1. Exerc. 8.20 (with $\alpha = 0.037$). Solution: Remember that the UMP test rejects $H_0$ if the likelihood ratio is large:

$$r(x) = \frac{f(x|H_1)}{f(x|H_2)} > k$$

and utilizes randomization if

$$r(x) = k$$

where $k$ is chosen to yield the given zise.

Let us look at $r(x)$,

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r(x)$</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>.84</td>
</tr>
</tbody>
</table>

Thus, we reject if $x < c$ and randomize if $x = c$. To get $\alpha = 0.037$ we set $c = 4$ (using trials and errors approach) and if $x = 4$ then reject with probability 0.7. In this case the critical function is

$$\phi(x) = \begin{cases} 
1 & \text{if } x < 4 \\
.7 & \text{if } x = 4 \\
0 & \text{if } x > 4.
\end{cases}$$

Now we check that

$$\alpha = P(\text{Reject}|H_0) = E(\phi(X)|H_0)$$

$$= \sum_{i=1}^{3} P(X = i|H_0) + (0.7)P(X = 4|H_0) = 0.01 + 0.01 + 0.01 + (0.7)(0.01) = 0.037.$$

Further,

$$\text{Type II Error} = \alpha_1 = P(\text{Accept}|H_1) = 1 - P(\text{Reject}|H_1) = 1 - E(\phi(X)|H_1)$$

$$= 1 - [0.06 + 0.05 + 0.04 +(0.7)(0.03)] = 1 - [0.15 + 0.021] = 1 - .171 = .829.$$ 

Please note that .171 is the power of the test.

2. Exerc. 8.22. Given: $X_1, \ldots, X_n$ are iid $Bernoulli(p)$.

(a) Test $H_0 : p = 1/2$ versus $p = 1/4$ with $\alpha = 0.0547$.

This is a distribution with the MSS $Y = \sum_{i=1}^{n} X_i$ (check this!), which is $Binomial(p, n)$, and the critical function of the UMP test is (see the graph)

$$\phi(Y) = \begin{cases} 
1 & \text{if } Y < c \\
\gamma & \text{if } Y = c \\
0 & \text{if } Y > c.
\end{cases}$$
Then, for \( n = 10 \) using \( \text{Binomial}(p = 1/2, n = 10) \) Table, we get \( c = 3, \gamma = 0 \). Check this:

\[
P(Y < 3|p = 1/2, n = 10) = .0547.
\]

This critical function (the UMP test) also yields the power

\[
\text{Power} = P(Y < 3|p = 1/4, n = 10) = 0.526
\]

b) Given that \( H_0 : p \leq 1/2 \) and \( H_1 : p > 1/2 \), and also that the critical function is

\[
\phi(x) = \begin{cases} 
1 & \text{if } Y \geq 6 \\
0 & \text{if } Y \leq 5.
\end{cases}
\]

The power function is

\[
\beta(p) = E(\phi(Y)|p) = P(Y \geq 6|p) = \sum_{k=6}^{10} \frac{10!}{k!(10-k)!} \left( \frac{p}{1-p} \right)^k (1-p)^{10-k}.
\]

Please note how the graphic also follows from the fact that the test is UMP.

Size is

\[
\text{Size} = \beta(1/2) = P(Y \geq 6|p = 1/2, n = 10) \approx .377
\]

(c) A nonrandomized test exists for \( \alpha \) equal to \( P(Y \leq j|p = 1/2, n) \). For \( n = 10 \) these are (from the Table)

\[
\alpha = 0, \frac{1}{1024}, \frac{11}{1024}, \frac{56}{1024}, \ldots, 1.
\]

3. Exers. 8.23 Given that \( X \sim \text{Beta}(\theta, 1) \), that is

\[
f_X(x|\theta) = \frac{\Gamma(\theta + 1)}{\Gamma(\theta)\Gamma(1)} x^{\theta-1}(1 - x)^{1-1}I(x \in [0, 1]), \quad \theta > 0.
\]

Also remember (p.99 in the text) that \( \Gamma(\theta + 1) = \theta \Gamma(\theta) \) and \( \Gamma(1) = 1 \).

(a) We test \( H_0 : \theta \leq 1 \) versus \( H_1 : \theta > 1 \) and the given critical function is \( \phi(x) = 1 \)

whenever \( X > 1/2 \) and it is zero otherwise.

