{2a_0\theta_0} \right\} \exp\left\{ \frac{\sum_{l=1}^{n} (X_l - \hat{\theta})^2}{2\hat{a}\hat{\theta}} \right\}.$$
\[ = \left( \frac{\sigma^2_{MLE}}{a_0 \theta_0} \right)^{\frac{n}{2}} e^{\frac{n}{2}} \exp\left\{-\sum_{i=1}^{n}(X_i - \hat{\theta}_0)^2 \right\} \exp\left\{-\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{2\sigma^2_{MLE}} \right\} \]

This is difficult to simplify further, so I stop here.

Based on the obtained results,

\[ \text{MLT : } \{ X : \text{Reject if } \lambda(X, a_0) \leq c(\alpha) \} \]

where \( c(\alpha) \) satisfies \( P(\lambda(X, a_0) \leq c(\alpha) | \theta, a_0) \leq \alpha. \)

Then the inverted confidence set is

\[ S_\alpha := \{ a : \lambda(X, a) > c(\alpha) \}. \]

Unfortunately, some research is needed (may be using numerical methods) to analyze the shape of the set. But the presented answer is sufficient.

2. Exerc. 9.9. Case 1. If \( X = \mu + Z, z \sim f_Z(z) \), then \( f_X(x) = f_Z(x - \mu) \). Thus \( \bar{X} - \mu \bar{Z} \) and \( \bar{Z} \) does not depend on \( \mu \).

Case 2. If \( X = \sigma Z, Z \sim f_Z(z) \), then \( f_X(x) = \frac{1}{\sigma} f_Z\left(\frac{x}{\sigma}\right) \). We conclude that

\[ \frac{\bar{X}}{\sigma} = \frac{\sigma \bar{Z}}{\sigma} = \bar{Z} \]

and the distribution of \( \bar{Z} \) does not depend on \( \sigma \). So it is a pivot.

Case 3. If \( X = \mu + \sigma Z \) then (you need to know how to prove this and we discussed this in class)

\[ f_X(x) = \frac{1}{\sigma} f_Z\left(\frac{x - \mu}{\sigma}\right). \]

Then

\[ \frac{\bar{X} - \mu}{\sigma} = \frac{\mu + \sigma \bar{Z} - \mu}{\sigma} = \bar{Z} \]

and the distribution of \( \bar{Z} \) does not depend on \((\mu, \sigma)\) and thus this is the pivot.

3. Exerc. 9.11 If \( \theta \) is an underlying parameter and \( T \sim F_T(t|\theta) \) and \( T \) is continuous random variable, then (remember the topic of generating a random variable) \( F_T(T|\theta) \sim Uniform(0, 1) \). From this we conclude that

\[ P(\alpha_1 < F_T(T|\theta) < 1 - \alpha_2|\theta) = 1 - (\alpha_1 + \alpha_2). \]

This implies that for \( H_0 : \theta = \theta_0 \) the \( 1 - \alpha \)-level acceptance region is (denote \( \alpha := \alpha_1 + \alpha_2 \))

\[ A(\theta_0) := \{ T : \alpha_1 < F_T(T|\theta) < 1 - \alpha_2 \} \]

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and the corresponding confidence $1 - \alpha$ set is

$$CS(T) = \{\theta : \alpha_1 < F_T(T|\theta) < 1 - \alpha_2\}.$$ 

4. Exerc. 9.13 $X \sim \text{Beta}(\theta, 1)$, that is

$$f_X(x|\theta) = \theta x^{\theta - 1} I(0 < x < 1).$$

Let $Y = -[\ln(X)]^{-1}$. Then $X = e^{-1/Y}$ and this implies

$$f_Y(y) = \theta y^{-2} e^{-\theta/y} = \frac{1}{\theta} y^{-2} e^{-1/y|\theta} I(0 < y < \infty).$$

We now see that $\theta$ is the scale parameter. Then

$$P_\theta(Y/2 \leq \theta \leq Y) = \int_\theta^{2\theta} \theta y^{-2} e^{-\theta/y} dy = e^{-\theta/y|\theta} = e^{-1/2} - e^{-1}.$$

We may conclude that the pivot is $T = X^\theta = e^{-\theta/Y}$. Then the confidence set is

$$CS(T) = \{\theta : a < x^\theta < b\} = \{\theta : \frac{\ln(b)}{\ln(X)} < \theta < \frac{\ln(a)}{\ln(X)}\}.$$ 

Finally, $F_X(x) = x^\theta$, $0 < x < 1$, so $X^\theta \sim \text{Uniform}(0, 1)$. By choosing $b - a = 1 - \alpha$ we get $(1 - \alpha)$-confidence interval.

