14.2 LIMITS + CONTINUITY

Recall if \( z = f(x,y) \) we can sketch the graph of \( f \) which is a surface in space.

Height of surface above \((x,y)\) is \( z = f(x,y)\).

LIMITS IN CALCULUS I

Suppose \( y = f(x) \) is defined for all \( x \) except maybe at \( x = 0 \).

\[ \lim_{x \to 0} f(x) = \frac{\sin x}{x} \]

Question: What happens to values of \( f \) as \( x \) gets close to 0?
We say \[ \lim_{x \to 0} f(x) = L \] exists if you can make values \( f(x) \) as close as you like to \( L \) provided you choose \( x \) close enough to 0.

\[ \text{I want to be this close to } L \]

\[ \text{need } x \text{ to be in this small interval about } 0. \]

**Calculus III**

The same idea works for \( z = f(xy) \):

We say \[ \lim_{(x,y) \to (0,0)} f(x,y) = L \] exists if you can make values \( f(x,y) \) as close as you like to \( L \) provided you choose point \( (x,y) \) close enough to \( (0,0) \).
\[ z = f(x, y) = x^2 + y^2 \]

\[ \lim_{(x, y) \to (0, 0)} f(x, y) = 0 \quad \text{since} \]

If unit values of \( f \) close to 0

Then need to pick \((x, y)\) in this disc.

The smaller we want \( f(x, y) \) to be \( \to 0 \)

The smaller we need to pick this disc.

**Calculus I**

Methods to show \( \lim_{x \to 0} f(x) = L \) exists

1. Suppose \( f(x) = \frac{g(x)}{h(x)} \) where \( g, h \) are continuous and \( h(0) \neq 0 \).

Then

\[ \lim_{x \to 0} \frac{g(x)}{h(x)} = \frac{g(0)}{h(0)} \quad \text{"plug in."} \]
works in calc III too:

\[ \lim \frac{x^2 + 3e^{-y}}{\cos x + y^2 + 5} = \frac{0^2 + 3 \cdot 0}{1 + 0^2 + 5} = \frac{3}{6} = \frac{1}{2}. \]

2) Suppose \( f(x) = \frac{g(x)}{h(x)} \) and \( f(0) = \frac{0}{0} \).

First Try + Factor + Cancel:

\[ \lim_{x \to 0} \frac{3x + 2x^2}{x} \]

\[ = \lim_{x \to 0} \frac{x(3 + 2x)}{x} \]

\[ = \lim_{x \to 0} (3 + 2x) \]

\[ = 3 + 2 \cdot 0 = 3 \]

Works in calc III too:

\[ \lim \frac{x^2 - y^2}{x+y} \]

\[ = \lim \frac{(x-y)(x+y)}{x+y} \]

\[ = \lim_{(x,y) \to (0,0)} x-y = 0 \]

\[ \text{Factor} \]

\[ \text{Cancel} \]

Now plug in
\[ \lim_{x \to 0} \frac{\sin(x^2)}{x} = \lim_{(x,y) \to (0,0)} \left[ \frac{\sin(x^2)}{x^2} \right] x \]

\[ = \left[ \lim_{(x,y) \to (0,0)} \frac{\sin(x^2)}{x^2} \right] \left[ \lim_{(x,y) \to (0,0)} \frac{x^2}{x} \right] \]

\[ = \left[ \lim_{u \to 0} \frac{\sin(u)}{u} \right] \left[ \lim_{(x,y) \to (0,0)} x \right] \text{ (with } u = x^2 \text{)} \]

\[ = 1 \times 0 = 0 \]

\[ \text{4) L'Hopital's Rule} \]

\(- \text{ NEVER USE THIS IN CALCULUS IT DOESN'T WORK.} \)
**Definition:** We say \( z = f(x,y) \) is continuous at \( (a,b) = (0,0) \) if
\[
\lim_{{(x,y) \to (0,0)}} f(x,y) = f(0,0)
\]

Functions \( z = f(x,y) \) built out of polynomials, trig, exponential functions involving \( x, y \) are continuous where they are defined.

\[
z = \frac{\sin(x+y) + e^{x^2 - 2}}{1 + x^2 + y^2}
\]

is continuous at all \((x,y)\) in \(\mathbb{R}^2\).

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**A Method to Show Limits DNE**

**Calculus I**

If \( \lim_{{x \to a^-}} f(x) \neq \lim_{{x \to a^+}} f(x) \) \( \text{Left} \neq \text{Right Limit} \)

then \( \lim_{{x \to a}} f(x) \) DNE.

\[
f(x) = \begin{cases} 
1 & x > 0 \\
0 & x \leq 0
\end{cases}
\]

\[
\lim_{{x \to 0^-}} f(x) = 0 \quad \lim_{{x \to 0^+}} f(x) = 1
\]

But \( \lim_{{x \to 0}} f(x) \nexists \).
**CALCULUS II**

For

\[ \lim_{(x,y) \to (0,0)} f(x,y) \]

There are \( \infty \) ways to go to \((0,0)\).

