SOLUTION FOR HOMEWORK 3, STAT 6332

1. Here \( g(\theta) = \theta(1 - \theta) \), \( X \sim Binom(n, \theta) \) and \( \theta \sim Beta(a,b) \). Remember that \( \delta_B := E\{g(\Theta)|X\} \).
   
   As we know (or check it directly) that \( p^\theta(x) \) is the density of \( Beta(a+x, b+n-x) \). For a Beta-distribution we know that if \( Z \sim Beta(\alpha, \beta) \) then
   
   \[ E(Z) = \frac{\alpha}{\alpha + \beta} \quad \text{Var}(Z) = \frac{\alpha \beta}{(\alpha + \beta)(\alpha + \beta + 1)}. \]
   
   Using these results we get that
   
   \[ E\{Z(1 - Z)\} = E(Z) - E(Z^2) = E(Z) - \text{Var}(Z) + [E(z)]^2 \]
   
   [using simple algebra]
   
   \[ = \frac{\alpha \beta}{(\alpha + \beta)(\alpha + \beta + 1)}. \]
   
   We conclude that
   
   \[ \delta_B = E\{\Theta(1 - \Theta)|X\} = \frac{(a + X)(b + n - X)}{(a + b + n)(a + b + n + 1)}. \]
   
   Note: (i) If \( n = 0, X = 0 \) then \( \delta_B^0 = \frac{ab}{(a+b)(a+b+1)} \).
   
   (ii) If \( n \to \infty, X \to \infty \), then \( \delta_B \to \delta^\infty_B = X(n - X)/[n(n + 1)] \).

2. UMVU for \( \theta \) is \( \delta_1 = X/n \) and for \( p^2 \) is \( \delta_2 = X(X - 1)/[n(n - 1)] \) (indeed, these are unbiased estimators and functions of the complete sufficient statistic).

   As a result, the UMVU of the estimand \( \theta(1 - \theta) \) is \( \delta^* = \delta_1 - \delta_2 = X(n - X)/[n(n - 1)] \). Comparison with \( \delta_B \) yields that asymptotically the estimators are identical.

3. (a) Here \( X^n := (X_1, \ldots, X_n) \) is a sample from \( Poisson(\lambda) \) and \( \Lambda \) is \( Gamma(g, 1/\alpha) \). Then a direct analysis shows that the following conditional probability density, as a function in \( \lambda \), is proportional to:

   \[ p^{\Lambda|X^n}(\lambda|x^n) \propto \left[ \prod_{i=1}^{n} e^{-\lambda} \lambda^{x_i} (x_i!)^{-1}\right] \left[ \frac{\lambda^g \lambda^{g-1} e^{-\lambda}}{\Gamma(g)} \right]
   
   \[ \propto \lambda^{(\sum_{i=1}^{n} x_i + g) - 1} e^{-\lambda(\alpha + m)}. \]

   This yields that the conditional probability density is again \( Gamma(n\bar{x} + g, (\alpha + m)^{-1}) \). Note that we have the conjugate prior here.

   Using this result we get for the squared loss function,

   \[ \delta_B = E(\Lambda|X_n) = \frac{n\bar{X} + g}{\alpha + n} \]
\[
= \frac{g}{\alpha} \frac{\alpha}{\alpha + n} + \bar{X} \frac{n}{\alpha + n}.
\]

Note that in the first term \( \frac{g}{\alpha} \) is the mean of the prior distribution and clearly \( \bar{X} \) is the UMVU and MLE.

(b) (i) if \( n \to \infty \) then \( \delta_B \to \bar{X} \).

(ii) if \( \alpha \to \infty \) then \( \delta_B \to g/\alpha \), and if \( g \to 0 \) then \( \delta_B \to \bar{X}n(\alpha + n)^{-1} \).