5. Exerc. 9.14(a) Natural separate $1 - \beta$ confidence intervals are

$$\{\mu : \bar{X} + t_{n-1,\beta/2} S/n^{1/2} \leq \mu \leq \bar{X} + t_{n-1,1-\beta/2} S/n^{1/2}\},$$

and

$$\{\sigma : \frac{(n - 1)S^2}{\chi^2_{n-1,\beta/2}} \leq \sigma^2 \leq \frac{(n - 1)S^2}{\chi^2_{n-1,1-\beta/2}}\}.$$ 

To use the Bonferroni inequality $P(A \cap B) \geq P(A) + P(B) - 1$, we choose $\beta + \gamma = \alpha$, $\beta > 0$, $\gamma > 0$ and consider a simultaneous set

$$CS_\alpha(\bar{x}, S) = \{\mu, \sigma : [\bar{X} + t_{n-1,\beta/2} S/n^{1/2} \leq \mu \leq \bar{X} + t_{n-1,1-\beta/2} S/n^{1/2}] \cap \frac{(n - 1)S^2}{\chi^2_{n-1,\gamma/2}} \leq \sigma^2 \leq \frac{(n - 1)S^2}{\chi^2_{n-1,1-\gamma/2}}\}.$$ 

6. Exerc. 9.25 Here

$$f_X(x|\theta) = \theta^n \prod_{l=1}^n x_l^{\theta - 1} I(0 < x_{(1)} \leq X_{(n)} < 1),$$

and

$$\pi(\theta) = \frac{1}{\Gamma(r)} \theta^{r - 1} e^{-\theta/\lambda} I(\theta > 0).$$
Denote $T := \prod_{l=1}^{n} X_l$ and note that $T \in (0, 1)$. Then the posterior density is

$$
\pi(\theta | x) = \frac{\pi(\theta) f_X(x|\theta)}{\int_0^\infty \pi(\theta) f_X(x|\theta) d\theta} 
\propto \theta^{r-1} e^{\theta/\lambda} - n T^{\theta-1} = \theta^{r-1+n} e^{-\theta/\lambda} - (\ln(T))^{-1}.
$$

This is $\text{Gamma}(r + n, (\lambda^{-1} - \ln(T))^{-1})$ distribution if the both parameters are positive. Assuming this, the $(1 - \alpha)$ confidence set is

$$
CS_{\alpha}(T) = \{\theta : \hat{\Gamma}_{1-\alpha} < \theta < \hat{\Gamma}_{1-\alpha-2}\}
$$

where $\hat{\Gamma}_\beta$ is the $\beta$-quantile of $\text{Gamma}(r + n, (\lambda^{-1} - \ln(T))^{-1})$ and $\alpha_1 + \alpha_2 = \alpha$.

7. Exerc. 9.29. Here $X_1, \ldots, X_n$ are iid $\text{Bernoulli}(p)$, and $p$ is $\text{Beta}(a, b)$. Then we do similarly to the previous problem. Denote $Y \sum_{l=1}^{n} X_l$ and note that it is MSS and furthermore $Y \sim \text{Binom}(n, p)$. Then

$$
\pi(p | Y) \propto p^Y (1 - p)^{n-Y} p^{a-1} (1 - p)^{b-1} = p^{(Y+a)-1}(1 - p)^{(n+b-Y)-1}.
$$

We conclude that $\pi(p | Y)$ is $\text{Beta}(Y + a, n - Y + b)$. As a result, $a(1 - \alpha)$ credible interval for $p$ is

$$
CS_{\alpha}(Y) = \{p : \beta_{Y+a,n-Y+b,\alpha_1} \leq p \leq \beta_{Y+a,n-Y+b,1-\alpha_2}\}
$$

with any $\alpha_1 + \alpha_2 = \alpha$. 