1. We have (for a general case)
\[ p(x|\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \]

Denote \( p'(x) = \frac{\partial p(x|\sigma)}{\partial \sigma} \). Then
\[ \frac{p'(x|\sigma)}{p(x|\sigma)} = \frac{1}{\sigma} + \frac{(x-\mu)^2}{\sigma^3}. \]

Then
\[ E\left\{ \frac{p'(x|\sigma)}{p(x|\sigma)} \right\} = \sigma^{-2} - 2\sigma^{-4} E(X-\mu)^2 + \sigma^{-6} E(X-\mu)^4 \]
\[ = \sigma^{-2} - 2\sigma^{-2} + 3\sigma^4/\sigma^6 = 2\sigma^{-2} =: I(\sigma). \]

Note that for an optimal \( \hat{\sigma} \) this implies that \( n^{1/2}(\hat{\sigma} - \sigma) \overset{L}{\to} N(0,\sigma^2/2) \).

Now let \( \theta = \sigma^2 \) be the estimand. Then, using the Delta method,
\[ n^{1/2}(\hat{\sigma}^2 - \sigma^2) \overset{L}{\to} N(0,(2\sigma)^2(\sigma^2/2)) = N(0,2\sigma^4). \]

But we know (or you can calculate it) that \( I(\theta) = 1/[2\sigma^4] \), that is, we got the efficient estimator.

2. Let \( p(x|\theta) = p(x-\theta), x \in (-\infty, \infty) \). Then
\[ I(\theta) = \int_{-\infty}^{\infty} [p'(x-\theta)]^2 p^{-1}(x-\theta) dx \]

[change of variable \( u = x - \theta \)]
\[ = \int_{-\infty}^{\infty} [p'(u)]^2 p^{-1}(u) du = C. \]

In other words, the Fisher information for location family is constant. Thus, for an unbiased estimate \( \hat{\theta}_n \) based on \( n \) observations we have
\[ E_\theta\{(\hat{\theta}_n - \theta)^2\} \geq [nC]^{-1}. \]

3. Consider \( p(x|\theta) = \theta^{-1} f(x/\theta), \theta > 0 \). Then
\[ p'(x|\theta) = -(1/\theta^2) f(x/\theta) - \theta^{-3} x f'(x/\theta). \]

We conclude that
\[ I(\theta) = \theta^{-2} \int_{-\infty}^{\infty} \frac{\theta^{-2} [f(x/\theta) + x\theta^{-1} f'(x/\theta)]}{\theta^{-1} f(x/\theta)} dx \]
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[make change of variable $u = x/\theta$ so $du = dx/\theta$]

$$= \theta^{-2} \int_{-\infty}^{\infty} \frac{[f(u) + uf'(u)]^2}{f(u)} du$$

$$= \theta^{-2} \int_{-\infty}^{\infty} [uf'(u)/f(u) + 1]^2 f(u) du.$$ 

4. Here the idea is that if $Y = \theta Z > 0$ then $\ln(Y) = \ln(\theta) + \ln(Z)$ becomes the location model for which, as we know from Problem 1, the Fisher information is constant.

Now let us prove the wished assertion. The density is $e^{-\xi} f(xe^{-\xi})$. Then

$$I(\xi) = \int_{-\infty}^{\infty} \frac{[-e^{-\xi} f(xe^{-\xi}) - xe^{-2\xi} f'(xe^{-\xi})]^2}{e^{-\xi} f(xe^{-\xi})} dx$$

[change of variable $u = xe^{-\xi}$, so $du = e^{-\xi} dx$]

$$= \int_{-\infty}^{\infty} \frac{[f(u) + uf'(u)]^2}{f(u)} du.$$ 

As you see, the last expression does not depend on $\xi$.

5. Let us recall the idea of the Information inequality,

$$0 = \int \left[ \frac{\partial p_\theta(x)}{\partial \theta} \right] dx = \int p'_\theta(x) dx.$$ 

Then

$$\frac{\partial}{\partial \theta} E_\theta \{ \delta(X) \} = \int p'_\theta(x) \delta(x) dx$$

$$= \int p'_\theta(x)(\delta(x) - g(\theta)) dx.$$ 

Then we use Cauchy-Schwarz for

$$\int \frac{p'_\theta \ast x}{\sqrt{p_\theta(x)}} \left[ \sqrt{p_\theta(x)}(\delta(x) - g(\theta)) \right] dx,$$

and get the Information inequality.

Now remember that we have equality in Cauchy-Schwarz iff

$$\frac{p'_\theta(x)}{\sqrt{p_\theta(x)}} = C \sqrt{p_\theta(x)}(\delta(x) - g(\theta))$$

which yields

$$\delta(x) = g(\theta) + C \frac{p'_\theta(x)}{p_\theta(x)}.$$
Finally
\[ E(\delta(X) - g(\theta))^2 = \frac{(g(\theta))^2}{I(\theta)}, \]
and this implies that \( C^2 = [g'(\theta)]^2/I^2(\theta) \).

