1. Exerc. 7.33. Here we just recall that

\[ \text{MSE}(\hat{p}_B) = \frac{np(1-p)}{(\alpha + \beta + n)^2} + \left(\frac{np + \alpha}{\alpha + \beta + n} - p\right)^2. \]

Then you plug in \( \alpha = \beta = (n/4)^{1/2} \). After simplifications

\[ \text{MSE}(\hat{p}_B) = \frac{n}{4(n^{1/2} + n)^2} \]

and this is constant in \( p \).

How do we get such \( \alpha \) and \( \beta \)? We take a partial derivative in \( p \) and then set it equal to zero.

2. Exerc. 7.37. Let \( X_1, \ldots, X_n \) be iid according to the pdf

\[ f(x|\theta) = (2\theta)^{-1}I(|x| < \theta), \ \theta \in \Omega = (0, \infty). \]

We discussed in class that \( |X|_{(n)} \) is the CSS. At the same time, \( (X_{(1)}, X_{(n)}) \) is not CSS because \(-E_\theta(X_{(1)}) = E_\theta(X_{(n)})\). (What do you think about the SS statistic \( (X_{(1)}, X_{(n)}) \) for the cases of \( \text{Unif}(\theta, 2\theta) \) or \( \text{Unif}(\theta - 1/2, \theta + 1/2) \)?)

Now, \( Y := |X|_{(n)} \) is SS by Factorization Theorem because

\[ f(x|\theta) = (2\theta)^{-n}I(|x|_{(n)} \leq \theta). \]

Further, let us find the pdf of \( Y \). Write,

\[ F_Y(y) = P(Y \leq y) = P(|X|_{(n)} \leq y) = \prod_{i=1}^{n} |X_i| \leq y = \theta^{-n}y^n I(y \in (0, \theta)). \]

Thus, the pdf of \( Y \) is

\[ f_Y(y) = \frac{d}{dz} F_Y(z) \bigg|_{z=y} = n\theta^{-n}y^{n-1}I(0 < y < \theta). \] (1)

Let us show that \( Y \) is CSS. Suppose that

\[ E_\theta(g(Y)) = \int_0^\infty g(y)n\theta^{-n}y^{n-1}dy \equiv 0, \quad \text{for all } \theta > 0. \] (2)

Then \( dE_\theta(g(Y))/d\theta \equiv 0 \) for all \( \theta > 0 \). This yields (remember the Leibnitz rule on p.69 of how to take the derivative)

\[ -n\theta^{-n-1}\int_0^\theta g(y)ny^{n-1}dy + g(\theta)n\theta^{-n}\theta^{n-1} \equiv 0, \quad \theta > 0. \]
But the integral in the above-written identity is zero due to (2). This implies that $g(\theta)\theta^{-1} \equiv 0$ for all $\theta > 0$, and this $g(\theta) \equiv 0$ for all $\theta > 0$. We proved (directly) that $Y$ is complete.

Our next step is to find an unbiased estimator $\delta(Y)$ which is also the UMVUE. Let us check, using (1), that

$$E_{\theta}(Y) = \int_{0}^{\theta} yn\theta^{-n}y^{n-1}dy\theta^{-n}n \int_{0}^{\theta} y^{n}dy$$

$$= \theta^{-n}n(n+1)^{-1}\theta^{n+1} = \frac{n}{n+1}\theta.$$ 

Thus, the UMVUE is

$$\delta^*(Y) = \frac{n+1}{n} \max_i |X_i|.$$ 

3. Exers. 7.40. Let $X_1, \ldots, X_n$ be a sample from Bernoulli$(p)$. We calculate the Fisher information for a single observation:

$$I(p) := -E_p\left\{ \frac{\partial^2}{\partial p^2}\ln(p^X(1-p)^{1-X}) \right\} = -E_p\left\{ \frac{\partial^2}{\partial p^2}[X\ln(p) + (1-X)\ln(1-p)] \right\}$$

$$= -E_p\left\{ \frac{\partial}{\partial p} \left[ \frac{x}{p} - \frac{1-X}{1-p} \right] \right\} = -E_p\left\{ -\frac{x}{p^2} - \frac{1-X}{(1-p)^2} \right\}$$

$$= \frac{(1-p)^2p + p^2(1-p)}{p^2(1-p)^2} = \frac{1-p + p}{p(1-p)} = \frac{1}{p(1-p)}.$$ 

The Cramér-Rao lower bound tells us that

$$\text{Var}_p(\delta(X)) \geq \frac{[\partial E_p(\delta(X))/\partial p]^2}{nI(p)}.$$ 

If $\delta^*(X)$ is unbiased, the numerator in the lower bound is 1, and this yields that

$$\text{Var}_p(\delta^*(X)) \geq \frac{p(1-p)}{n},$$

and because $\text{Var}_p(\bar{X}) = p(1-p)/n$, the sample mean is the best unbiased estimator of $p$.

4. Exerc. 7.41. A sequence of iid RVs $X_1, \ldots, X_n$ is observed. It is known that $E_{\mu,\sigma^2}(X) = \mu$, $\text{Var}_{\mu,\sigma^2} = \sigma^2$.

