Lecture 3 “Introduction to material processing” is covered in a separate document (power point slides).

Lecture 4 Basic Plasma Equations
In our first lecture, we dealt with some simple properties of fluids. While plasmas have many of the qualities of fluids, they also have charged particles which are subject to electromagnetic forces.

In undergraduate electromagnetism, we learned that the E-M forces were governed by a set of equations known as Maxwell’s Equations. (Here we also include the equation for charge conservation.)

\[ \nabla \times \mathbf{E} = -\partial_t \mathbf{B} \]
\[ \nabla \times \mathbf{H} = \mathbf{J}_{\text{free}} + \partial_t \mathbf{D} \]

**Maxwell’s Equations Point Form:**
\[ \nabla \cdot \mathbf{D} = \rho_{\text{free}} \]
\[ \nabla \cdot \mathbf{B} = 0 \]
\[ \nabla \cdot \mathbf{J} = -\partial_t \rho \]
\[ \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \iint \mathbf{B} \cdot d\mathbf{S} \quad \text{Enclosing curve} \]
\[ \oint \mathbf{H} \cdot d\mathbf{l} = \iint \mathbf{J}_{\text{free}} \cdot d\mathbf{S} + \frac{d}{dt} \iint \mathbf{D} \cdot d\mathbf{S} \quad \text{Enclosing curve} \]

**Maxwell’s Equations Integral Form:**
\[ \iint \mathbf{D} \cdot \mathbf{\Sigma dS} = \iiint \rho_{\text{free}} d\tau \quad \text{Enclosing surface} \]
\[ \iint \mathbf{B} \cdot \mathbf{\Sigma dS} = 0 \quad \text{Enclosing surface} \]
\[ \iint \mathbf{J} \cdot \mathbf{\Sigma dS} = -\frac{d}{dt} \iiint \mathbf{\rho d\tau} \quad \text{Enclosing surface} \]

We can readily transfer from the point form of Maxwell’s equations to the integral from using the Divergence and Stoke’s Theorems.

**Divergence Theorem**
\[ \iiint \nabla \cdot \mathbf{V} d\tau = \iint \mathbf{V} \cdot n d\sigma \quad \text{Volume enclosig surface} \]

**Stoke’s Theorem**
\[ \iint (\nabla \times \mathbf{V}) \cdot n d\sigma = \oint \mathbf{V} \cdot d\mathbf{r} \quad \text{Enclosing curve} \]

In addition, we also found **Lorentz Force Equation:** \( \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \), and **Ohm’s Law:** \( \mathbf{J} = \sigma \mathbf{E} \).
To keep things simple, we will at first just focus on plasma-induced fields. Of particular interest is the first equation in our list of Maxwell’s equations, $\nabla \times E = -\partial_t B$ (Faraday’s Law). We know that we can let $B = \nabla \times A$, where $A$ is known as the magnetic vector potential. Thus, rewriting Faraday’s law, we find
\[
\nabla \times (E + \partial_t A) = 0
\]
\[
\Downarrow
\]
\[
E = -\nabla \Phi - \partial_t A
\]
where $\Phi$ is the scalar potential. Often in laboratory plasmas there is no time varying magnetic field and hence $\partial_t A = 0$

\[
E = \nabla \Phi
\]

NOTE THAT THIS IS NOT TRUE IN FUSION PLASMAS NOR IS IT TRUE IN SOME ARC TYPE PLASMAS! IN BOTH CASES THERE EXISTS A STRONG SELF-INDUCED TIME-VARYING MAGNETIC FIELDS.