We conclude that
\[ \delta(X) = g(\theta) + \frac{g'(\theta)}{I(\theta)} \partial p_\theta(X) / \partial \theta \]
\[ = g(\theta) + \frac{g'(\theta)}{I(\theta)} \partial \ln(p_\theta(X)) / \partial \theta. \]

6. Here \( X \sim \text{Norm}(\theta, 1) \), and then
\[ p_\theta^{X|X \in (a,b)}(x) = \frac{(2\pi)^{1/2}e^{-(x-\theta)^2/2}}{\int_a^b (2\pi)^{1/2}e^{-(u-\theta)^2/2}du}. \]

Further,
\[ P_\theta(X \in (b - \epsilon, b)|X \in (a, b)) \]
\[ = \frac{F_\theta(b) - F_\theta(b - \epsilon)}{F_\theta(b) - F_\theta(a)} = \frac{F(b - \theta) - F(b - \epsilon - \theta)}{F(b - \theta) - F(a - \theta)}. \]

Now, if \( \theta \to \infty \) then \( F(x - \theta) \) becomes small. It is known that
\[ F(x) = \frac{e^{-x^2}}{\sqrt{2\pi|x|}}(1 + o_x(1)), \ x \to -\infty \]
Thus
\[ \frac{F(b - \theta)}{F(a - \theta)} = O(e^{2(b-a)\theta}) \to \infty \]
as \( \theta \to \infty \).

Also, from
\[ P_\theta(X \in (b - \epsilon, b)|X \in (a, b)) = 1 - \frac{F(b - \epsilon - \theta)}{F(b - \theta)} \]
and (1) we get
\[ \frac{F(b - \epsilon - \theta)}{F(b - \theta)} = O(e^{-2\theta}) \to 0 \]
as \( \theta \to \infty \).

7. Here
\[ P_\theta(X_i = x) = \theta x e^{-\theta} / [x!(1 - e^{-\theta})], \ x = 1, 2, \ldots \]
We are seeking the maximum of
\[ L(x^n, \theta) = \sum_{i=1}^{n} \ln(P_\theta(X = x_i)) = \sum_{i=1}^{n} x_i \ln(\theta) - n\theta - \sum_{i=1}^{n} \ln(x_i!) - n \ln(1 - e^{-\theta}). \]
Set the derivative to zero,

\[ 0 = L'(X^n, \theta) = \sum_{i=1}^{n} \frac{X_i}{\theta} - n - \frac{n e^{-\theta}}{1 - e^{-\theta}} \]

\[ = \sum_{i=1}^{n} \frac{x_i}{\theta} - n[1 + \frac{e^{-\theta}}{1 - e^{-\theta}}] = \sum_{i=1}^{n} \frac{x_i}{\theta} - n/(1 - e^{-\theta}) = 0. \]

We get \( \bar{x} = \theta/(1 - e^{-\theta}) \). Now consider a function

\[ f(\theta) := \frac{\theta}{1 - e^{-\theta}}. \]

We write

\[ f'(\theta) = \frac{(1 - e^{-\theta}) - \theta e^{-\theta}}{(1 - e^{-\theta})^2} = \frac{1 - (1 + \theta)e^{-\theta}}{(1 - e^{-\theta})^2}. \]

As we see, \( f'(\theta) \) is positive for both small and large \( \theta \). Let us check the possibility of \( f'(\theta) = 0 \). In this case \( e^\theta = 1 + \theta \), and this occurs only if \( \theta = 0 \), and this case is not of interest to us. We conclude that \( f(\theta) \) is increasing in \( \theta \) and the solution \( \bar{x} = f(\theta) \) is unique.

8. Here \( L(\theta) := L(X^n, \theta) = \ln(p(X^n|\theta)) \). Then

\[ L(\theta_0 + 1/\sqrt{n}) - L(\theta_0) + (1/2) I(\theta_0) \]

[using Taylor’s formula]

\[ = n^{-1/2} L'(\theta_0) + 2n^{-1} L''(\theta_0) + [6n^{3/2}]^{-1} L'''(\tilde{\theta}) + (1/2) I(\theta_0). \]

By the CLT

\[ n^{-1/2} L'(\theta_0) = n^{-1/2} \sum_{i=1}^{n} \frac{p'(X_i|\theta)}{p(X_i|\theta)} \xrightarrow{\mathbb{P}} N(0, I(\theta)). \]

By the LLN

\[ (2n)^{-1} L''(\theta_0) \xrightarrow{\mathbb{P}} (1/2) E_{\theta_0} \left\{ \frac{p''(X|\theta_0)p(X|\theta_0) - (p'(X|\theta_0))^2}{p^2(X|\theta_0)} \right\} = -(1/2) I(\theta_0). \]

Also note that \( n^{-1}|L'''(\tilde{\theta})| < C \). These relations imply that

\[ [L(\theta_0 + 1/n^{1/2}) - L(\theta_0) + (1/2) I(\theta_0)]/I^{1/2}(\theta_0) \xrightarrow{\mathbb{P}} N(0, 1). \]

9. Assume that \( f(x) \) is differentiable. Then

\[ \ln(f(x)) = \psi(x) \]

and we get for the considered function that \( \psi''(x) < 0 \). Further, \( f'(x)/f(x) = \psi'(x) \). We need to ask about \( f(x) \to 0 \) as \( |x| \to \infty \). Then \( f'(x) \) has the sign of \( \psi'(x) \). Further, \( \psi'(x) \)
cannot strictly positive or negative, so \( f(x) \) increases and then decreases. This yields one mode if we assume that \( \psi''(x) < 0 \).

10. Let us consider just the case of one observation. Then \( L'(x, \theta) = 0 \) implies \( f'(x)/f(x) = 0 \) because strictly monotone in \( \theta \) implies unique root. If \( f'(x)/f(x) \) is decreasing then \( L(x, \theta) \) is convex (or the second derivative is positive).

For \( n \) observations, \( L(X^n, \theta) = \sum_{i=1}^{n} \ln(p(X_i|\theta)) \), and \( L'(X_n, \theta) = -\sum_{i=1}^{n} f'(X_i-\theta)/f(X_i-\theta) \). Then you just repeat the above-provided arguments.