- **Every curve in \((x,y)\)-plane that goes to \((0,0)\) gives you a way.**

So if you can find 2 curves \(C_1, C_2\) so that

\[ \lim_{(x,y) \to (0,0)} f(x,y) = L_1 \]

along \(C_1\),

\[ \lim_{(x,y) \to (0,0)} f(x,y) = L_2 \]

along \(C_2\),

and \( L_1 \neq L_2 \) then \( \lim_{(x,y) \to (0,0)} f(x,y) \) does not exist.

But if \( L_1 = L_2 \) that does not mean limit exists.

Maybe there is another curve that gives different limit.
\[ \lim_{(x,y) \to (0,0)} \frac{2xy}{x^2+y^2} \]

Approach \((0,0)\) along lines \(y = kx\) for different \(k\).

\[ \lim_{(x,y) \to (0,0)} \frac{2xy}{x^2+y^2} = \lim_{x \to 0} \frac{2x(kx)}{x^2+(kx)^2} \]

Along \(y = kx\)

\[ = \lim_{x \to 0} \frac{2kx^2}{x^2(1+k^2)} = \lim_{x \to 0} \frac{2k}{1+k^2} \]

\[ = \frac{2k}{1+k^2}. \]

So different lines give different limits.

WHAT ABOUT

\[ \lim_{(x,y) \to (0,0)} \frac{2x^2y}{x^4+y^2} \]

WHAT DOES GRAPH OF \(f\) LOOK LIKE? I HAVE A IDEA OF IT!
Try \( y = kx \)

\[
\lim_{{(x,y) \to (0,0)}} \frac{2x^2y}{x^4 + y^2} = \lim_{{x \to 0}} \frac{2kx^3}{x^4 + k^2x^2}
\]
Along \( y = kx \)

\[
= \lim_{{x \to 0}} \frac{2kx^3}{x^2(x^2 + 1)} = \lim_{{x \to 0}} \frac{2kx}{x^2 + k^2}
\]

Plug \( x = 0 \)

\[
= \frac{2k \cdot 0}{0^2 + k^2} = 0 = 0 \quad \text{provided} \; k \neq 0.
\]

And if \( k = 0 \) Then

\[
\lim_{{(x,y) \to (0,0)}} \frac{2x^2y}{x^4 + y^2} = \lim_{{x \to 0}} \frac{0}{x^4} = 0
\]

Along \( y = 0 \)

So NO MATTER WHICH LINE you come into \((0,0)\), ALONG \( \overline{\text{ALWAYS GET TO height}} \; 0 \).

Does this tell us LIMIT is 0?

\( \overline{\text{No!}} \) Maybe if we come in along another curve we will end up at a different height.
In fact:

$$\lim_{(x,y) \to (0,0)} \frac{2x^2y}{x^4+y^2} = \lim_{x \to 0} \frac{2x^2}{x^4 + k^2x^4}$$

Along $y = kx^2$

$$= \lim_{x \to 0} \frac{2k}{1 + k^2} = \frac{2k}{1 + k^2}$$

Depends on $k$.

So limit DNE.

Along $C_1$, $L_1 = \frac{2x^2}{1 + 1} = 1$

But along $C_2$, $L_2 = \frac{2x^2}{1 + 2} = \frac{4}{5}$.

Since $L_1 \neq L_2$

$$\lim_{(x,y) \to (0,0)} \frac{2x^2y}{x^4+y^2} \text{ DNE.}$$

What does graph of $f$ look like?

I have a model of it!
ONE FINA METHOD CONVERT TO POLAR COORDS

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]

No matter how \((x, y) \to (0, 0)\) we know
\[ r = \sqrt{x^2 + y^2} \to 0 \quad \text{holds.} \]

EX
\[ z = f(x, y) = \frac{2xy}{x^2 + y^2} \]

\[ \lim_{(x, y) \to (0, 0)} \frac{2xy}{x^2 + y^2} = \lim_{r \to 0} \frac{2r^3 \cos \theta \sin \theta}{r^2} \]
\[ = \lim_{r \to 0} 2r \cos \theta \sin \theta = 0 \]

So \(-2r \leq 2r \cos^2 \theta \leq 2r\)

Apply Sandwich Theorem

But:
\[ \lim_{(x, y) \to (0, 0)} \frac{2xy}{x^2 + y^2} = \lim_{r \to 0} \frac{2r^2 \cos \theta \sin \theta}{r^2} \]
\[ = \lim_{r \to 0} (2\cos 2\theta) \quad \text{depends on } \theta \quad \text{and} \quad \text{so limit DNE.} \]