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Problem 1

Show that the Generalized Geography problem belongs to PSPACE.

Let

\[ GG = \{ \langle G, b \rangle | \text{Player I, which plays first, has a winning strategy for the generalized geography game played on graph } G = (V, E) \text{ starting at node } b \} \].

The following algorithm \( M \) decides \( GG \)

\[ M = \text{" On input } \langle G, b \rangle, \text{ where } G = (V, E) \text{ is a directed graph and } b \in V :\]

1. If \( b \) has outdegree 0, then reject, since there are no moves available for Player 1.
2. Remove \( b \) and all of its incoming and outgoing edges from \( G \) to get \( G' \)
3. For each node \( b_1, b_2, \ldots, b_k \) where \( b \) has a directed edge to, call \( M \) on \( \langle G', b_i \rangle \) recursively for each \( i \in \{1, 2, \ldots, k\} \).
4. If all of these accept, then no matter which decision Player 1 makes, Player 2 has a winning strategy, so reject. Otherwise, Player 1 has a choice that will deny any successful strategies of Player 2, therefore return accept. "

**Lemma.** The algorithm \( M \) runs in polynomial space; hence \( GG \) is in PSPACE.

**Proof.** The only non-trivial space required by this algorithm is for storing the recursion stack. Each level of the recursion adds a single node to the stack, and at most \( m \) levels occur, where \( m \) is the number of nodes in \( G \). Hence the algorithm runs in linear space.
Problem 2

Show that the Planar Generalized Geography (PGG) problem belongs to PSPACE-complete. Give detailed explanation.

Consider the graph that is used in reducing TQBF to GG (see Figure 1 in Ref. [1] in the last page). We will show that every constructed graph can be transformed into an equivalent PGG graph as follows: Draw the graph in a plane and consider all the 2 crossing edges and the end incident vertices:

![Graph Diagram](image)

Replace that section of the graph by the following subgraph.

![Subgraph Diagram](image)

**PROPOSITION.** $\exists$ has a win in the new graph with the indicated replacement iff $\exists$ has a win in the original graph.

**PROOF.** We will show that if the play enters through node 1, the play has to exit from the node 3. Similarly, if the play enters through node 2, the play has to exit from node 4. Let the first move be made by $\exists$, without loss of generality.

1) **Case 1 (Play enters through vertex 1):** $\exists$ will follow the edge (1,5) since it is the only option. Similarly, $\forall$ will follow the edge (5,6) since it is the only option. Then, $\exists$ has to follow the edge (6,7), otherwise he will lose because if he follows the other option, which is the edge (6,9), then $\forall$ will follow the edge (9,12) and win. Now, $\forall$ has two options: (7,8) or (7,13). $\forall$ has to follow the edge (7,8) otherwise he will lose because if it follows the edge (7,13), then $\exists$ will follow the edge (13,11) and win. Now, the game is at node 8 and $\exists$ will follow the edge (8,3). Hence, the first case analysis is complete: If play enters through node 1, it should exit through the node 6.

2) **Case 2 (Play enters through vertex 2):** $\exists$ will follow the edge (2,11) since it is the only option. Similarly, $\forall$ will follow the edge (11,6). Then, $\exists$ has to follow the edge (6,9), otherwise he will lose because if he follows the other option, which is the edge (6,7), then $\forall$ will follow the edge (7,13) and win. Now $\forall$ has two options: (9,10) and (9,12). $\forall$ has to follow the edge (9,10) otherwise he will lose because if it follows the edge (9,12), then $\exists$ will follow the edge (12,5) and win. Now, the game is at node 10 and $\exists$ will follow the edge (10,4). Hence, the second case analysis is complete: If play enters through node 2, it should exit through node 4.

**Conclusion.** PGG is in PSPACE (using the algorithm $M$ given in Problem 1). Since TQBF is PSPACE-complete and can be reduced to PGG in polynomial time, PGG is PSPACE-hard. Since PGG is in PSPACE and PGG is PSPACE-hard, PGG is PSPACE-complete.
Problem 3

Show that the Planar Generalized Geography problem in bipartite graphs is PSPACE-complete. Give detailed explanation.

Consider the constructed graph in the previous problem, we will make a slight modification to that graph to make it bipartite. We enlarge the diamonds representing the existential variables to the following:

\[ v_{i,0}, \overline{v}_{i,0}, v_{i,1}, \overline{v}_{i,1}, v_{i,2}, \overline{v}_{i,2}, v_{i,3}, \overline{v}_{i,3}, v_{i,4}, \overline{v}_{i,4} \]

and the back arcs representing clauses are then attached at \( v_{i,2} \) or \( \overline{v}_{i,2} \) instead of \( v_{i,1} \) or \( \overline{v}_{i,1} \) which allows the graph to remain bipartite.