(a) If $\delta := \sum_{i=1}^{n} a_i X_i$ then

$$E_{\mu,\sigma^2}(\delta) = \sum_{i=1}^{n} a_i \mu = mu \sum_{i=1}^{n} a_i = \mu \text{if } \sum_{i=1}^{n} a_i = 1.$$ 

(b) Let us find $\{a_i\}$ that minimize the variance.

$$\text{Var}_{\mu,\sigma^2}(\delta) = E_{\mu,\sigma^2}(\sum_{i=1}^{n} (a_i X_i - \mu)^2) = \text{Var}_{\mu,\sigma^2}(\sum_{i=1}^{n} a_i (X_i - \mu))$$
[because observations are iid we continue]

\[\sum_{i=1}^{n} a_i^2 \text{Var}_{\mu, \sigma^2}(X_i) = \sigma^2 \sum_{i=1}^{n} a_i^2.\]

Now we should find \(\{a_i\}\) that minimize \(\sum_{i=1}^{n} a_i^2\) given \(\sum_{i=1}^{n} a_i = 1\). Let us check that \(a_i \equiv 1/n\) are the extreme (what else can we try?). Write (in what follows the summation is over \(i \in \{1, 2, \ldots, n\}\)),

\[\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} (a_i - n^{-1} + n^{-1})^2 = \sum_{i=1}^{n} (a_i - n^{-1})^2 + 2n^{-1} \sum_{i=1}^{n} (a_i - n^{-1}) + n^{-1}.\]

The last sum is zero because \(\sum a_i = 1\). As a result, we get that

\[\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{m} (a_i - n^{-1})^2 + n^{-1} \geq n^{-1} \text{ with the equality iff } a_i \equiv n^{-1}.\]

5. Exerc. 7.47 We have \(X = \mu + \epsilon\) where \(\epsilon \sim N(0, \sigma^2)\) and \(\mu\) is an underlying radius. A sample of size \(n\) is observed. What is the UMVUE of \(a = \pi \mu^2\)?

Here \(\bar{X}\) is CSS (due to the exponential family) so I just note that \(E_{\mu, \sigma^2}(\bar{X}^2) - n^{-1} \sigma^2 = \mu^2\), so

\[\hat{a}_{\text{unb}} = \pi(\bar{X}^2 - n^{-1} \sigma^2)\).

Because it is a function of the CSS, it is UMVUE.

Here \(\sigma^2\) is known, but what if it is also unknown? Consider the following estimator:

\[\hat{a} := \pi(\bar{X} - n^{-1} S_n^2)\]

What do you think about its properties?

6. Exerc. 7.48 Here \(X_1, \ldots, X_n\) are iid Bernoulli(\(p\)).

(a). We found in Exerc. 7.40 that \(I(p) = 1/[p(1-p)]\), so \(\text{Var}_p(\bar{X}) = n^{-1} p(10p)\) attains the lower bound \([nI(p)]^{-1} = n^{-1} p(1 - p)\).

(b) Well, we can write using iid,

\[E_p\{X_1X_2X_3X - 4\} = \prod_{l=1}^{4} E_p\{X_l\} = p^4.\]

Then, because \(\sum_{l=1}^{n} X_l\) is CSS (remember that we are dealing with an exponential class), the statistic

\[\delta(\sum_{l=1}^{n} X_l) := E_p\{X_1X_2X_3X_4 | \sum_{l=1}^{n} X_l\}\]

is the UMVUE of \(p^4\). Can you calculate it? Try by yourself and then look at this:

\[\delta(t) = E_p\{X_1X_2X_3X_4 | \sum_{l=1}^{n} X_l = t\} = P_p(\{X_1 = 1, X_2 = 1, X_3 = 1, X_4 = 1 | \sum_{l=1}^{n} X_l = t\}\)
\[
\frac{P_p(X_1 = 1, X_2 = 1, X_3 = 1, X_4 = 1, \sum_{i=1}^n X_i = t)}{P_p(\sum_{i=1}^n X_i = t)} = \frac{p^4((n-4)!/(t-4)!(n-t)!)p^{t-4}(1-p)^{n-t}}{[n!/t!(n-t)!]p^t(1-p)^{n-t}} = \frac{(n-4)!t!}{n!(t-4)!}.
\]

7. Exerc. 7.49 Let \(X_1, \ldots, X_n\) be a sample from \(\text{Expon}(\lambda)\).
(a) Find an UE of \(\lambda\) based on \(X_{(1)}\).

Well, because \(f_X(x|\lambda) = \lambda^{-1}e^{-x/\lambda}I(x > 0)\) we use our technique to find the density of the first ordered observation. Remember how we do this:

\[
F_{X_{(1)}}(x|\lambda) = \Pr(X_{(1)} \leq x) = 1 - \Pr(X_{(1)} > x) = 1 - [\lambda^{-1} \int_x^{\infty} e^{-z/\lambda} dz] = 1 - e^{-nx/\lambda}.
\]

Take derivative and get \(f_{X_{(1)}}(x|\lambda) = n\lambda^{-1}e^{-nx/\lambda}\).

