Problem 1. The log-likelihood is

\[ L(\theta) = n \ln(\theta) + (\theta - 1) \sum_{i=1}^{n} \ln(X_i) - \sum_{i=1}^{n} X_i^\theta. \]

Based on this we get

\[ L'(\theta) = n\theta^{-1} + \sum_{i=1}^{n} \ln(X_i) - \sum_{i=1}^{n} X_i^\theta \ln(X_i). \]

Furthermore,

\[ L''(\theta) = -n\theta^{-2} - \sum_{i=1}^{n} X_i^\theta (\ln(X_i))^2. \]

Hence, if \( \hat{\theta}_0 \) is an appropriate initial estimator, the one-step Newton-Raphson method gives us

\[ \hat{\theta}_1 := \hat{\theta}_0 \left[ 1 + \frac{n + \hat{\theta}_0 (\sum_{i=1}^{n} \ln X_i - \sum_{i=1}^{n} X_i^{\hat{\theta}_0} \ln(X_i))}{n + (\hat{\theta}_0)^2 \sum_{i=1}^{n} X_i^{\hat{\theta}_0} (\ln(X_i))^2} \right]. \]

Remark: Note that the second derivative is negative, and as a result, if the likelihood has a root, it is unique and equals the MLE. As \( \theta \to 0 \), we have \( L'(\theta) \to \infty \) while for sufficiently large \( \theta \) we have negative \( L'(\theta) \). This proves the existence of the root. Note that this type of speculation is typical in the analysis.

Usually one can use a moment estimator as the initial estimator (it is typically \( \sqrt{n} \)-consistent). In our case a method of moment estimator of \( \theta \) is the solution of \( \bar{X} = \Gamma(\theta^{-1} + 1) \).

Problem 2.

(i) Let us make some preliminary calculations:

\[ \frac{\partial \ln(f(x))}{\partial \mu} = -\sigma^{-1} + 2/[\sigma(1 + e^{-(x-\mu)/\sigma})], \]

\[ \frac{\partial^2 \ln(f(x))}{\partial \mu^2} = -\frac{2e^{-(x-\mu)/\sigma}}{\sigma^2(1 + e^{-(x-\mu)/\sigma})^2}, \]

and we also calculate the Fisher information

\[ -E\left\{ \frac{\partial^2 \ln(f(x))}{\partial \mu^2} \right\} = \frac{2}{\sigma^2} \int_{-\infty}^{\infty} \frac{e^{2(x-\mu)/\sigma}}{\sigma[1 + e^{(x-\mu)/\sigma}]^4} \]

[make the change of variable \( y = -(x-\mu)/\sigma \)]

\[ = \frac{2}{\sigma^2} \int_{-\infty}^{\infty} \frac{e^{2y}}{(1 + e^y)^4} dy \]
[make the change of variable \( t = (1 + e^y)^{-1} \) or \( y = \ln(1 - t) - \ln(t) \)]

\[
\frac{2}{\sigma^2} \int_0^1 \frac{(1 - t)^2}{t^2} t^4 \left( \frac{1}{1 - t} + \frac{1}{t} \right) dt
\]

\[
= 2\sigma^{-2} \int_0^1 (1 - t) t dt = \frac{1}{3\sigma^2}.
\]

Finally, we are ready to write down one-step MLEs:

\[
\hat{\mu}_1 = \hat{\mu}_0 - \left[ L'(\hat{\mu}_0) / L''(\hat{\mu}_0) \right]
\]

for the Newton-Raphson method and

\[
\hat{\mu}_1 = \hat{\mu}_0 + 3\sigma^2 n^{-1} L'(\hat{\mu}_0)
\]

by the Fisher-scoring method.

As an initial estimate, we can choose the moment estimator \( \hat{\mu}_0 := \bar{X} \) because \( \mu \) is the theoretical mean of the logistic distribution.

(ii) Let us again begin with some preliminary calculations. Write,

Differentiating \( \ln(f(x)) \) with respect to \( \sigma \) yields

\[
\frac{\partial \ln(f(x))}{\partial \sigma} = -\frac{1}{\sigma} - \frac{x - \mu}{\sigma} + \frac{2(x - \mu)}{\sigma^2 [1 + e^{-(x - \mu)/\sigma}]},
\]

and further

\[
\frac{\partial^2 \ln(f(x))}{\partial \sigma^2} = \frac{1}{\sigma^2} + \frac{2(x - \mu)}{\sigma^3}
\]

\[
- \frac{4(x - \mu)}{\sigma^3 [1 + e^{-(x - \mu)/\sigma}]} - \frac{2(x - \mu)^2 e^{-(x - \mu)/\sigma}}{\sigma^4 [1 + e^{-(x - \mu)/\sigma}]^2}.
\]

Next step is the Fisher information. Note that \( E(X) = \mu \) and using the change of variable \( y = -(x - \mu)/\sigma \) write

\[
-E\{\partial^2 \ln(f(x))/\partial \sigma^2\} = -\frac{1}{\sigma^2} + \frac{4}{\sigma^2} \int_{-\infty}^\infty ye^{2y}(1 + e^y)^{-3} dy + 2\sigma^{-2} \int_{-\infty}^\infty y^2 e^{2y}(1 + e^y)^4 dy.
\]