Why is the constructed graph bipartite? Note that bipartite graph is a graph whose vertices can be divided into two disjoint sets \( U \) and \( T \) such that every edge connects a vertex in \( U \) to one in \( T \). First of all, the construction of the planar graph is preserving the bipartiteness. Consider the modification that we made in the previous question to make the graph planar. Let the node 1 be in \( U \), without loss of generality, then node 2 has to be in \( U \), too. Node 3 and node 4 has to be in \( T \). Hence we can focus on the non-planar graph to simplify the explanation. I will use \( V_{i,j} \) to denote the set of vertices in the \( i^{th} \) diamond the play is possibly at, after the \( j^{th} \) selection at diamond \( i \). In the above figure \( V_{i,1} = \{v_{i,1}, \overline{v}_{i,1}\} \) and \( V_{i,4} = \{v_{i,4}\} \). In the constructed graph, assume that there are \( k \) variables, i.e., \( \{v_0, v_1, \ldots, v_{k-1}\} \). Assume also that, without loss of generality, the first set of nodes in the first diamond, which is denoted as \( V_{0,0} \), belongs to \( U \). Then, \( V_{i,0}, V_{i,2} \) and \( V_{i,4} \) will be in \( U \) for \( i = \{0, 2, 4, k-2\} \). Additionally, \( V_{j,1} \), will be in \( U \) for \( j = \{1, 3, 5, k-1\} \). Furthermore, nodes for clauses will be in \( U \). The rest of the nodes \( V_{i,1} \) and \( V_{i,3} \) for \( i = \{0, 2, 4, k-2\} \), and \( V_{j,0} \) and \( V_{j,2} \) for \( j = \{1, 3, 5, k-1\} \) will be in \( V \). Additionally, nodes for variables in clauses will be in \( V \). Since these nodes are connected to the \( V_{i,2} \) in the \( \exists - diamonds \) and \( V_{i,1} \) in the \( \forall - diamonds \), which are in \( U \), and \( \exists - diamonds \) and \( \forall - diamonds \) alternate, the graph is bipartite.

Conclusion. PGG in bipartite graphs (BPGG) is in PSPACE (using the algorithm \( M \) given in Problem 1). Since TQBF is PSPACE-complete and can be reduced to BPGG in polynomial time, BPGG is PSPACE-hard. Since BPGG is in PSPACE and BPGG is PSPACE-hard, BPGG is PSPACE-complete. \( \square \)
**Problem 4**

Show that the following variation of the Generalized Geography (GG) problem is PSPACE-complete: Given a directed graph and a first move of the 1st player, determine whether the second player has a winning strategy.

Let’s name the problem as modified Generalized Geography (MGG) problem. We need to show that MGG is in PSPACE and MGG is PSPACE-hard by reducing a PSPACE-complete problem to MGG problem.

**MGG ∈ PSPACE:** We can modify the algorithm in Problem 1 slightly such that the vertex $b$, which is the starting vertex, and all of its incident edges are ignored and the algorithm $M$ runs where new starting point is the endpoint of the first move of Player 1, if it accepts, accept, otherwise reject. Algorithm runs in linear space since $M$ runs in linear space.

**MGG is PSPACE-hard:** We will reduce GG problem, which is known to be PSPACE-complete, to MGG problem to show that MGG problem is PSPACE-hard.

**GG =** \{ \langle G, b \rangle | \text{Player I, which plays first, has a winning strategy for the generalized geography game played on graph } G = (V, E) \text{ starting at node } b \}.

Given an input $< G, b >$ of any instance GG problem, we will convert its input to an input of an instance of a MGG problem, i.e., $< G', b, nextb >$, such that instance of GG is a YES instance iff instance of MGG is a YES instance. Here is the reduction. Set $nextb = b$. Add a new vertex to $G$ and make it the starting vertex $b \in G'$. Finally add an edge between vertex $b$ and $nextb$.

Note that if Player 1 has a winning strategy in $G$, Player 2 should have a winning strategy in $G'$. $b \in G'$ can not be visited again, so rest of the game is played on a subgraph of $G'$ which is same as $G$. The only difference is that Player 2 will be the first player of the game. Conversely, if Player 2 has a winning strategy in $G'$, then Player 1 can use the same strategy and win; hence Player 1 has a winning strategy in $G$. 

\[\begin{array}{c}
\text{b} \\
\text{2} \\
\text{3} \\
\text{5} \\
\text{4} \\
\text{6} \\
\text{7} \\
\end{array}\quad \begin{array}{c}
\text{b} \\
\text{nextb} \\
\text{2} \\
\text{3} \\
\text{5} \\
\text{4} \\
\text{6} \\
\text{7} \\
\end{array}\]

$< G, b >$ $< G', b, nextb >$
Problem 5

Is there a fixed integer $k > 0$ such that the GO problem in $k \times k$ board is PSPACE-hard? Explain why your answer is correct.

The answer is NO. When $k$ is fixed, then the game is a finite game for which a large table (constant space) containing a winning strategy could, in principle, be given.
References