The power function is

\[
\beta(\theta) = E_{\theta}\{\phi(X)\} = \int_{1/2}^{1} \frac{\Gamma(\theta + 1)}{\Gamma(\theta)\Gamma(1)} x^{\theta-1} dx
\]
The graphic is

Further,

\[
\text{Size} = \max_{\theta \in [0,1]} \beta(\theta) = \beta(1) = 1/2.
\]

(b) Here we test \( H_0 : \theta = 1 \) versus \( H_1 : \theta = 2 \) with a given \( \alpha \). Then the UMP test rejects whenever

\[
r(x) = \frac{f_X(x|\theta = 2)}{f_X(x|\theta = 1)} > k.
\]

This yields

\[
r(x) = \frac{\Gamma(3)x^{2-1}\Gamma(1)\Gamma(1)}{\Gamma(2)\Gamma(1)\Gamma(2)x^{2-1}} = \frac{\Gamma(3)}{(\Gamma(2))^2}x > k.
\]

Remember that \( \Gamma(n) = (n-1)! \) and then \( r(x) = 2x > k \) yields the rejection region \( x > c \). Because \( \alpha \) is given we write

\[
\alpha = P(X > c|\theta = 1) = \int_c^1 \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}x^0dx = 1 - c.
\]

We conclude that \( c = 1 - \alpha \) and the UMP rejects whenever \( X > 1 - \alpha \).

(c) Here we test \( H_0 : \theta \leq 1 \) versus \( H_a : \theta > 1 \). The issue is about the UMP. For any pair \( \theta_2 > \theta_1 \) of two values of the parameter, we notice that the function

\[
r(x|\theta_1, \theta_2) = \frac{f(x|\theta_2)}{f(x|\theta_1)} = [\theta_2/\theta_1]x^{\theta_2-\theta_1}
\]

is monotonic in \( x \). This yields MLR and consequently the UMP test exists, and it is \( X > c_\alpha \).

4. Exerc. 8.24 The LRT rejects \( H_0 \) when

\[
\lambda(x) = \frac{f(x|H_0)}{\max(f(x|H_1), f(x|H_0))} < c.
\]

The MP test rejects when

\[
r(x) = \frac{f(x|H_1)}{f(x|H_0)} > k.
\]
As a result, we can write that
\[ \lambda(x) = \begin{cases} 
 1/r(x) & \text{if } f(x|H_1) \geq f(x|H_0), \text{ that is } \lambda(x) \leq 1 \\
 1 & \text{if } f(x|H_1) \leq f(x|H_0), \text{ that is } \lambda(x) = 1. 
\end{cases} \]

Thus, in general the two tests are different. At the same time, in practice typically \( c \) is smaller than 1 so they are often identical. Also note that if \( c = 1 \) then the power of the LRT is 1 and the size is 1.

5. Exerc. 8.25 Show that a family has an MLR. (Remember that a family with a monotone likelihood ratio has a UMP test — this is why the MLR is important.)

In what follows \( \theta_1 < \theta_2 \).
(a) \( \text{Normal}(\theta, \sigma^2) \) with \( \sigma^2 \) known. Write
\[ r(x) = \frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{e^{-(x-\theta_2)^2/2\sigma^2}}{e^{-(x-\theta_1)^2/2\sigma^2}} = e^{(\theta_2-\theta_1)/\sigma^2} e^{(\theta_1^2-\theta_2^2)/2\sigma^2}. \]

The ratio is increasing in \( x \), and this implies the MLR. Please note that the same result holds for a sample of size \( n \).
(b) \( \text{Poisson}(\theta) \). Write,
\[ r(x) = \frac{p(x|\theta_2)}{p(x|\theta_1)} = \frac{e^{-\theta_2} \theta_2^x x!}{x! e^{-\theta_1} \theta_1^x} = e^{\theta_2 - \theta_1} [\theta_2/\theta_1]^x. \]
The ratio is increasing in \( x \), and this yields the MLR.
(c) \( \text{Binomial}(\theta, n) \). Write
\[ r(x) = \frac{p(x|\theta_2)}{p(x|\theta_1)} = \frac{n!}{x!(n-x)!} \frac{\theta_2^x (1 - \theta_2)^{n-x}}{\theta_1^x (1 - \theta_1)^{n-x}} \]
\[ = \left[ \frac{\theta_2 (1 - \theta_1)}{\theta_1 (1 - \theta_2)} \right]^x \frac{1 - \theta_2}{1 - \theta_1}. \]
Again, the ratio is increasing in \( x \), so the family has the MLR.

Remark: Have you noticed that the three distributions are from the exponential family? See also the next problem.