We can now plug our scalar potential back into Gauss’ Law to get
\[
\nabla \sum E = -\nabla \sum \nabla \Phi
\]
\[
= -\nabla^2 \Phi
\]
\[
= \frac{\rho}{\varepsilon}
\]
This is known as Poisson’s equation. Assume that all of our charged particles are singly charged positive ions, Ion+, and negatively charged electrons, e−. (Electron-positron, Ion+-Ion-, etc., plasmas also exist. We are just looking at the most typical.) Then Poisson’s equation becomes
\[
\nabla^2 \Phi = -\frac{\rho}{\varepsilon}
\]
\[
= \frac{e}{\varepsilon} (n_e - n_i)
\]
where $n_e$ is the electron density and $n_i$ is the ion density. This tells us that a plasma can induce an electric field that depends on the local densities of the charge carrier. Because of this, we will begin to look at the velocity distribution function $f(r, v, t) = f(x, y, z, v_x, v_y, v_z, t)$. The number of particles that are inside a volume of $dx dy dz dv_x dv_y dv_z$ is simply
\[
f(x, y, z, v_x, v_y, v_z, t) dx dy dz dv_x dv_y dv_z
\]
Often the six coordinates are considered to be independent. Then the rate of change of particles at point is
\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r} + \frac{\partial f}{\partial v} = \left( \nabla_r f \right) \Sigma v + \left( \nabla_v f \right) \Sigma a + \frac{\partial f}{\partial t}
\]

This is the Boltzmann equation. If particles are not sourced/sunk at the point then the total number does not change and hence
\[
\frac{df}{dt} = 0 = \left( \nabla_r f \right) \Sigma v + \left( \nabla_v f \right) \Sigma a + \frac{\partial f}{\partial t}
\]
This is known as the collisionless Boltzmann equation or the Vlasov equation.

**The velocity distribution and macroscopic quantities.**

In the very first class, we talked about the velocity distribution of a gas, and an assortment of different velocity distributions. Further, we related the velocity distribution, \( f(v) \), to a probability distribution, \( p(y) \). Now, we want to return to the concept.

**Expectation value**

Let us assume that we can randomly pick a \( y \) from a set, having a probability \( p(y_0) \) of picking \( y_0 \). What value would we expect to get – on average – for \( y \)? If I were to draw 100 \( y \)'s from a basket, we would expect \( p(y_0) * 100 \) to be \( y_0 \). Assume that I can also pick \( y_1, y_2, \) and \( y_3 \). Then the average of my 100 picks would be

\[
\langle y \rangle = \frac{y_0p(y_0) + y_1p(y_1) + y_2p(y_2) + y_3p(y_3)}{p(y_0) + p(y_1) + p(y_2) + p(y_3)}
\]

\[
= \frac{\sum_i y_ip(y_i)}{\sum_i p(y_i)}
\]

If I were to have a continuous set of \( y \)'s then the equation becomes

\[
\langle y \rangle = \frac{\int yp(y)dy}{\int p(y)dy}
\]

as we have already used. (Often we normalize the probability such that \( \int p(y)dy = 1 \).) This averaging is important as when we make a measurement, we are typically making a series of measurements – even if we don’t know it – and our result is the average of our series of measurements.

\( f(r,v,t) \) is our velocity and density distribution function and it is in a very real sense a probability distribution – with 6 dimensions \( (r,v) \) and one free parameter \( (t) \)! (Because of this definition, \( r \) and \( v \) are truly independent quantities – \( v \) does not depend on the location only \( f \). Thus, if we where to desire to know the average velocity then we look at

\[
\langle v \rangle = \frac{\int vf(r,v,t)dv}{\int f(r,v,t)dv}
\]
Likewise the average of the $v^2$ is simply
\[ \langle v^2 \rangle = \frac{\int v^2 f(r,v,t)dv}{\int f(r,v,t)dv}. \]

We can do this for all sorts of averages
\[
\begin{align*}
\int r^0 f(r,v,t)dr &/ \int f(r,v,t)dr; & \int r^1 f(r,v,t)dr &/ \int f(r,v,t)dr; & \int r^2 f(r,v,t)dr &/ \int f(r,v,t)dr; & \ldots \\
\int v^0 f(r,v,t)dv &/ \int f(r,v,t)dv; & \int v^1 f(r,v,t)dv &/ \int f(r,v,t)dv; & \int v^2 f(r,v,t)dv &/ \int f(r,v,t)dv; & \ldots 
\end{align*}
\]