4. Let \( X_1, \ldots, X_n \) be iid Normal \((0, \sigma^2)\). The joint density is then \( f^{X_n}(x^n) = C r^{-\tau} e^{-r \sum X^2_i} \) where \( \tau = 1/(2\sigma^2) \) and \( r = n/2 \). Writing \( Y = \sum_{i=1}^n X_i^2 \) we can see that the posterior density of \( \tau \) given \( X^n \) is
\[
C(Y)\tau^{r+g-1}e^{-\tau(\alpha+Y)},
\]
which is \( Gamma(r + g, 1/(\alpha + Y)) \).

If the loss is squared error, the Bayes estimator of \( 2\sigma^2 = 1/\tau \) is the posterior expectation of \( 1/\tau \), which is (using the formula that I gave you) is
\[
\frac{\alpha + Y}{r + g - 1}.
\]

Then the Bayes estimator of \( \sigma^2 = 1/2\tau \) is
\[
\frac{\alpha + Y}{n + 2g - 2}.
\]

For the loss function \( \frac{(\delta - \sigma^2)^2}{\sigma^2} \), which is rather natural equivariant loss for the scale parameter [note that if you multiply the scale and the estimator by a constant then the loss remains the same] the Bayes risk is
\[
E\{(\frac{\delta(X^n)}{\sigma^2} - 1)^2\} = E\{E\{(\frac{\delta(X^n)}{\sigma^2} - 1)^2|X^n\}\}.
\]

Denote again \( \tau = 1/(2\sigma^2) \) then we get that
\[
E\{(\frac{\delta(X^n)}{\sigma^2} - 1)^2|X^n\} = E\{(\frac{2\tau}{\delta - 1} - 1)^2|X^n\}.
\]

Now remember that a constant \( c \), which minimizes \( E\{\frac{Z}{c} - 1\}^2 \) is \( c = E\{Z^2\}/E\{Z\} \) [you can check this by taking derivative]. As a result,
\[
[\delta^*]^{-1} = \frac{E\{(2\tau Z)^2|X^n\}}{E\{2\tau|X^n\}}.
\]

Finally, remember the posterior \( Gamma(r + g, 1/(\alpha + Y)) \) distribution of \( \tau \) given \( X^n \), \( Y = \sum_{i=1}^n X_i^2 \) and conclude that
\[
\delta^* = \frac{E\{\tau|X\}}{2E\{\tau^2|X\}} = \frac{\alpha + Y}{n + 2g + 2}.
\]
5. (i) Consider the squared loss function, $P^{X|\xi}$ and $P_{Y|\eta}$.
As we know, in this case $\delta_B = E\{\Theta|X\}$, so for $\Theta = \eta - \xi$ we have

$$\delta_B(X, Y) = E\{(\eta - \xi)|X, Y\} = E(\eta|X, Y) - E(\xi|X, Y).$$

To continue we note that $\eta$ and $\xi$ are independent given $X$, and $\xi$ is independent of $Y$. Using this we can continue,

$$\delta_B(X, Y) = E\{\eta|Y\} - E\{\xi|X\} = \delta'_{\Lambda} - \delta_{\Lambda}.$$

(ii) Here $\eta > 0$, and $\Theta = \xi/\eta$. Write

$$\delta_B(X, Y) = E\{\xi/\eta|X, Y\}.$$

Now let us note that $\xi$ and $\eta$ are independent given $X$ and $Y$ because

$$P_{\xi,\eta|X,Y} = \frac{P_{\xi,\eta,X,Y}}{P_{X,Y}}$$

$$= \frac{P_{X} P_{\xi|X} P_{\eta|Y}}{P_{X} P_{Y}} = P_{\xi|X} P_{\eta|Y}.$$ 

Using this observation we conclude that

$$\delta_B(X, Y) = E\{\xi/\eta|X, Y\} = E(\xi|X) E\left(\frac{1}{\eta}|Y\right) = \delta_{\Lambda} \delta'_{\Lambda}.$$