6. Exerc. 8.27 Let \( g(t|\theta) = c(\theta) h(t) e^{w(\theta)t} \) be a one-parameter exponential family. Then for \( \theta_2 > \theta_1 \) we can write
\[ r(t|\theta_1, \theta_2) = \frac{g(t|\theta_2)}{g(t|\theta_1)} = e^{(w(\theta_2) - w(\theta_1)) t} \frac{c(\theta_2)}{c(\theta_1)}. \]
If \( w(\theta) \) is increasing then the likelihood ratio \( r \) is also increasing in \( t \), and we establish the MLR. Problem 8.25 gives us 3 examples.
7. Exerc. 8.28. Given: The logistic distribution

\[ f(x|\theta) = \frac{e^{x-\theta}}{(1 + e^{x-\theta})^2} I(x \in (-\infty, \infty)), \quad \theta \in (-\infty, \infty). \]

(a) Suppose that \( \theta_2 > \theta_1 \) and write

\[ r(x) = \frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{e^{x-\theta_2} - e^{x-\theta_1} (1 + e^{x-\theta_1})^2}{(1 + e^{x-\theta_2})^2}. \]

To study the monotonicity, let us look at the derivative

\[ \frac{dr(x)}{dx} = e^{\theta_1 - \theta_2} \left[ \frac{1 + e^{x-\theta_1} e^{x-\theta_1} (1 + e^{x-\theta_2}) - e^{x-\theta_2} (1 + e^{x-\theta_1})}{(1 + e^{x-\theta_2})^2} \right] \]

\[ = 2e^{\theta_1 - \theta_2} \frac{e^{x-\theta_1} - e^{x-\theta_2}}{(1 + e^{x-\theta_2})^3} > 0 \]

because \( e^{-\theta_1} > e^{-\theta_2} \). The MLR is established.

(b) We test \( H_0: \theta = 0 \) versus \( H_a: \theta = 1 \). According to part (a) we have \( r(x) > k \) iff \( x > c \). Then

\[ \alpha = P(X > c|\theta = 0) = \int_c^{\infty} \frac{e^x}{(1 + e^x)^2} dx = 1 - F_0(c) \]

where the logistic cdf \( F_0(c) := e^{-\theta}/[1 + e^{-\theta}] \). We conclude that

\[ \alpha = 1 - \frac{e^c}{1 + e^c} = [1 + e^c]^{-1}, \]

and the last relation yields

\[ c = \ln([1 - \alpha]/\alpha). \]

Similarly,

\[ \beta = F(c|\theta = 1) = \frac{e^{c-1}}{1 + e^{c-1}} \]

and you can plug-in the \( c \) expressed as \( \alpha \) (I skip this).

For the given number, if \( \alpha = .2 \) then \( c = 1.39 \) and \( \beta = .6 \).

(c) This follows from the MLR property (Karlin-Rubin Theorem).

8. Exerc. 8.29 Given that \( X \sim Cauchy(\theta), \theta_2 > \theta_1 \).

(a). This is a very famous result. Write

\[ r(x) = \frac{1 + (x - \theta_1)^2}{1 + (x - \theta_2)^2}, \]

and note that \( r(x) \uparrow 1 \) as \( x \to \infty \) and \( r(x) \downarrow 1 \) as \( x \to \infty \). See the graphic below. So \( r(x) \) is not MLR.
(b) Given \( f(x|\theta) = 1/[\pi(1+(x-\theta)^2)] I(x \in (-\infty, \infty)) \). Test \( H_0 : \theta = 0 \) versus \( H_1 : \theta = 1 \). Remember that the MP test rejects when \( r(X) > k \). Here

\[
 r(x) = \frac{1 + x^2}{1 + (x - 1)^2}
\]

and

\[
 \frac{dr(x)}{dx} = \frac{2x(1 + (x - 1)^2) - 2(x - 1)(1 + x^2)}{(1 + (x - 1)^2)^2} = \frac{2(1 + x - x^2)}{(1 + (x - 1)^2)^2}
\]

This allows us to plot the graphic. Note that the roots of \( 1 + x - x^2 = 0 \) are \( x_{12} = [1 \pm \sqrt{5}]/2 \).