However, we are not interested in most of them. In fact we typically only look at

\[
\begin{align*}
n(r,t) &= \int v^0 f(r,v,t)dv \\
\Gamma(r,t) &= \langle v^1 \rangle = \int v^1 f(r,v,t)dv \\
E_{\text{total}} &= \frac{1}{2} mn\langle v^2 \rangle = \frac{1}{2} m \int v^2 f(r,v,t)dv \\
E_{\text{total}} &= \frac{1}{2} P = \frac{3}{2} nkT = \frac{1}{2} mn\left(\langle v \rangle - \langle v \rangle \right)^2 \\
E_{\text{random}} &= \frac{1}{2} m \int \left(\langle v \rangle \right)^2 f(r,v,t)dv \\
E_{\text{directed}} &= E_{\text{total}} - E_{\text{random}} = \frac{1}{2} mn\langle v \rangle^2
\end{align*}
\]

(These are known as the zeroth, first, and second moments of the distribution function.)

Why are we bringing this up again? Well we can do the same thing to Boltzmann’s Equation.

Zeroth moment of the Boltzmann Equation – The Equation of Continuity (Particle conservation)
\[
\begin{align*}
\int \left( \frac{df}{dt} \right)dv &= \text{Gain} - \text{Loss} = f|_t = \int \left( v \sum \nabla_v f + \frac{F}{m} \sum \nabla_v f + \frac{\partial f}{\partial t} \right)dv \\
&= \left( \int \left( v \sum \nabla_v f \right)dv + \int \left( \frac{F}{m} \sum \nabla_v f \right)dv + \int \left( \frac{\partial f}{\partial t} \right)dv \right) \\
&= \left( \nabla_v \sum \int (vf)dv + \int \left( \frac{F}{m} \sum \nabla_v f \right)dv + \frac{\partial}{\partial t} \int f dv \right) \\
&= \left( \nabla_v \sum n\left( v \right) + \int \left( \frac{F}{m} \sum \nabla_v f \right)dv + \frac{\partial n}{\partial t} \right)
\end{align*}
\]

Now what happens to the central term on the right hand side? First let us assume that force, $F$, is independent of the velocity. (Strictly speaking, this is not true as $F = e(v \wedge B)$.) Then the integral becomes
\[
\int \left( \frac{F}{m} \nabla (v \cdot f) \right) dv = \frac{F}{m} \int \nabla (v \cdot f) dv
\]
\[
= \left[ \frac{F}{m} f \right]_0^\infty = \frac{F}{m} (0 - 0) = 0
\]

The distribution goes to zero for physical reasons – we don’t want an infinite density. If the force is dependent on velocity, it is typically just to first order as in the magnetic force. Then
\[
\int \left( \frac{F}{m} \nabla (v \cdot f) \right) dv = \frac{q}{m} \int ((v \times B) \nabla \times f) dv
\]

To solve this, we must deal with some vector identities. First,
\[
A \times (B \times C) = B(A \cdot C) - C(A \cdot B) \quad \text{and}
\]
\[
A \cdot (B \times C) = (A \cdot B) C - (A \cdot C) B
\]

Thus,
\[
(v \times B) \nabla \times f = (v \times B) \nabla \times [f]
\]
\[
= \nabla \times [f] \nabla (v \times B)
\]
\[
= \nabla \times [f] \times v \nabla B
\]
\[
= \nabla \times [f \times v] \times B - (f \nabla \times [v]) \nabla B
\]
\[
= \nabla \times [f \times v] \times B - f \nabla \times \nabla (v \times B)
\]

First we will look at the second term on the right.
\[
\nabla \times (v \times B) = \partial_y (V_x B_z - V_z B_x) + \partial_z (V_y B_x - V_x B_y) + \partial_x (V_y B_z - V_z B_y)
\]
\[
= 0
\]

Now we can go back to the integral and we find
\[
\int \left( \frac{F}{m} \nabla (v \cdot f) \right) dv = -\frac{q}{m} \int ((v \times B) \nabla \times f) dv
\]
\[
= -\frac{q}{m} \int (\nabla \times [f \times v]) dv
\]
\[
= -\frac{q}{m} \oint_{\text{surface at } \infty} (\nabla \times [f \times v]) dv
\]
\[
= 0
\]

Here \(f \times v\) goes to zero on the surface for physical reasons – we don’t want an infinite energy so \(f\) must go to zero faster than \(v\).

