1. Here $X$ is $\text{Binom}(p, n)$. Set $\delta^* := x/N$ with probability $1 - \epsilon$ and $\delta^* := 1/2$ with probability $\epsilon$. Then we can write

$$E_p(\delta^*(X) - p)^2 = (1 - \epsilon)E_p(X/n - p)^2 + \epsilon(1/2 - p)^2 = (1 - \epsilon)[p(1 - p)/n] + \epsilon(1/2 - p)^2.$$ 

Substituting $\epsilon = 1/(n + 1)$ we get

$$E_p(\delta^*(X) - p)^2 = [n/(n + 1)][p(1 - p)/n] + (1/(n + 1))(1 - 2p)^2/4 = 1/[4(n + 1)].$$

We know that $E_p(X/n - p)^2 = [p(1 - p)/n]$ and then $\max_{p \in (0,1)}[p(1 - p)/n] = 1/4n$. Compare $1/[4(n + 1)]$ with $1/4n$ and conclude that

$$\sup_p E_p(\delta^*(X) - p)^2 < \sup_p E_p(X/n = p)^2.$$ 

2. If $X$ is $\text{Binom}(p, n)$ and we use Bayes approach with prior being $\text{Beta}(n_1/2, n_1/2)$ then the Bayes estimate with constant risk (and thus the minimax estimate) is

$$\delta = [X/n][n_1/2/(1 + n_1/2)] + [1/(2(1 + n_1/2))]. \tag{1}$$

Remember that this we discussed in class.

Now let us look at the bias of the minimax estimator:

$$E_p(\delta - p) = p[n_1/2/(1 + n_1/2)] + [1/(2(1 + n_1/2))] - p = p[n_1/2/(1 + n_1/2) - 1] + 1/(2(1 + n_1/2)) = -p/(1 + n_1/2) + 1/2[1 + n_1/2] = (1 - 2p)/[2(1 + n_1/2)].$$

Thus, the bias tends to zero as $n \to \infty$, the bias is positive for $p < 1/2$ and negative for $p > 1/2$ with being zero at $p = 1/2$.

As a result, you can appreciate the idea of Problem 1 where the estimator is chosen to be closer to 1/2.

3. Here we have a sample from unknown cdf $F$. The estimand is $F(0) = P(X \leq 0)$, loss function is squared. We want to find the minimax estimator. The solution is the minimax estimator for the Binomial case $\text{Binom}(Y, F(0))$ where $Y = \sum_{t=1}^n I(X_t \leq 0)$, which is

$$\delta = \frac{Y}{n^{1/2}} \frac{1}{1 + n^{1/2}} + \frac{1}{2(1 + n^{1/2})}.$$ 

Why? This is the reason (we touched it in class). Consider a subfamily $\mathcal{F}_0$ of distributions with given $F(0) = p$, $0 < p < 1$. Note that

$$\sup_{F} E_F(\delta - F(0))^2 \geq \sup_{F \in \mathcal{F}_0} E_F(\delta - F(0))^2.$$
For $F \in \mathcal{F}_0$ we converted our problem into $Z \sim Binom(p, n)$ where $Z = 1$ if $X_i < 0$ and $Z = 0$ otherwise. A minimax estimate in this case is exactly our $\delta$ and its risk is (I calculated this in class) is $1/[4(1 + n^{1/2})^2]$. Note that it does not depend on $p = F(0)$. We can write

$$\sup_{F \in \mathcal{F}_0} E_F(\delta - F(0))^2 = 1/[4(1 + n^{1/2})].$$

We conclude, via lower bound and the risk of the estimator, that the estimate is minimax.

4. We know (or check it) that $\bar{Y}$ and $\bar{X}$ are minimax both for the case (a) when $\sigma$ and $\tau$ are given and when (b) $\sigma$ and $\tau$ are restricted.

Then an identical reasoning leads us to the wished conclusion that $\bar{Y} - \bar{X}$ is minimax for estimation of $\theta = \eta - \xi$ because

$$E_{\eta, \xi}(\bar{Y} - \bar{X} - \theta)^2 = E_{\eta}(\bar{Y} - \eta)^2 + E_{\xi}(\bar{X} - \xi)^2$$

and we can define prior for $\theta$ as the distribution of $\Theta := \eta - \xi$ with $\eta$ and $\xi$ being independent with distributions:

(a) in this case they may have the same $\text{Norm}(0, \sigma_k^2)$ where $\sigma_k \to \infty$.

(b) Consider $\sigma^2 = A$, $\tau^2 = B$ and repeat case (a).