As we see, \(X_{(1)} \sim \text{Expon}(\lambda/n)\) so \(\bar{E}(X_{(1)}) = \lambda/n\) and

\[
\tilde{\lambda} := nX_{(1)}
\]

is UE.

(b) Let us find UMVUE. Here \(Y := \sum_{i=1}^n X_i\) is CSS (again due to the exponential family).

Because \(E_{\lambda}(Y) = n\lambda\) we get

\[
\hat{\lambda}_{\text{UMVUE}} = \bar{X}.
\]

8. Exerc. 7.52 Here \(X_1, \ldots, X_n\) are iid from \(\text{Poisson}(\lambda)\).
(a) Write

\[
f_X(x|\lambda) = \prod_{l=1}^n \frac{e^{-\lambda\lambda x_i}}{x_i!} = \frac{e^{-n\lambda\sum_{i=1}^n x_i}}{\prod_{l=1}^n x_i!}.
\]

As we see, this is an exponential family with \(Y := \sum_{i=1}^n X_i\) being the CSS.

We conclude, using our theory, that \(\bar{X} = Y/n\) is the UMVUE of \(\lambda\).

(b) To analyze directly

\[
E_{\lambda}(S^2|\bar{X}) = E_{\lambda}\{(n-1)^{-1} \sum_{i=1}^n (X_i - \bar{X})|\bar{X}\}
\]

is possible but rather complicated.

Let be smart and use the theory. We know that \(E_{\lambda}(S^2) = \lambda\) because \(\lambda\) is the variance and \(S^2\) is UE of the variance. But \(\lambda\) is also the mean for poisson distribution, so \(E_{\lambda}(S^2|\bar{X})\) is the UMVUE of the mean. Further, \(\bar{X}\) is also UMVUE of \(\lambda\) and \(\bar{X}\) is the CSS, so by uniqueness of the UMVUE we have

\[
E_{\lambda}(S^2|\bar{X}) = \bar{X}
\]

for Poisson distribution!
Then we also can write that
\[ \text{Var}_\lambda(S^2) = \text{Var}_\lambda(E_\lambda(S^2|\bar{X})) + E_\lambda\{\text{Var}_\lambda(S^2|\bar{X})\} > \text{Var}_\lambda(E_\lambda(S^2|\bar{X})) = \text{Var}_\lambda(\bar{X}). \]

(c) If \( Y \) is CSS and \( Z \) is any other statistic such that \( E_\theta(Y) \equiv E_\theta(Z) \) for all \( \theta \in \Omega \) then \( E_\theta(Z|Y) =: g(Y) \), then \( E_\theta(g(Y) - Y) \equiv 0 \) for all \( \theta \in \Omega \) because \( E_\theta g(Y) = E_\theta(Z) = E_\theta(Y) \) for all \( \theta \in \Omega \). Because \( Y \) is complete this yields that \( g(Y) = Y \) a.s.

Finally, we know that conditioning on a CSS reduces the variance, so
\[ \text{Var}_\theta(Z) > \text{Var}_\theta(E_\theta(Z|Y)) = \text{Var}_\theta(Y). \]

9. Exerc. 7.55
(a) Given that the pdf is \( f(x|\theta) = \theta^{-1}I(0 < x < \theta) \).

Here \( Y := X_{(n)} \) is the CSS and \( f_Y(y|\theta) = n\theta^{-n}y^{n-1}. \) Then
\[
E_\theta(Y^r) = \int_0^\theta y^r n\theta^{-n}y^{n-1}dy = n\theta^{-n} \int_0^\theta y^{n+r-1}dy = \frac{n}{n+r} \theta^{-n} \theta^{n+r} = \frac{n}{n+r} \theta^r.
\]

As a result,
\[
\hat{\theta}_{UMVU} := \frac{n + r}{n} X_{(n)}^r.
\]

10. Exerc. 7.59. Here \( X_1, \ldots, X_n \) are iid from \( N(\mu, \sigma^2) \). Find UMVUE for \( \sigma^p, p > 0 \).

Well, it is reasonable to try to analyze \( (S^2)^{p/2} \) because \( S^2 \) is a good estimate of \( \sigma^2 \). We know that \( S^2 \overset{D}{=} \sigma^2(n-1)X_{n-1}^2 \), so
\[
E_{\mu,\sigma^2}(S^2)^{p/2} = \sigma^p [(n-1)^{-p/2}E\{(X_{n-1}^2)^{p/2}\}].
\]

Denote \( K_p := (n-1)^{-p/2}E\{(X_{n-1}^2)^{p/2}\} \). This is a function in \( p \) which can be calculated for all \( p \) (I skip its calculation but you can think about moment generating function for chi-squared RV), and then
\[
\hat{\delta}_p := \frac{(S^2)^{p/2}}{K_p}
\]
is the UMVUE. Indeed, \((\bar{X}, S^2)\) is the CSS for the normal distribution and the mean of the estimator is \( \sigma^p \).