This leaves the continuity equation
\[
\frac{\partial n}{\partial t} = \left( \nabla_r \nabla (n(v)) + \frac{\partial n}{\partial t} \right)
\]

First moment of the Boltzmann equation – Momentum Conservation
Now we multiple Boltzmann’s equation
\[
\frac{df}{dt} = v \nabla (v \cdot f) + \frac{F}{m} \nabla (v \cdot f) + \frac{df}{dt}
\]
by \( mv \) and integrate over velocity to get
\[
m \int \left( \frac{df}{dt} \right) dv = \Delta \text{Momentum} = m \int v \left( v \Sigma f + \frac{F}{m} \Sigma v f + \frac{\partial f}{\partial t} \right) dv
\]
\[
= m \left( \int v (v \Sigma f) dv + \int v \left( \frac{F}{m} \Sigma v f \right) dv + \int v \left( \frac{\partial f}{\partial t} \right) dv \right)
\]

First we will look at the third part of the right-hand side
\[
\int v \left( \frac{\partial f}{\partial t} \right) dv = \frac{\partial}{\partial t} \left[ \int v f dv \right] - \int \frac{\partial}{\partial t} (f) dv
\]
\[
= \frac{\partial}{\partial t} \left[ \int v f dv \right] - \int a(f) dv
\]
\[
= \frac{\partial}{\partial t} \left[ \int v f dv \right]
\]
\[
= \frac{\partial}{\partial t} [n(v)]
\]

Now for the second term
\[
\int v \left( \frac{F}{m} \Sigma v f \right) dv = \frac{q}{m} \int v [E + v \wedge B] \Sigma v f dv \text{ using the chain rule}
\]
\[
= \frac{q}{m} \int v \Sigma \left[ f v (E + v \wedge B) \right] dv
\]
\[
- \frac{q}{m} \int f v \Sigma (E + v \wedge B) dv - \frac{q}{m} \int [f (E + v \wedge B)] \Sigma v dv
\]
\[
= \frac{q}{m} \int f v (E + v \wedge B) dv
\]
\[
= 0 \text{ from before}
\]
\[
- \frac{q}{m} \int f v \Sigma (E + v \wedge B) dv - \frac{q}{m} \int [f (E + v \wedge B)] \Sigma \Sigma v dv
\]
\[
= - \frac{q}{m} \int [f (E + v \wedge B)] \Sigma I dv
\]
\[
= - \frac{q}{m} \int [f (E + v \wedge B)] dv
\]
\[
= - \frac{q}{m} m (E + \langle v \rangle \wedge B)
\]

Finally, we can deal with the first term…

**(HOLD ON THIS GETS EASIER!**)
\[ \int v(\nabla f) dv = \nabla \sum \int f v dv \]
\[ = \nabla \sum n(vv) \]

What precisely is this last term? Well let us consider what \( v \) is…

\[ v = v_{\text{directed}} + v_{\text{random}} \]

Thus

\[ \nabla \sum [n(vv)] = \nabla \sum \left[ n(\langle v_{\text{dir}} + v_{\text{rand}} \rangle (v_{\text{dir}} + v_{\text{rand}})) \right] \]

\[ = \nabla \sum \left[ n((v_{\text{dir}} v_{\text{dir}} + v_{\text{dir}} v_{\text{rand}} + v_{\text{rand}} v_{\text{dir}} + v_{\text{rand}} v_{\text{rand}})) \right] \quad \text{but} \]

\[ \langle v_{\text{dir}} \rangle = v_{\text{dir}} \quad \text{so} \]

\[ = \nabla \sum (nv_{\text{dir}} v_{\text{dir}}) + \nabla \sum \left( 2nv_{\text{dir}} \frac{v_{\text{rand}}}{2} \right) + \nabla \sum n(v_{\text{rand}} v_{\text{rand}}) \]