We have \( r(1) = 2/1 \) and \( r(3) = [1 + 9]/[1 + 4] = 2 \). As a result, according to these calculations and the graphic, the interval \([1, 3]\) is the MP of its size. Let us make some calculations:

\[
\alpha = P(X \in [1, 3]|\theta = 0) = \int_1^3 \frac{1}{\pi(1 + x^2)} \, dx = \pi^{-1} \arctan(x) \bigg|_{x=1}^{x=3} \approx .15.
\]

Further,

\[
\alpha_1 = \text{second type error} = 1 - P(x \in [1, 3]|\theta = 1) = 1 - \frac{1}{\pi} \int_1^3 \frac{1}{1 + (x - 1)^2} \, dx = 1 - \frac{1}{\pi} \arctan(x - 1) \bigg|_{x=1}^{x=3} \approx .65
\]

(c) Clearly this is not the UMP test. Further, note that the rejection region can be a tail, or two tails, etc.

9. Exerc. 8.32 Given \( X_1, \ldots, X_n \) are iid Normal(\( \theta, 1 \)), and \( \theta_0 \) is given.

(a) Test \( H_0 : \theta > \theta_0 \) versus \( H_1 : \theta < \theta_0 \). Since normal family is a MLR family, we know that the UMP test exists and, via a familiar calculation the rejection region is (below \( \theta_1 < \theta_0 \))

\[
r(x) = \frac{f(x|\theta_1)}{f(x|\theta_0)} > k \Rightarrow \bar{x} < c.
\]
This yields \( c = \theta_0 - z_{1-\alpha} \sigma / n^{1/2} \) (check this famous result!).

(b) We discussed this in class (remember that here left and right tails are the UMP for parameters smaller and larger than \( \theta_0 \). Due to this fact and importance of the normal distribution, the notion of an unbiased test has been introduced.

10. Exercise 8.33. Given: \( X_1, \ldots, X_n \) is a sample from the \( \text{Uniform}(\theta, \theta + 1) \). We test \( H_0 : \theta = 0 \) versus \( H_1 : \theta > 0 \). The proposed test has the critical function \( \phi(x) = 1 \) if \( X_{(1)} \geq 1 \) or \( X_{(n)} \geq k \) and it is zero otherwise.

(a) To have the size \( \alpha \), we need (note that under the null hypothesis all observations are at most 1)

\[
\alpha = P(X_{(n)} > k | \theta = 0) = \int_k^1 n(1-y)^{n-1}dy = (1-k)^n.
\]

This yields \( k = 1 - \alpha^{1/n} \).

(b) Write using \( F_{X_{(1)},X_{(n)}}(u,v|\theta) = n(n-1)(v-u)^{n-2}I(\theta < u < v < \theta + 1) \),

\[
\beta(\theta) = P(\{X_{(1)} \geq 1 \cup X_{(n)} > k\} | \theta)
\]

\[
= \begin{cases} 
0 & \text{if } \theta \leq k - 1 \\
\int_k^\theta n(1-(y_1-\theta))^{n-1}dy_1 = (1-k+\theta)^n & \text{if } k - 1 < \theta \leq 0 \\
\int_k^{\theta+1} n(1-(y_1-\theta))^{n-1}dy_1 + \int_\theta^{\theta+1} n(n-1)(y_n-y_1)^{n-2}dy_ndy_1 = \alpha + 1 - (1-\theta)^n & \text{if } 0 < \theta < k \\
1 & \text{if } \theta > k.
\end{cases}
\]

(c) Note that \( X_{(1)} \) and \( X_{(n)} \) are MSS. Then for \( 0 < \theta < 1 \) we can write

\[
\frac{f_{X_{(1)},X_{(n)}}(y_1,y_n|\theta)}{f_{X_{(1)},X_{(n)}}(y_1,y_n|0)} = \begin{cases} 
0 & \text{if } 0 < y_1 \leq \theta, \ y_1 < y_n < 1 \\
1 & \text{if } \theta < y_1 < y_n < 1 \\
\infty & \text{if } 1 \leq y_n < \theta + 1, \ \theta < y_1 < y_n.
\end{cases}
\]

For \( \theta > 1 \) we have

\[
\frac{f_{X_{(1)},X_{(n)}}(y_1,y_n|\theta)}{f_{X_{(1),X_{(n)}}}(y_1,y_n|0)} = \begin{cases} 
0 & \text{if } y_1 < y_n < 1 \\
\infty & \text{if } \theta < y_1 < y_n < \theta + 1.
\end{cases}
\]

From these results follows the UMP.

(d) According to part (b), \( \beta(\theta) = 1 \) for all \( \theta \geq k = 1 - \alpha^{1/n} \). So these conditions are satisfied for all \( n \).