12.6 QUADRATIC SURFACES

DEF A QUADRATIC SURFACE is the set of points \((x, y, z)\) in \(\mathbb{R}^3\) which satisfy a QUADRATIC EQUATION in \(x, y, z\):

\[ Q(x, y, z) = 0 \] (A LEVEL SET EQUATION)

Different choices of coefficients in \(Q\) give different shaped surfaces.

EX 1 \[ x^2 + y^2 + z^2 - 4 = 0 \] SPHERE

2 \[ -z + 4x^2 + y^2 = 0 \] ELLIPTIC PARABOLOID (BOWL)

3 \[ y^2 - x^2 - z = 0 \] HYPERBOLIC PARABOLOID (SADDLE SURFACE)

4 \[ x^2 + y^2 = 4 \] CYLINDER
WARM UP  Families of Quadratic Curves in $\mathbb{R}^2$

**Ex 1**  \[ y = x^2 + k \] for different parameters $k$

**Parabolas**

\[ k = \pm 1 \]

"Translate $y = x^2$ up by $k"$

**Ex 2**  \[ x^2 + \left( \frac{y}{2} \right)^2 = k^2 \]

Intercepts
- If $y = 0$, $x = \pm k$
- If $x = 0$, $y = \pm 2k$

Concentric Ellipses
3) Hyperbola: \( y^2 - 4x^2 = k \)

If \( k = 0 \) get \( y = \pm 2x \) asymptotes.

If \( k = 1 \) then \( x = \infty, y = \pm 1 \)

If \( k = -1 \) then \( y = 0, x = \pm \frac{1}{2} \)

As \( |k| \) increases, the asymptotes move out along axes and the eccentricity remains same.

Exs of Quadric Surface

1) \( x^2 + y^2 + z^2 = r^2 \)

Sphere: radius \( r \), center origin.
\( z = 4x^2 + y^2 \)

Elliptic Paraboloid

To make this picture, I sliced the surface with planes parallel to the coordinate planes to get curves in these planes.

\[
\begin{align*}
    z &= k \\
    4x^2 + y^2 &= k \\
    x^2 + \left( \frac{y}{2} \right)^2 &= \frac{k}{4}
\end{align*}
\]

\( k > 0 \)

\( k = 0 \)

\( k < 0 \)

Ellipses
\[ x = \frac{1}{k} \quad \text{and} \quad y = \frac{1}{k} \]

\[ Z = 4k^2 + y^2 \quad \text{for} \quad k = k \]

\[ Z = 4x^2 + z^2 \quad \text{for} \quad k = 0 \]

Steep paraboloids \( z \equiv k \)

3. Hyperboloid Paraboloid (Saddle Surface)

\[ Z = \frac{y^2}{k^2} - \frac{x^2}{k^2} \quad \text{for} \quad k > 0 \]

\[ y^2 - x^2 = k^2 \]

\[ \left( \frac{y}{k} \right)^2 - \left( \frac{x}{k} \right)^2 = 1 \]

\[ Z = -\frac{1}{k^2} \quad \text{for} \quad k > 0 \]

\[ \left( \frac{x}{k} \right)^2 - \left( \frac{y}{k} \right)^2 = 1 \]
Putting it all together

See slides too

4. Ellipsoid

\[ \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 + \left( \frac{z}{c} \right)^2 = 1 \]

Slices in \( x = \pm a, y = \pm b, z = \pm c \) are ellipses.
\( z^2 = 9x^2 + y^2 \)

**DOUBLE ELLIPTICAL CONE**

\[ z = k, \quad 9x^2 + y^2 = k^2 \]

\[ \left( \frac{3x}{k} \right)^2 + \left( \frac{y}{k} \right)^2 = 1 \]

\[ y = 0 \rightarrow z = \pm 3x \]

\[ z^2 = y^2 \rightarrow z = \pm y \]

**Ellipse**

\( \in \Theta \)
\[
x^2 + y^2 = 1
\]

\[
x^2 + y^2 = 4
\]

Slice at \(x = k\) and get

\[
y^2 + k^2 = 1
\]

For \(k = 0\), \(k > 0\) and \(k < 0\):

\[
x^2 + y^2 - (x^2 - 1) = 1
\]

For \(k > 1\), \(k < 1\):

\[
-x^2 + y^2 - (x^2 - 1) = -1
\]

What is this?