The last term we have dealt with before. We know for a Maxwellian that

\[ \langle E \rangle = \frac{1}{2} m \langle v^2 \rangle = \frac{1}{2} kT = \frac{1}{2} \frac{P}{n} \quad \text{but for our calculations} \]

\[ v = v_{\text{rand}} ! \]

Thus we find

\[ P = mn \langle v_{\text{rand}} v_{\text{rand}} \rangle \]

(Notice that \( P \) is a tensor – as might make sense – Pressure is direction dependent)

Now going back to our momentum conservation equation we find

\[ \Delta \text{Momentum} \equiv \Delta M = m \left( \int v(\nabla f) dv + \int v \left( \frac{F}{m} \sum \nabla f \right) dv + \int v \left( \frac{\partial f}{\partial t} \right) dv \right) \]

\[ = m \left( \nabla \sum n(v) + \nabla \sum \frac{P}{m} + \frac{q}{m} n(E + \langle v \rangle \times B) + \frac{\partial}{\partial t} [n(v)] \right) \]

\[ = \left( m \langle v \rangle \nabla \sum n(v) + mn \langle v \rangle \sum \nabla \sum (\langle v \rangle) + m \nabla \sum P - qn(E + \langle v \rangle \times B) + m \frac{\partial}{\partial t} [n(v)] \right) \]

using the continuity equation

\[ f = \left( \nabla \sum n(v) + \frac{\partial n}{\partial t} \right) \]

and rearranging…
\[\Delta \mathbf{M}_{k} = \left( m\langle v \rangle \nabla_{r} \Sigma n(v) + mn\langle v \rangle \Sigma \nabla_{r}((v)) + \nabla_{r} \Sigma \mathbf{P} - qn(E + \langle v \rangle \wedge B) + m \frac{\partial}{\partial t} [n\langle v \rangle] \right)\]

\[\Downarrow\]

\[mn \left( \frac{\partial \langle v \rangle}{\partial t} + \langle v \rangle \nabla_{r} \langle v \rangle \right) = \Delta \mathbf{M}_{k} - m\langle v \rangle f_{k} - \nabla_{r} \Sigma \mathbf{P} + qn(E + \langle v \rangle \wedge B)\]

This is slightly different than Lieberman but it is correct.

This is also known as the fluid equation of motion.

**Energy Conservation (Heat flow equation)**

Now we multiply Boltzmann’s equation

\[\frac{df}{dt} = v \Sigma \nabla_{r} f + \frac{F}{m} \Sigma \nabla_{r} f + \frac{\partial f}{\partial t}\]

by \(\frac{1}{2} m\mathbf{v}^{2}\) and integrate over velocity to get

\[\frac{1}{2} m \int \mathbf{v}^{2} \left( \frac{df}{dt} \right) d\mathbf{v} = \Delta \text{Energy} = \frac{1}{2} m \left( \int \mathbf{v}^{2} (v \Sigma \nabla_{r} f) d\mathbf{v} + \int \mathbf{v}^{2} \left( \frac{F}{m} \Sigma \nabla_{r} f \right) d\mathbf{v} + \int \mathbf{v}^{2} \left( \frac{\partial f}{\partial t} \right) d\mathbf{v} \right)\]

This is trivial and is left to the reader (yeah right!)

Rather than go through this derivation, we will accept Lieberman’s formula. (We don’t have time to do this in class now anyway and the first two formula are more important.)