Now we are using integration by parts and convert integrals to the known moments of the logistic distribution:

\[
\int_{-\infty}^\infty ye^{2y}(1 + e^y)^{-3} dy = ye^y [2(1 + e^y)^2]^{-1} \big|_{-\infty}^\infty + 2^{-1} \int_{-\infty}^\infty \frac{e^y + ye^y}{(1 + e^y)^2} dy
\]

\[
= 2^{-1} \int_{-\infty}^\infty \frac{e^y}{(1 + e^y)^2} dy + 2^{-1} \int_{-\infty}^\infty \frac{ye^y}{(1 + e^y)^2} dy = 2^{-1}.
\]

Well, now the more complicated term. Write

\[
\int_{-\infty}^\infty \frac{y^2 e^{2y}}{(1 + e^y)^4} dy = 2 \int_0^\infty \frac{y^2 e^{2y}}{(1 + e^y)^3} dy
\]
\[
\frac{2y^2e^y}{3(1 + e^y)^3} \bigg|_0^\infty + \frac{(2/3) \int_0^\infty (2y + y^2)e^y}{(1 + e^y)^3} \ dy
\]

\[
= \frac{2y + y^2}{3(1 + e^y)^2} \bigg|_0^\infty + \frac{(2/3) \int_0^\infty 1 + y}{(1 + e^y)^2} \ dy
\]

\[
= \frac{2(1 + y)e^{-y}}{3(1 + e^y)} \bigg|_0^\infty - \frac{(2/3) \int_0^\infty ye^{-y}}{1 + e^y} \ dy
\]

\[
= (1/3) - \frac{(2/3) \int_0^\infty ye^{-2y}}{1 + e^{-y}} \ dy.
\]

Using the series \((1 + e^{-y})^{-1} = \sum_{j=0}^\infty (-e^{-y})^j\) write

\[
\int_0^\infty \frac{ye^{-2y}}{1 + e^{-y}} \ dy = \int_0^\infty ye^{-2y} \sum_{j=0}^\infty (-1)^j e^{-jy} \ dy
\]

\[
= \sum_{j=0}^\infty (-1)^j \int_0^\infty ye^{-(j+2)y} \ dy = \sum_{j=0}^\infty (-1)^j (j + 2)^{-2}.
\]

Note that \((-1)^j\) changes sign when \(j\) is even or odd, we continue

\[
\sum_{j=0}^\infty (-1)^j (j + 2)^{-2} = \sum_{k=1}^\infty (2k)^{-2} - \sum_{k=1}^\infty (2k + 1)^{-2}
\]

\[
= \sum_{k=1}^\infty (2k)^{-2} - \sum_{k=1}^\infty \left[ \frac{1}{(2k + 1)^2} + \frac{1}{(2k)^2} - \frac{1}{(2k)^2} \right]
\]

\[
= 2 \sum_{k=1}^\infty \frac{1}{(2k)^2} - \sum_{j=2}^\infty j^{-2} = 2^{-1} \sum_{k=1}^\infty k^{-2} - \sum_{j=1}^\infty j^{-2} + 1
\]

\[
= 1 - 2^{-1} \sum_{k=1}^\infty k^{-2} = 1 - \pi^2/12.
\]

Combining the results we conclude that

\[-E\{\partial^2 \ln(f(\theta))/\partial \sigma^2\} = -\sigma^{-2} + 2\sigma^{-2} + 2\sigma^{-2}[(1/3) - (2/3)(1 - \pi^2/12)] = \sigma^{-2}[(1/3) + \pi^2/9].\]

Now we can use these calculations in writing down one-step MLEs:

\[
\hat{\sigma}_1 = \hat{\sigma}_0 - \left[ L'(\hat{\sigma}_0)/L''(\hat{\sigma}_0) \right]
\]

for the Newton-Raphson method, and

\[
\hat{\sigma}_1 = \hat{\sigma}_0 + \frac{\hat{\sigma}_0^2}{\pi^2/9 + 1/3} L'(\hat{\sigma}_0)
\]

for the Fisher-scoring method.
The initial \( \sqrt{n} \)-consistent estimator can be chosen as the moment estimator. Using the fact that the logistic distribution has variance \( \sigma^2 \frac{\pi^2}{3} \), we get that

\[
\hat{\sigma}_0 = \frac{3^{1/2}}{\pi} [n^{-1} \sum_{i=1}^{n} (X_i - \mu)^2]^{1/2}.
\]

Problem 3. Note that in the considered Grouped case of indirect observations the \( n \) observations constitute \( n \) trinomial trials with probabilities \( p_1 = p_1(\theta) = F(a - \theta) \), \( p_2 = p_2(\theta) = F(b - \theta) - F(a - \theta) \), and \( p_3 = p_3(\theta) = 1 - F(b - a) \) for the three outcomes.

Then you can work with any outcome. For instance, let \( Z \) denotes the number of observations less than \( a \), then

\[
n^{1/2} \left[ \frac{Z}{n} - p_1 \right] \overset{\mathcal{L}}{\rightarrow} N(0, p_1(1 - p_1)).
\]

As a result, using the Delta-method, you can check that the estimate

\[
\tilde{\theta} = a - F^{-1}(Z/n)
\]

is an \( \sqrt{n} \)-consistent estimate of \( \theta \).