**SIDE NOTE ON VECTORS**

We know that in matrix notation the inner product is

\[\mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} A_{x} \\ A_{y} \\ A_{z} \end{pmatrix} \cdot \begin{pmatrix} B_{x} \\ B_{y} \\ B_{z} \end{pmatrix}\]

\[= (A_{x} \rightarrow \mathbf{B}) \cdot \begin{pmatrix} B_{x} \\ B_{y} \\ B_{z} \end{pmatrix}\]

\[= A_{x} B_{x} + A_{y} B_{y} + A_{z} B_{z}\]

What we have above is the form
\[ \mathbf{A} \mathbf{B} = \begin{pmatrix} A_x & B_x \\ A_y & B_y \\ A_z & B_z \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \]

\begin{align*}
\text{multiply sequential} \\
\text{row elements of } \mathbf{A} \text{ with} \\
\text{column elements of } \mathbf{B} \\
\begin{pmatrix}
A_x B_x & A_x B_y & A_x B_z \\
A_y B_x & A_y B_y & A_y B_z \\
A_z B_x & A_z B_y & A_z B_z 
\end{pmatrix}
\end{align*}

**Equilibrium Properties**

**Boltzmann’s Relation**

Boltzmann’s relation makes use of the momentum conservation equation

\[ mn \left( \frac{\partial \langle \mathbf{v} \rangle}{\partial t} + \langle \mathbf{v} \rangle \nabla \langle \mathbf{v} \rangle \right) = \Delta \mathbf{M} - m \langle \mathbf{v} \rangle f - \nabla_r \Sigma \mathbf{P} + q n (\mathbf{E} + \langle \mathbf{v} \rangle \times \mathbf{B}). \]

Under many conditions, most of these terms are small, particularly for the electron. This is particularly true for the electrons. If we assume that the electron is massless, \( m_e \approx 0 \), then our equation reduces to

\[ 0 = -\nabla_r \Sigma \mathbf{P} + q n (\mathbf{E} + \langle \mathbf{v} \rangle \times \mathbf{B}). \]

If we further assume that the electron is either traveling along the \( \mathbf{B} \) field or that \( \mathbf{B}=0 \) then

\[ \nabla_r \Sigma \mathbf{P} = q n \mathbf{E} \]

but
∇_r \sum P = \nabla \sum \begin{pmatrix} P_x & 0 & 0 \\ 0 & P_y & 0 \\ 0 & 0 & P_z \end{pmatrix}

\partial_x P_x + \partial_y P_y + \partial_z P_z = \partial_r \sum P = \partial_r (n k T_e) = k T_e \partial_r (n) = q n E = q n (-\nabla \Phi)

\downarrow

\partial_r \Phi = \frac{k T_e}{e} \frac{1}{n} \partial_r (n)

\downarrow

\Phi = \frac{k T_e}{e} \ln (n) + \frac{\text{const}}{n} \ln (n_0)

\downarrow

n = n_0 \exp \left( \frac{e \Phi}{k T_e} \right)

\text{Boltzmann Relation}

\begin{array}{c}
\hline
n = n_0 \exp \left( \frac{e \Phi}{k T_e} \right)
\hline
\end{array}

\text{Debye Length}

We can now calculate the Debye length – an effective length over which a plasma will shield an electric field. (The length is the 1/e distance for reducing a potential.)

First, we have Poisson’s equation

\nabla^2 \Phi = -\frac{P}{\varepsilon} = \frac{e}{\varepsilon} (n_e - n_i)

We make the further assumption that the density of the electrons in absence of the potential is the same as the ion density. (We make this assumption because if either the electrons or the ions were to leave an area, a significant electric field would be setup to try to pull them back together.) Thus,

n_i = n_0

Plugging this and the Boltzmann relation into Poisson’s equation gives

\nabla^2 \Phi = \frac{e n_0}{\varepsilon} \left[ \exp \left( \frac{e \Phi}{k T} \right) - 1 \right]

Now, using the Taylor series expansion of the exponential, which we assume is approximately 1, gives
∇\Phi = \frac{en_0}{\varepsilon} \left( 1 + \frac{e\Phi}{kT} + O\left( \frac{e\Phi}{kT} \right)^2 - 1 \right)

\approx \frac{e^2 n_0}{\varepsilon kT} \Phi

Solving the differential equation leaves

\Phi \approx \Phi_0 \exp \left( \frac{-|r|}{\lambda_{\text{Debye}}} \right) \text{ where}

\lambda_{\text{Debye}} = \left( \frac{e^2 n_0}{\varepsilon kT} \right)^{-\frac{1}{2}}